STATISTICS IN TRANSITION-new series, March 2012
Vol. 13, No. 1, pp. 21—36

ESTIMATION OF POPULATION MEAN USING TWO AUXILIARY SOURCES IN SAMPLE SURVEYS

Diwakar Shukla[1], Sharad Pathak[1] and Narendra Singh Thakur[2]

ABSTRACT

This paper proposes families for estimation of population mean of the main variable under study using the information on two different auxiliary variables under simple random sampling without replacement (SRSWOR) scheme. Three different classes of estimators are constructed, examined with a complete study with other existing estimators. The expression for bias and mean squared error of the proposed families are obtained up to first order of approximation. Usual ratio estimator, product estimator, dual to ratio estimator, ratio-cum-product type estimator and many more estimators are identified as particular members of the suggested family. Expressions of optimization are derived and theoretical results are supported by numerical examples.

Key words: Family of estimators, SRSWOR, Bias and Mean squared error. **AMS Subject Classification:** 94A20, 62D05

1. Introduction

To improve the exactitude in sample surveys theory the use of two auxiliary variables for estimation of population mean of a variable under study has played an influential role. A number of estimators are accessible in the literature of sample surveys where supporting information is the contributor to improve the methodology. Out of all ratio and product estimators are good examples as evidence to state this. The ratio estimation method is practical when the correlation coefficient between the study and auxiliary variable is positive [Cochran (1940, 42)]. If the correlation coefficient between the study and auxiliary variable is negative then the use of product estimation will make the study valuable [Robson (1957) and Murthy (1964)].

[1] Department of Mathematics and Statistics. Dr. Hari Singh Gour Central University, Sagar, M.P., India - 470003. E-mail: sharadpathakstats@yahoo.com, diwakarshukla@rediffmail.com.
2 Center for Mathematical Sciences, Banasthali University, Rajasthan, India.

There are so many situations in survey sampling where the record of more auxiliary variable is available for the investigators (at least for two variables). There are so many researchers who used the information of more than two auxiliary variables to contribute in the field. Dalabehara and Sahoo (1994) presented a class of estimators in stratified sampling with two auxiliary variables for estimation of mean. In another contribution Dalabehara and Sahoo (2000) proposed an unbiased estimator in two-phase sampling using two auxiliary variables.

Abu-Dayyeh et al. (2003) used auxiliary variables to show estimators of finite population mean. Sahoo and Sahoo (1993) suggested a class of estimators in two-phase sampling using two auxiliary variables. In another work Sahoo and Sahoo (2001) discussed about predictive estimation of finite population mean in two-phase sampling using two auxiliary variables. Singh and Shukla (1987) have a discussion on one parameter family of factor type ratio estimator. In a study Shukla et al. (1991) transformed factor type estimator to make the estimation more effective. Shukla (2002) Studied F-T estimator and sampling procedure undertaken was two-phase sampling. In this sequence Singh and Singh (1991) provided Chain type estimator with two auxiliary variables under double sampling scheme. In another study Singh et al. (1994) suggested a class of chain-ratio estimator with two auxiliary variables and the study completed under double sampling scheme. Kadilar and Cingi (2004) took two auxiliary variables in simple random sampling to find population mean. Moreover, Kadilar and Cingi (2005) derived a new estimator using two auxiliary variables. Perri (2007) analysed the work of Singh (1965, 1967b) and suggested a new improved work on ratio-cum-product type estimators with the application of Srivenkataramana, T. (1980) estimator on pervious proposed work of Singh (1967b).

Many authors including Srivastava (1971), Srivastava and Jhajj (1983), Ray and Sahai (1980), Khare and Srivastava (1981), Hansen et al.(1953) and Desraj (1965) used more than one supporting information to make the study more impressive. Some other useful contributions over applications of auxiliary information are due to Mukhopadhyay (2000), Cochran (2005), Murthy (1976), Sukhatme et al. (1984), Naik and Gupta (1991), Singh and Shukla (1993) and Shukla et al (2009) etc.

2. Notations and Assumptions

Notations for the study are:

$\bar{Y}, \bar{X}_1, and\ \bar{X}_2$: Population Parameters

$\bar{y}, \bar{x}_1\ and\ \bar{x}_2$: Mean per unit estimates for a simple random sample of size n.

n : Sample size

f : Sampling friction ($f = n/N$)

N : Population size

ρ_{01} : Correlation between variable *Y* and X_1

ρ_{02} : Correlation between variable *Y* and X_2

ρ_{12} : Correlation between variable X_1 and X_2

$C_Y = S_Y / \bar{Y}$: Coefficient of variation for variable Y (C_0)

$C_{X_1} = S_{X_1} / \bar{X}_1$: Coefficient of variation for variable X_1 (C_1)

$C_{X_2} = S_{X_2} / \bar{X}_2$: Coefficient of variation for variable X_2 (C_2)

3. Some Estimators

In the literature of survey sampling so many estimators and estimation procedures exist. This literature is the basic motivation to work in this direction and contribution in this area. Let Y is the main variable and X_1, X_2 are two auxiliary variable then some well known estimators are as follows.

