STATISTICS IN TRANSITION-new series, December 2012
Vol. 13, No. 3, pp. 519—536

A CLASS OF CHAIN RATIO-CUM-DUAL TO RATIO TYPE ESTIMATOR WITH TWO AUXILIARY CHARACTERS UNDER DOUBLE SAMPLING IN SAMPLE SURVEYS

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ABSTRACT

This paper considers a chain ratio-cum-dual to ratio type estimator for estimating population mean of the study variate using two auxiliary variates under double (two-phase) sampling procedure, when the information on another additional auxiliary variate is available along with the main auxiliary variate. The asymptotically optimum estimators (AOEs) in the class are identified in two different cases with their bias and variances. The optimum values of the first phase and second phase sample sizes have been obtained for the fixed cost of survey. Theoretical and empirical studies have also been done to demonstrate the efficiency of the proposed estimator with respect to strategies which utilized the information on two auxiliary variates.

Key words: auxiliary variate, double sampling, chain ratio-cum-dual to ratio estimator, bias, variance, efficiency.

1. Introduction

The use of auxiliary information in the estimation of population values of the study variate has been a common phenomenon in sampling theory of surveys. Auxiliary information may be fruitfully utilized either at planning stage or at design stage or at the information stage to arrive at improved estimator compared to those, not utilizing auxiliary information. The use of ratio and product strategies in survey sampling solely depends upon the knowledge of population mean $\bar{X}$ of the auxiliary character x. In many situations of practical importance, the population mean $\bar{X}$ is not known before the start of a survey. In such a situation, the usual thing to do is to estimate it by the sample mean $\bar{x}_1$ based on

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a preliminary sample of size n_1 of which n is a subsample $\left(n < n_1\right)$. If the population mean $\bar{Z}$ of another auxiliary variate z, closely related to auxiliary variate x but compared to x remotely related to study variate y is known, it is advisable to estimate $\bar{X}$ by $\bar{X} = \bar{x}_1 \bar{Z} / \bar{z}'$, which would provide better estimate of $\bar{X}$ than $\bar{x}_1$ to the terms of order $o\left(n^{-1}\right)$ if $\rho_{xz} C_x / C_z > 1/2$.

Chand (1975) and Sukhatme and Chand (1977) proposed a technique of chaining the available information on auxiliary characteristics with the main characteristics. Kiregyera (1980, 1984), Singh et al. (2006) also proposed some chain type ratio and regression estimators based on two auxiliary variables. Using prior information on parameters of auxiliary variate/variates, Srivastava and Jhajj (1980, 1995), Isaki (1983), Singh and Kataria (1990), Prasad and Singh (1990, 1992), Ahmed *et al.* (2000) defined estimators or classes of estimators of S_y^2. Using double sampling technique, Singh and Singh (2001) defined a ratio-type estimator of S_y^2. Al-Jararha and Ahmed (2002) defined two classes of estimators of S_y^2 by using prior information on parameter of one of the two auxiliary variables under double sampling scheme. Ahmed *et al.* (2003) gave some chain ratio-type as well as chain product-type estimators of S_y^2, under two-phase sampling scheme. Singh and Tailor (2005) and Tailor and Sharma (2009) worked on ratio-cum-product estimators. Singh *et al.* (2006) proposed chain ratio and regression type estimators for median estimation. Use of a known coefficient of kurtosis of second auxiliary variable in double sampling Singh *et al.* (2009) defined a chain-type estimator of population variance.

Consider a finite population $U = \left(U_1, U_2, ..., U_N\right)$ of N units, y be the study variate, x and z are the two auxiliary variates. If $\bar{X}$ is not known, but $\bar{Z}$ the population mean of another cheaper auxiliary variate z closely related to x but compared to x remotely related to y (i.e. $\rho_{yx} > \rho_{yz}$) is also available . In this case, Chand (1975) defined the chain ratio estimator

$$\bar{y}_R^{(dc)} = \bar{y} \frac{\bar{x}_1}{\bar{x}} \frac{\bar{Z}}{\bar{z}'}$$

where $\bar{x}$ and $\bar{y}$ are the sample mean of x and y respectively based on the sample size n out of the population N units and $\bar{x}_1 = \left(1/n_1\right) \sum_{i=1}^{n_1} x_i$ denote the sample mean of x based on the first-phase sample of the size n_1 and

$\bar{z}' = (1/n_1)\sum_{i=1}^{n_1} z_i$ denote the sample mean based on $n_1 > n$ units of the auxiliary variate z.

Using the transformation $\bar{x}_i^{\sigma} = \left(N\bar{X} - nx_i\right)/(N-n), \left(i = 1, 2, 3, ..., N\right)$, Srivenkataramana (1980) obtained dual to ratio estimator as

$$\bar{y}_R = \bar{y}\frac{\bar{x}^{\sigma}}{\bar{X}}$$

where $\bar{x}^{\sigma} = \left(N\bar{X} - n\bar{x}\right)/(N-n)$.

Kumar *et al.* (2006) proposed a dual to ratio estimator in double sampling

$$\bar{y}_k^{(d)} = \bar{y}\frac{\bar{x}^{\tau}}{\bar{x}_1}$$

where $\bar{x}^{\tau} = \left(n_1\bar{x}_1 - n\bar{x}\right)/(n_1 - n)$ is an unbiased estimator of $\bar{X}$ by using the transformation $\bar{x}_i^{*} = \left(n_1\bar{x}_1 - nx_i\right)/(n_1 - n), \left(i = 1, 2, 3, ..., N\right)$.

Using an additional auxiliary variate z in dual to ratio estimator in double sampling,

Kumar *et al.* (2006) estimator reduces to chain dual to ratio estimator in double sampling as

$$\bar{y}_k^{(dc)} = \bar{y}\frac{\bar{x}^{*}}{\bar{x}_1}\frac{\bar{z}'}{\bar{Z}}$$

where $\bar{x}^{*} = \left(n_1\bar{x}_1\bar{Z}/\bar{z}' - n\bar{x}\right)/(n_1 - n)$.

For estimating the population mean $\bar{Y}$ of the study variate y, recently Sharma *et al.* (2010) considered an estimator of ratio-cum-dual to ratio estimator given by

$$\bar{y}_{RdR} = \bar{y}\left[\alpha\frac{\bar{X}}{\bar{x}} + (1-\alpha)\frac{\bar{x}^{\sigma}}{\bar{X}}\right].$$

where $\bar{X}$ is the known population mean of x and α is suitably chosen constant.

Motivated by Sharma *et al.* (2010), we have proposed a class of chain ratio-cum-dual to ratio type estimator in double sampling for estimating population mean $\bar{Y}$ using two auxiliary characters. Numerical illustrations are given in the support of the present study.

2. The proposed class of estimator

Let instead of $\bar{X}$ the population mean $\bar{Z}$ of another auxiliary variable z which has a positive correlation with x (that is, $\rho_{xz} > 0$) be known. We further assume that $\rho_{yx} > \rho_{yz} > 0$. Let $\bar{x}_1$ and $\bar{z}'$ be the sample means of x and z respectively based on a preliminary sample of size n_1 drawn with simple random sampling without replacement (SRSWOR) strategy in order to get an estimate of $\bar{X}$. Then the proposed class of estimator for estimating $\bar{Y}$ is

$$\bar{y}_{RdR}^{(dc)} = \bar{y}\left[\alpha \frac{\bar{x}_1}{\bar{x}}\frac{\bar{Z}}{\bar{z}'} + (1-\alpha)\frac{\bar{x}^* \bar{z}'}{\bar{x}_1 \bar{Z}}\right] \qquad (1)$$

where α is determined so as to minimize the variance (V) of $\bar{y}_{RdR}^{(dc)}$.

To obtain the bias (B) and variance of $\bar{y}_{RdR}^{(dc)}$, we write

$e_0 = \left(\bar{y} - \bar{Y}\right)/\bar{Y}$, $e_1 = \left(\bar{x} - \bar{X}\right)/\bar{X}$, $e_1' = \left(\bar{x}_1 - \bar{X}\right)/\bar{X}$ and $e_2' = \left(\bar{z}' - \bar{Z}\right)/\bar{Z}$

Expressing $\bar{y}_{RdR}^{(dc)}$ in terms of e's, we have

$$\bar{y}_{RdR}^{(dc)} = \bar{Y}(1+e_0)\Big[\alpha(1+e_1')\left\{1+(e_1+e_2'+e_1e_2')\right\}^{-1}$$
$$+(1-\alpha)\left\{N^* - n^*(1+e_1)\left(1+e_1'-e_2'-e_1'e_2'+e_2'^2\right)^{-1}\right\}\Big] \qquad (2)$$

where $N^* = n_1/(n_1-n)$ and $n^* = n/(n_1-n)$.