3.1. Ratio estimator

$$\bar{y}_R = \bar{y}\left(\frac{\bar{X}}{\bar{x}}\right) \tag{3.1a}$$

$$Bias(\bar{y}_R) = E(\bar{y}_R - \bar{Y}) = \bar{Y}M_1\left[C_X^2 - \rho C_Y C_X\right] \tag{3.1b}$$

$$MSE(\bar{y}_R) = \bar{Y}^2 M_1\left[C_Y^2 + C_X^2 - 2\rho C_Y C_X\right] \; ; \; M_1 = \left(\frac{1}{n} - \frac{1}{N}\right) \tag{3.1c}$$

3.2. Product estimator

$$\bar{y}_P = \bar{y}\left(\frac{\bar{x}}{\bar{X}}\right) \tag{3.2a}$$

$$Bias(\bar{y}_P) = E(\bar{y}_P - \bar{Y}) = \bar{Y}M_1 \rho C_Y C_X \tag{3.2b}$$

$$MSE(\bar{y}_R) = \bar{Y}^2 M_1\left[C_Y^2 + C_X^2 + 2\rho C_Y C_X\right] \; ; \; M_1 = \left(\frac{1}{n} - \frac{1}{N}\right) \tag{3.2c}$$

3.3. Dual to ratio estimator [By Srivenkataramana, T. (1980)]

$$\bar{y}_{VR} = \bar{y}\frac{N\bar{X} - n\bar{x}}{(N-n)\bar{X}} \tag{3.3a}$$

$$Bias(\bar{y}_{VR}) = E(\bar{y}_{VR} - \bar{Y}) = -\frac{\bar{Y}}{N}\rho C_Y C_X \tag{3.3b}$$

$$MSE(\bar{y}_{VR}) = \bar{Y}^2 M_1\left[C_Y^2 + \alpha^2 C_X^2 - 2\alpha\rho C_Y C_X\right] ;M_1 = \left(\frac{1}{n} - \frac{1}{N}\right), \alpha = n/(N-n)$$
… (3.3c)

3.4. Ratio-cum-product type estimator

Singh (1965, 1967b) proposed some ratio-cum-product type estimators as

$$\bar{y}_{R1} = \bar{y}\frac{\bar{X}_1}{\bar{x}_1}\frac{\bar{x}_2}{\bar{X}_2} \tag{3.4a}$$

$$Bias(\bar{y}_{R1}) = E(\bar{y}_R - \bar{Y}) = \bar{Y}M_1\left[C_1^2 + \rho_{02}\, C_0 C_2 - \rho_{01}\, C_0 C_1 - \rho_{12}\, C_1 C_2\right] \tag{3.4b}$$

$$MSE(\bar{y}_{R1}) = \bar{Y}^2 M_1\left[C_0^2 + C_1^2 + C_2^2 - 2\rho_{01}\, C_0 C_1 + 2\rho_{02}\, C_0 C_2 - 2\rho_{12}\, C_1 C_2\right] \tag{3.4c}$$

$$\bar{y}_{R2} = \bar{y}\frac{\bar{X}_1}{\bar{x}_1}\frac{\bar{X}_2}{\bar{x}_2} \tag{3.5a}$$

$$Bias(\bar{y}_{R2}) = E(\bar{y}_{R2} - \bar{Y}) = \bar{Y}M_1\left[C_1^2 + C_2^2 - \rho_{02}\, C_0 C_2 - \rho_{01}\, C_0 C_1 + \rho_{12}\, C_1 C_2\right] \tag{3.5b}$$

$$MSE(\bar{y}_{R2}) = \bar{Y}^2 M_1\left[C_0^2 + C_1^2 + C_2^2 - 2\rho_{01}\, C_0 C_1 - 2\rho_{02}\, C_0 C_2 + 2\rho_{12}\, C_1 C_2\right] \tag{3.5c}$$

$$\bar{y}_{P1} = \bar{y}\frac{\bar{x}_1}{\bar{X}_1}\frac{\bar{x}_2}{\bar{X}_2} \tag{3.6a}$$

$$Bias(\bar{y}_{P1}) = E(\bar{y}_{P1} - \bar{Y}) = \bar{Y}M_1\left[\rho_{01}\, C_0 C_1 + \rho_{02}\, C_0 C_2 + \rho_{12}\, C_1 C_2\right] \tag{3.6b}$$

$$MSE(\bar{y}_{P1}) = \bar{Y}^2 M_1\left[C_0^2 + C_1^2 + C_2^2 + 2\rho_{01}\, C_0 C_1 + 2\rho_{02}\, C_0 C_2 + 2\rho_{12}\, C_1 C_2\right] \tag{3.6c}$$

$$\bar{y}_{P2} = \bar{y}\frac{\bar{x}_1}{\bar{X}_1}\frac{\bar{X}_2}{\bar{x}_2} \tag{3.7a}$$

$$Bias(\bar{y}_{P2}) = E(\bar{y}_{P2} - \bar{Y}) = \bar{Y}M_1\left[C_2^2 + \rho_{01}\, C_0 C_1 - \rho_{02}\, C_0 C_2 - \rho_{12}\, C_1 C_2\right] \tag{3.7b}$$

$$MSE(\bar{y}_{P2}) = \bar{Y}^2 M_1\left[C_0^2 + C_1^2 + C_2^2 + 2\rho_{01}\, C_0 C_1 - 2\rho_{02}\, C_0 C_2 - 2\rho_{12}\, C_1 C_2\right] \tag{3.7c}$$

where $M_1 = \left(\frac{1}{n} - \frac{1}{N}\right)$

4. Proposed Estimator(s)

Singh and Shukla (1987) discussed a family of factor-type *(F-T)* ratio estimator for estimating population mean. In another contribution Singh and Shukla (1993) derived efficient factor-type estimator for estimating the same population parameter. Deriving motivation from both some proposed estimators are given below.

$$\left.\begin{aligned} (\bar{y}_{F-T})_1 &= \bar{y}\,T_1 T_2 \\ (\bar{y}_{F-T})_2 &= \bar{y}\frac{T_1}{T_2} \\ (\bar{y}_{F-T})_3 &= \bar{y}\frac{T_2}{T_1} \end{aligned}\right\} \qquad (4.1)$$

Where $T_i = \dfrac{(A_i + C_i)\bar{X}_i + fB\,\bar{x}_i}{(A_i + fB_i)\bar{X}_i + C_i\bar{x}_i}$ (4.2)

$$A_i = (K_i - 1)(K_i - 2); B_i = (K_i - 1)(K_i - 4); C_i = (K_i - 2)(K_i - 3)(K_i - 4) \qquad \ldots (4.3)$$

Remark 4.1 Here we have a combination of K_i where $i = (1,2)$. Some of the factors are shown in the following table where $(K_1 = K_2)$. As above concerned K_i where $i = (1,2)$ is constant to choose suitably so that the resulting mean squared error of proposed estimators may become least. For example let $K_i = 1$ then the values of T_1 and T_2 will be $\dfrac{\bar{X}_1}{\bar{x}_1}$ and $\dfrac{\bar{X}_2}{\bar{x}_2}$ respectively and so on.

Remark 4.2 By proposed estimator we can obtain so many different estimators. For each combination of (K_1, K_2) an estimator exists.

Table 4.1. Some Members of the proposed estimation.

$t_1 = \bar{y}\dfrac{\bar{X}_1}{\bar{x}_1}\dfrac{\bar{X}_2}{\bar{x}_2}$ (At $K_1 = K_2 = 1$)	$t_2 = \bar{y}\dfrac{\bar{X}_1}{\bar{x}_1}\dfrac{\bar{x}_2}{\bar{X}_2}$ (At $K_1 = 1, K_2 = 2$)	$t_3 = \bar{y}\dfrac{\bar{X}_1}{\bar{x}_1}\dfrac{N\bar{X}_2 - n\bar{x}_2}{(N-n)\bar{X}_2}$ (At $K_1 = 1, K_2 = 3$)	$t_4 = \bar{y}\dfrac{\bar{X}_1}{\bar{x}_1}$ (At $K_1 = 1, K_2 = 4$)
$t_5 = \bar{y}\dfrac{\bar{x}_1}{\bar{X}_1}\dfrac{\bar{X}_2}{\bar{x}_2}$ (At $K_1 = 2, K_2 = 1$	$t_6 = \bar{y}\dfrac{\bar{x}_1}{\bar{X}_1}\dfrac{\bar{x}_2}{\bar{X}_2}$ (At $K_1 = K_2 = 2$)	$t_7 = \bar{y}\dfrac{\bar{x}_1}{\bar{X}_1}\dfrac{N\bar{X}_2 - n\bar{x}_2}{(N-n)\bar{X}_2}$ (At $K_1 = 2, K_2 = 3$)	$t_8 = \bar{y}\dfrac{\bar{x}_1}{\bar{X}_1}$ (At $K_1 = 2, K_2 = 4$)

Table 4.1. Some Members of the proposed estimation (cont.).