We assume that $\left|e_1+e_2'+e_1e_2'\right|<1$ and $\left|e_1'-e_2'-e_1'e_2'+e_2'^2\right|<1$ so that $\left\{1+(e_1+e_2'+e_1e_2')\right\}^{-1}$ and $\left\{1+\left(e_1'-e_2'-e_1'e_2'+e_2'^2\right)\right\}^{-1}$ are expandable.

Expanding the right hand side of (2), multiplying out and retaining terms of e's up to the second degree we obtain

$$\bar{y}_{RdR}^{(dc)} - \bar{Y} \cong \bar{Y}\Big[e_0 - n^*\left(e_1 - e_1' + e_2' + e_0e_1 - e_0e_1' + e_0e_2' - e_1e_1' + e_1e_2' - e_1'e_2' + e_1'^2\right)$$

$$+\alpha\left\{e_1^2 + e_2'^2 + n^*e_1'^2 - N^*\left(e_1 - e_1' + e_2' + e_0e_1 - e_0e_1' + e_0e_2'\right) - N^{**}\left(e_1e_1' - e_1e_2' + e_1'e_2'\right)\right\} \qquad (3)$$

where $N^{**} = (n_1-2n)/(n_1-n)$.

Taking the expectation in (3) and noting that

$$E(e_0) = E(e_1) = E(e_1') = E(e_2') = 0$$

and that the expectations of the second degree terms of order n^{-1}, we obtain

$$E\left\{\bar{y}_{RdR}^{(dc)}\right\} = \bar{Y} + o(n^{-1}).$$

Thus, the bias of the estimator $\bar{y}_{RdR}^{(dc)}$ is of the order n^{-1} and hence its contribution to the variance will of the order of n^{-2}.

To find the bias and variance of $\overline{y}_{RdR}^{(dc)}$, let $C_y^2 = S_y^2 / \overline{Y}^2$, $C_x^2 = S_x^2 / \overline{X}^2$, $C_z^2 = S_z^2 / \overline{Z}^2$, $\rho_{xy} = S_{xy} / S_x S_y$, $\rho_{yz} = S_{yz} / S_y S_z$ and $\rho_{zx} = S_{zx} / S_z S_x$.

where

$S_x^2 = \{1/(N-1)\} \sum_{i=1}^{N} \left(x_i - \overline{X}\right)^2$, $S_y^2 = \{1/(N-1)\} \sum_{i=1}^{N} \left(y_i - \overline{Y}\right)^2$, $S_z^2 = \{1/(N-1)\} \sum_{i=1}^{N} \left(z_i - \overline{Z}\right)^2$,

$S_{yx} = \{1/(N-1)\} \sum_{i=1}^{N} \left(y_i - \overline{Y}\right)\left(x_i - \overline{X}\right)$, $S_{yz} = \{1/(N-1)\} \sum_{i=1}^{N} \left(y_i - \overline{Y}\right)\left(z_i - \overline{Z}\right)$ and $S_{xz} = \{1/(N-1)\} \sum_{i=1}^{N} \left(x_i - \overline{X}\right)\left(z_i - \overline{Z}\right)$.

The following two cases will be considered separately.

Case I: When the second phase sample of size n is a subsample of the first phase of size n_1. With this background, we get the following results

$$\left.\begin{aligned}
&E(e_0) = E(e_1) = E(e_1') = E(e_2') = 0,\\
&E(e_0^2) = \{(1-f)/n\} C_y^2 = M_{0,0}, \quad E(e_1^2) = \{(1-f)/n\} C_x^2 = M_{1,0},\\
&E(e_1'^2) = \{(1-f_1)/n_1\} C_x^2 = M_{2,0}, \quad E(e_2'^2) = \{(1-f_1)/n_1\} C_z^2 = M_{3,0},\\
&E(e_0 e_1) = \{(1-f)/n\} C_{yx} C_x^2 = M_{0,1}, \quad E(e_0 e_1') = \{(1-f_1)/n_1\} C_{yx} C_x^2 = M_{0,2},\\
&E(e_0 e_2') = \{(1-f_1)/n_1\} C_{yz} C_z^2 = M_{0,3}, \quad E(e_1 e_1') = \{(1-f_1)/n_1\} C_x^2 = M_{1,2},\\
&E(e_1 e_2') = \{(1-f_1)/n_1\} C_{xz} C_z^2 = M_{1,3}, \quad E(e_1' e_2') = \{(1-f_1)/n_1\} C_{xz} C_z^2 = M_{2,3}.
\end{aligned}\right\} \tag{4}$$