$t_9 = \bar{y}\dfrac{N\bar{X}_1 - n\bar{x}_1}{(N-n)\bar{X}_1}\dfrac{\bar{X}_2}{\bar{x}_2}$ (At $K_1=3, K_2=1$)	$t_{10} = \bar{y}\dfrac{N\bar{X}_1 - n\bar{x}_1}{(N-n)\bar{X}_1}\dfrac{\bar{x}_2}{\bar{X}_2}$ At $K_1=3, K_2=2$	$t_{11} = \bar{y}\dfrac{N\bar{X}_1 - n\bar{x}_1}{(N-n)\bar{X}_1}\dfrac{N\bar{X}_2 - n\bar{x}_2}{(N-n)\bar{X}_2}$ (At $K_1=K_2=3$)	$t_{12} = \bar{y}\dfrac{N\bar{X}_1 - n\bar{x}_1}{(N-n)\bar{X}_1}$ (At $K_1=3, K_2=4$)
$t_{13} = \bar{y}\dfrac{\bar{X}_2}{\bar{x}_2}$ (At $K_1=4, K_2=1$)	$t_{14} = \bar{y}\dfrac{\bar{x}_2}{\bar{X}_2}$ (At $K_1=4, K_2=2$)	$t_{15} = \bar{y}\dfrac{N\bar{X}_2 - n\bar{x}_2}{(N-n)\bar{X}_2}$ (At $K_1=4, K_2=3$)	$\bar{y}$ (At $K_1=K_2=4$)

5. Properties of Proposed Estimator

For large sample approximation we assume that

$$\bar{y} = \bar{Y}(1+e_0);\ \bar{x}_1 = \bar{X}_1(1+e_1)\ ;\ \bar{x}_2 = \bar{X}_2(1+e_2)\,;\alpha_i = \frac{fB_i}{A_i + fB_i + C_i}\ ;$$

$$\beta_i = \frac{C_i}{A_i + fB_i + C_i}$$

$$E(e_0) = E(e_1) = E(e_2) = 0;\ E(e_0^2) = M_1C_0^2;\ E(e_1^2) = M_1C_1^2;$$

$$E(e_2^2) = M_1C_2^2;\delta_{1i} = \alpha_i - \beta_i$$

$$E(e_0e_1) = M_1\rho_{01}C_0C_1; E(e_0e_2) = M_1\rho_{02}C_0C_2; E(e_1e_2) = M_1\rho_{12}C_1C_2;$$

$$M_1 = \left(\frac{1}{n} - \frac{1}{N}\right)$$

THEOREM 5.1:

[1]: The estimator $(\bar{y}_{F-T})_1$ in terms of e_0, e_1 and e_2 up to first order of approximation could be expressed as:

$$(\bar{y}_{F-T})_1 = \bar{Y}\left[1 + e_0 + \delta_1(e_1 + e_0e_1 - \beta_1{e_1}^2) + \delta_2(e_2 + e_0e_2 - \beta_2{e_2}^2) + \delta_1\delta_2e_1e_2\right] \tag{5.1}$$

[2]: Bias of $(\bar{y}_{F-T})_1$ up to first order approximation is:

$$B(\bar{y}_{F-T})_1 = \bar{Y}M_1\left[\delta_1(\rho_{01}C_0C_1 - \beta_1{C_1}^2) + \delta_2(\rho_{02}C_0C_2 - \beta_2{C_2}^2) + \delta_1\delta_2\rho_{12}C_1C_2\right] \tag{5.2}$$

[3]: Mean squared error of $(\bar{y}_{F-T})_1$ up to first order approximation is:

$$M(\bar{y}_{F-T})_1 = \bar{Y}^2M_1\left[{C_0}^2 + {\delta_1}^2{C_1}^2 + {\delta_2}^2{C_2}^2 + 2\delta_1\rho_{01}C_0C_1 + 2\delta_2\rho_{02}C_0C_2 + 2\delta_1\delta_2\rho_{12}C_1C_2\right]$$
… (5.3)

Proof 5.1:

[1]:

$$(\bar{y}_{F-T})_1 = \bar{y}\,\frac{(A_1+C_1)\bar{X}_1 + fB_1\,\bar{x}_1}{(A_1+fB_1)\bar{X}_1 + C_1\bar{x}_1}\,\frac{(A_2+C_2)\bar{X}_2 + fB_2\,\bar{x}_2}{(A_2+fB_2)\bar{X}_2 + C_2\bar{x}_2}$$

$$(\bar{y}_{F-T})_1 = \bar{Y}\,(1+e_0)(1+\alpha_1 e_1)(1+\alpha_2 e_2)(1+\beta_1 e_1)^{-1}(1+\beta_2 e_2)^{-1}$$

$$(\bar{y}_{F-T})_1 = \bar{Y}\left[1+e_0+\delta_1(e_1+e_0e_1-\beta_1 e_1{}^2)+\delta_2(e_2+e_0e_2-\beta_2 e_2{}^2)+\delta_1\delta_2 e_1 e_2\right]$$

[2]:

$$E\left[(\bar{y}_{F-T})_1-\bar{Y}\right]=E\left[\bar{Y}\,\{e_0+\delta_1(e_1+e_0e_1-\beta_1 e_1{}^2)+\delta_2(e_2+e_0e_2-\beta_2 e_2{}^2)+\delta_1\delta_2 e_1 e_2\}\right]$$

$$B(\bar{y}_{F-T})_1=\bar{Y}M_1\left[\delta_1\,(\rho_{01}C_0C_1-\beta_1 C_1{}^2)+\delta_2\,(\rho_{02}C_0C_2-\beta_2 C_2{}^2)+\delta_1\delta_2\rho_{12}C_1C_2\right]$$

[3]:

$$\left[(\bar{y}_{F-T})_1-\bar{Y}\right]^2=\bar{Y}\,[e_0+\delta_1(e_1+e_0e_1-\beta_1 e_1{}^2)+\delta_2(e_2+e_0e_2-\beta_2 e_2{}^2)+\delta_1\delta_2 e_1 e_2]^2$$

$$M(\bar{y}_{F-T})_1=\bar{Y}^2M_1[C_0{}^2+\delta_1{}^2C_1{}^2+\delta_2{}^2C_2{}^2+2\delta_1\rho_{01}C_0C_1+2\delta_2\rho_{02}C_0C_2+2\delta_1\delta_2\rho_{12}C_1C_2]$$

THEOREM 5.2:

[4]: The estimator $(\bar{y}_{F-T})_2$ in terms of e_0, e_1 and e_2 up to first order of approximation could be expressed as:

$$(\bar{y}_{F-T})_2=\bar{Y}\left[1+e_0+\delta_1(e_1+e_0e_1-\beta_1 e_1{}^2)-\delta_2(e_2+e_0e_2-\alpha_2 e_2{}^2)-\delta_1\delta_2 e_1 e_2\right] \tag{5.4}$$

[5]: Bias of $(\bar{y}_{F-T})_2$ up to first order approximation is:

$$B(\bar{y}_{F-T})_2=\bar{Y}\,M_1\left[\delta_1 C_1(\rho_{01}C_0-\beta_1 C_1)+\delta_2 C_2(\alpha_2 C_2-\rho_{02}C_0)-\delta_1\delta_2\rho_{12}C_1C_2\right] \tag{5.5}$$

[6]: Mean squared error of $(\bar{y}_{F-T})_2$ up to first order approximation is:

$$M(\bar{y}_{F-T})_2=\bar{Y}^2M_1\left[C_0^2+\delta_1^2C_1^2+\delta_2^2C_2^2+2\delta_1\,\rho_{01}C_0C_1-2\delta_2\,\rho_{02}C_0C_2-2\delta_1\,\delta_2\,\rho_{12}C_1\,C_2\right] \quad \ldots (5.6)$$

Proof 5.2:

[4]:

$$(\bar{y}_{F-T})_2=\bar{y}\,\frac{(A_1+C_1)\bar{X}_1+fB_1\,\bar{x}_1}{(A_1+fB_1)\bar{X}_1+C_1\bar{x}_1}\,\frac{(A_2+fB_2)\bar{X}_2+C_2\bar{x}_2}{(A_2+C_2)\bar{X}_2+fB_2\,\bar{x}_2}$$

$$(\bar{y}_{F-T})_2=\bar{Y}\,(1+e_0)\,(1+\alpha_1 e_1)(1+\beta_2 e_2)(1+\beta_1 e_1)^{-1}(1+\alpha_2 e_2)^{-1}$$

$$(\bar{y}_{F-T})_2=\bar{Y}\left[1+e_0+\delta_1(e_1+e_0e_1-\beta_1 e_1{}^2)-\delta_2(e_2+e_0e_2-\alpha_2 e_2{}^2)-\delta_1\delta_2 e_1 e_2\right]$$

[5]:

$$E\left[(\bar{y}_{F-T})_2 - \bar{Y}\right] = \bar{Y}\,E\left[e_0 + \delta_1(e_1 + e_0e_1 - \beta_1 e_1{}^2) - \delta_2(e_2 + e_0e_2 - \alpha_2 e_2{}^2) - \delta_1\delta_2 e_1 e_2\right]$$

$$B(\bar{y}_{F-T})_2 = \bar{Y}M_1\left[\delta_1 C_1(\rho_{01}C_0 - \beta_1 C_1\ ) + \delta_2 C_2(\alpha_2 C_2 - \rho_{02}C_0) - \delta_1\delta_2\rho_{12}C_1C_2\right]$$

[6]:

$$M(\bar{y}_{F-T})_2 = E\left[(\bar{y}_{F-T})_2 - \bar{Y}\right]^2$$

$$M(\bar{y}_{F-T})_2 = \bar{Y}^2 M_1\left[C_0^2 + \delta_1^2 C_1^2 + \delta_2^2 C_2^2 + 2\delta_1\,\rho_{01}C_0C_1 - 2\delta_2\,\rho_{02}C_0C_2 - 2\delta_1\,\delta_2\,\rho_{12}C_1\,C_2\right]$$

THEOREM 5.3:

[7]: The estimator $(\bar{y}_{F-T})_3$ in terms of e_0, e_1 and e_2 up to first order of approximation could be expressed as:

$$(\bar{y}_{F-T})_3 = \bar{Y}\left[1 + e_0 + \delta_1(\alpha_1 e_1^2 - e_1 - e_0e_1) + \delta_2(e_2 - \beta_2 e_2^2 + e_0e_2) - \delta_1\delta_2 e_1e_2\right] \tag{5.7}$$

[8]: Bias of $(\bar{y}_{F-T})_3$ up to first order approximation is:

$$B(\bar{y}_{F-T})_3 = \bar{Y}M_1\left[\delta_1(\alpha_1 C_1^2 - \rho_{01}C_0C_1) + \delta_2(\rho_{02}C_0C_2 - \beta_2 C_2^2) - \delta_1\delta_2\rho_{12}C_1C_2\right] \tag{5.8}$$

[9]: Mean squared error of $(\bar{y}_{F-T})_3$ up to first order approximation is:

$$M(\bar{y}_{F-T})_3 = \bar{Y}^2 M_1\left[C_0^2 + \delta_1^2 C_1^2 + \delta_2^2 C_2^2 - 2\rho_{01}C_0C_1\delta_1 + 2\rho_{02}C_0C_2\delta_2 - 2\delta_1\delta_2\rho_{12}C_1C_2\right] \tag{5.9}$$

Proof 5.3:

[7]:

$$(\bar{y}_{F-T})_3 = \bar{y}\,\frac{(A_2 + C_2)\bar{X}_2 + fB_2\,\bar{x}_2}{(A_2 + fB_2)\bar{X}_2 + C_2\bar{x}_2}\,\frac{(A_1 + fB_1)\bar{X}_1 + C_1\bar{x}_1}{(A_1 + C_1)\bar{X}_1 + fB_1\,\bar{x}_1}$$

$$(\bar{y}_{F-T})_3 = \bar{Y}(1+e_0)(1+\alpha_2 e_2)(1+\beta_1 e_1)(1+\beta_2 e_2)^{-1}(1+\alpha_1 e_1)^{-1}$$

$$(\bar{y}_{F-T})_3 = \bar{Y}\left[1 + e_0 + \delta_1(\alpha_1 e_1^2 - e_1 - e_0e_1) + \delta_2(e_2 - \beta_2 e_2^2 + e_0e_2) - \delta_1\delta_2 e_1e_2\right]$$

[8]:

$$(\bar{y}_{F-T})_3 = \bar{Y}\left[1 + e_0 + \delta_1(\alpha_1 e_1^2 - e_1 - e_0e_1) + \delta_2(e_2 - \beta_2 e_2^2 + e_0e_2) - \delta_1\delta_2 e_1e_2\right]$$

$$B(\bar{y}_{F-T})_3 = \bar{Y}M_1\left[\delta_1(\alpha_1 C_1^2 - \rho_{01}C_0C_1) + \delta_2(\rho_{02}C_0C_2 - \beta_2 C_2^2) - \delta_1\delta_2\rho_{12}C_1C_2\right]$$

[9]:

$$E\left[(\bar{y}_{F-T})_3 - \bar{Y}\right]^2 = E\left[\bar{Y}\left\{e_0 + \delta_1(\alpha_1 e_1^2 - e_1 - e_0 e_1) + \delta_2(e_2 - \beta_2 e_2^2 + e_0 e_2) - \delta_1\delta_2 e_1 e_2\right\}\right]^2$$

$$M(\bar{y}_{F-T})_3 = \bar{Y}^2 M_1\left[C_0^2 + \delta_1^2 C_1^2 + \delta_2^2 C_2^2 - 2\rho_{01}C_0C_1\delta_1 + 2\rho_{02}C_0C_2\delta_2 - 2\delta_1\delta_2\rho_{12}C_1C_2\right]$$

6. Minimum Mean Squared Error & Optimal Choices for Proposed Estimator(s)

In this proposed estimator we have multiple choices of the combination $K_i; i=(1,2)$ and optimal conditions obtained by mean squared error of all proposed designs.

For minimum mean squared error by $(\bar{y}_{F-T})_1$ differentiating (5.3) with respect to δ_1 and δ_2 respectively and equating to zero.

$$\left.\begin{aligned} C_1^2\delta_1 + C_1C_2\rho_{12}\delta_2 + \rho_{01}C_0C_1 = 0 \\ \rho_{12}C_1C_2\delta_1 + C_2^2\delta_2 + \rho_{02}C_0C_2 = 0 \end{aligned}\right\} \tag{6.1}$$

By solving these simultaneous equations, we have

$$\delta_1 = \frac{C_0}{C_1}\frac{\rho_{02}\rho_{12} - \rho_{01}}{(1-\rho_{12}^2)} = \hat{\delta}_{11} \text{ and } \delta_2 = \frac{C_0}{C_2}\frac{\rho_{01}\rho_{12} - \rho_{02}}{(1-\rho_{12}^2)} = \hat{\delta}_{12} \quad \ldots \tag{6.2}$$

At these values of $\hat{\delta}_{11}$ and $\hat{\delta}_{12}$ the minimum mean square error of the proposed estimator is

$$MSE(\bar{y}_{F-T})_1\Big|_{Min} = \bar{Y}^2 C_0^2 M_1\left[1 + V(V + 2\rho_{01}) + U(U + 2\rho_{02}) + 2UV\rho_{12}\right] \tag{6.3}$$

where $U = \dfrac{\rho_{01}\rho_{12} - \rho_{02}}{(1-\rho_{12}^2)}$ and $V = \dfrac{\rho_{02}\rho_{12} - \rho_{01}}{(1-\rho_{12}^2)}$

By adopting the same procedure we can obtain the minimum mean squared error corresponding to $(\bar{y}_{F-T})_2$ and $(\bar{y}_{F-T})_3$ by (5.6) and (5.9).

The information of optimization regarding $(\bar{y}_{F-T})_2$ and $(\bar{y}_{F-T})_3$ is

$$\hat{\delta}_{21} = \hat{\delta}_{11};\ \hat{\delta}_{22} = -\hat{\delta}_{12};\hat{\delta}_{21} = \hat{\delta}_{11} \text{and}\, \hat{\delta}_{22} = -\hat{\delta}_{12} \tag{6.4}$$

Rewriting (6.2), as

$$\left.\begin{aligned} \hat{\delta}_{11} &= \frac{C_0}{C_1}\frac{\rho_{02}\rho_{12} - \rho_{01}}{(1-\rho_{12}^2)} = \Delta_1(say) \\ \hat{\delta}_{12} &= \frac{C_0}{C_2}\frac{\rho_{01}\rho_{12} - \rho_{02}}{(1-\rho_{12}^2)} = \Delta_2(say) \end{aligned}\right] \tag{6.5}$$

From (6.5) we can obtain the relation in the form of characterizing scalar as follows

$$\left.\begin{aligned} &(\Delta_1+1)K_1^3 + (f\Delta_1 - f - 8\Delta_1 - 9)K_1^2 + (23\Delta_1 - 5f\Delta_1 + 5f + 26)K_1 \\ &\qquad + (4f\Delta_1 - 22\Delta_1 - 4f - 24) = 0 \\ &(\Delta_2+1)K_2^3 + (f\Delta_2 - f - 8\Delta_2 - 9)K_2^2 + (23\Delta_2 - 5f\Delta_2 + 5f + 26)K_2 \\ &\qquad + (4f\Delta_2 - 22\Delta_2 - 4f - 24) = 0 \end{aligned}\right] \quad \ldots (6.6)$$

Above polynomial (6.6) provides three choices of K_1 and K_2 for the minimum mean squared errors of proposed estimators.

In the similar way $\hat{\delta}_{21} = \Delta_1;\ \hat{\delta}_{22} = -\Delta_2;\hat{\delta}_{21} = -\Delta_1$ and $\hat{\delta}_{22} = \Delta_2$ will also provide the polynomials of degree three *i.e.* in each case we have three different choices of constant $K_i; i = 1, 2$ to improve the estimator.

7. Empirical study

The target in this section is to evaluate the gain in efficiencies (in terms of mse) obtained by the proposed estimators. To see the performance of the various estimators discussed here, we are considering two different population data used earlier by other researchers. The empirical analysis is discussed below.

Population - 1 [sources: Anderson (1958)]

y : *Head length of second son*

x_1 : *Head length of first son*

x_2 : *Head breadth of first son*

The required information is given in Table 7.1.

Table 7.1. Population – 1 Parameters.

Parameter	**Value**	**Parameter**	**Value**	**Parameter**	**Value**	**Parameter**	**Value**
$\bar{Y}$	183.84	n	7	C_0	0.0546	ρ_{01}	0.7108
$\bar{X}_1$	185.72	N	25	C_1	0.0526	ρ_{02}	0.6932
$\bar{X}_2$	151.12	f	0.28	C_2	0.0488	ρ_{12}	0.7346

Table 7.2. Percent Relative Efficiency of various estimators with respect to mean per unit estimator for Population – 1.

Estimator(s)	$PRE(\bullet)$ with respect to $\bar{y}$		
	$(\bar{y}_{F-T})_1$	$(\bar{y}_{F-T})_2$	$(\bar{y}_{F-T})_3$
$\bar{y}$	100	100	100
t_1	72.29	75.1	62.8
t_2	75.10	72.29	15.15
t_3	145.04	149.41	40.9
t_4	179.03	179.03	30.32
t_5	62.8	15.15	72.29
t_6	15.15	62.8	75.1
t_7	40.9	22.76	145.04
t_8	30.32	30.32	179.03
t_9	151.64	46.43	135.00
t_{10}	46.43	151.64	23.79
t_{11}	211.67	98.12	89.24
t_{12}	164.53	164.53	59.77
t_{13}	178.66	32.91	178.66
t_{14}	32.91	178.66	32.91
t_{15}	156.51	62.39	156.51
$(\bar{y}_{F-T})_1^*$	**231.90468**	101.6880601	83.46525626
$(\bar{y}_{F-T})_2^*$	101.68806	**231.9046782**	36.9499305
$(\bar{y}_{F-T})_3^*$	83.465256	36.9499305	**231.9046782**

Population - 2 [sources: Steel and Torrie (1960, p.282)]

y : Log of leaf burn in sec

x_1 : Potassium percentage

x_2 : Chlorine percentage

The information regarding population -2 is given in Table 7.3.

Table 7.3. Population - 2 Parameters.

Parameter	Value	Parameter	Value	Parameter	Value	Parameter	Value
$\bar{Y}$	0.6860	n	6	C_0	0.4803	ρ_{01}	0.1794
$\bar{X}_1$	4.6537	N	30	C_1	0.2295	ρ_{02}	-0.4996
$\bar{X}_2$	0.8077	f	0.20	C_2	0.7493	ρ_{12}	0.4074

Table 7.4. Percent Relative Efficiency of various estimators with respect to mean per unit estimator for Population – 1.

Estimator(s)	$\%RE(\bullet)$ with respect to $\bar{y}$		
	$(\bar{y}_{F-T})_1$	$(\bar{y}_{F-T})_2$	$(\bar{y}_{F-T})_3$
$\bar{y}$	100	100	100
t_1	17.67	75.5	20.89
t_2	75.50	17.67	34.69
t_3	57.125	149.82	55.87
t_4	94.61	94.61	71.44
t_5	20.89	34.69	17.67
t_6	34.69	20.89	75.50
t_7	55.87	76.10	57.12
t_8	71.44	71.44	94.61
t_9	19.54	59.01	20.41
t_{10}	59.01	19.54	47.98
t_{11}	64.46	143.70	64.06
t_{12}	102.94	102.94	94.59
t_{13}	20.02	53.33	20.02
t_{14}	53.33	20.02	53.33
t_{15}	64.858	131.16	64.85
$(\bar{y}_{F-T})_1^*$	**174.04**	40.64	70.53
$(\bar{y}_{F-T})_2^*$	40.64	**174.04**	43.93
$(\bar{y}_{F-T})_3^*$	70.53	43.93	**174.04**

8. Discussion & Conclusion

For population-1 the choices to optimization of mean squared error of $(\bar{y}_{F-T})_1$ can be derived from (6.5) which give a polynomial of degree three (6.6). On solution we have $[K_1]_1 = 6.0098$; $[K_1]_2 = 2.9586$; $[K_1]_3 = 1.6115$; $[K_2]_1 = 5.7733$; $[K_2]_2 = 2.9825$ and $[K_2]_3 = 1.634$. For $(\bar{y}_{F-T})_2$ the values are $[K_1]_4 = [K_1]_1$, $[K_1]_5 = [K_1]_2$; $[K_1]_6 = [K_1]_3$ and $[K_2]_4 = 1.9132$. Similarly for $(\bar{y}_{F-T})_3$ values are $[K_1]_7 = 1.9206$; $[K_2]_7 = [K_2]_1$; $[K_2]_8 = [K_2]_2$ and $[K_2]_9 = [K_2]_3$ whereas other roots are imaginary.

For population-2 the choices of the constant scalar K_i to reduce the mean squared error of $(\bar{y}_{F-T})_1$ are $[K_1]_1 = 39.9225$; $[K_1]_2 = 2.5859$; $[K_1]_3 = 1.0972$ and $[K_2]_1 = 1.939$. For $(\bar{y}_{F-T})_2$ the values are $[K_1]_4 = [K_1]_1$, $[K_1]_5 = [K_1]_2$; $[K_1]_6 = [K_1]_3$; $[K_2]_4 = 5.7698$; $[K_2]_5 = 2.8515$ and $[K_2]_6 = 1.6794$. Similarly for $(\bar{y}_{F-T})_3$ values are $[K_1]_7 = 1.9968$ and $[K_2]_7 = [K_2]_1$. The remaining roots are imaginary. $(\bar{y}_{F-T})_1^*$, $(\bar{y}_{F-T})_2^*$ and $(\bar{y}_{F-T})_3^*$ denotes the optimal efficiency gain with respect to mean per unit estimator in the above mentioned tables.

From these results it is certain that the proposed estimators submit a wide ground for the optimization by multiple choices of the characterizing scalar K_i. Since the generation of the estimators by the proposed classes is easy, a number of estimators can be able to achieve for more study. The proposed estimator proposed a wide choice for the characterizing scalar, which is the beauty of the proposed analysis.

By the compilation of the percentage relative efficiencies corresponding to population-1 and 2 shown in table-7.2 and table-7.4 it is clear that the proposed estimators are more efficient than the other existing estimators as ratio estimator, product estimator, dual to ratio estimator, mean per unit estimator, ratio-cum-product type estimator, etc., and many more chain type estimators which are discussed above, with considerable gain in terms of mean square error. Thus, the proposed estimators are recommended for use in practice.

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