where $f = n/N$, $f_1 = n_1/N$, $C_{yx} = \rho_{yx} C_y / C_x$, $C_{yz} = \rho_{yz} C_y / C_z$ and $C_{xz} = \rho_{xz} C_x / C_z$.

Taking the expectation in (3) and using results in (4) we get the bias of the estimator $\overline{y}_{RdR}^{(dc)}$ to the first order of approximation as

$$B\left\{\overline{y}_{RdR}^{(dc)}\right\}_I = \overline{Y}\left(\alpha P - \omega Q\right) \tag{5}$$

where $\omega = n^* + \alpha N^{**}$, $P = M_{1,0} - M_{2,0} + M_{3,0}$ and $Q = M_{0,1} - M_{0,2} + M_{0,3}$.

Again from (3), we have

$$\overline{y}_{RdR}^{(dc)} - \overline{Y} \cong \overline{Y}\left[e_0 - N_1^{**}\left(e_1 - e_1' + e_2'\right)\right]. \tag{6}$$

Squaring both sides in (6), then taking the expectation and using the results in (4), we obtained the variance of the estimator $\overline{y}_{RdR}^{(dc)}$ to terms of order n^{-1}, as

$$V\left\{\overline{y}_{RdR}^{(dc)}\right\}_I = \overline{Y}^2\left(M_{0,0} - 2\omega Q + \omega^2 P\right) \tag{7}$$

The variance of $\overline{y}_{RdR}^{(dc)}$ is minimum when

$$\alpha = \frac{Q}{PN^{**}} - n_1^* = \alpha_{Iopt.} \text{(say)} \tag{8}$$

where $n_1^* = n/(n_1 - 2n)$.

Putting (8) in (1) yields the 'asymptotically optimum estimator' (AOE) as

$$\left\{\overline{y}_{RdR}^{(dc)}\right\}_{Iopt.} = \overline{y}\left[\alpha_{Iopt.}\frac{\overline{x}_1}{\overline{x}}\frac{\overline{Z}}{\overline{z}'} + \left(1-\alpha_{Iopt.}\right)\frac{\overline{x}^*\overline{z}'}{\overline{x}_1\overline{Z}}\right]$$

Thus, the resulting variance of $\left\{\overline{y}_{RdR}^{(dc)}\right\}_{Iopt.}$ is given by

$$V\left\{\overline{y}_{RdR}^{(dc)}\right\}_{Iopt.} = \overline{Y}^2\left(M_{0,0} - Q^2/P\right). \tag{9}$$

Remark 2.1. For $\alpha = 1$, the estimator $\overline{y}_{RdR}^{(dc)}$ in (1) boils down to the chain ratio estimator $\overline{y}_{R}^{(dc)}$ suggested by Chand (1975) in double sampling. The bias and variance of $\overline{y}_{R}^{(dc)}$ can be obtained by putting $\alpha = 1$ in (5) and (7) respectively as

$$B\left\{\overline{y}_{R}^{(dc)}\right\}_I = \overline{Y}\left(P - Q\right).$$

and

$$V\left\{\overline{y}_{R}^{(dc)}\right\}_I = \overline{Y}^2\left(M_{0,0} - 2Q + P\right). \tag{10}$$

Remark 2.2. Setting $\alpha = 0$, the estimator $\overline{y}_{RdR}^{(dc)}$ in (1) reduces to the chain dual to ratio estimator $\overline{y}_{k}^{(dc)}$ in double sampling. The bias and variance of $\overline{y}_{k}^{(dc)}$ can be obtained by putting $\alpha = 0$ in (5) and (7) respectively as

$$B\left\{\overline{y}_{k}^{(dc)}\right\}_I = -\overline{Y}n^*Q$$

and

$$V\left\{\overline{y}_{k}^{(dc)}\right\}_I = \overline{Y}^2\left(M_{0,0} - 2n^*Q + n^{*2}P\right). \tag{11}$$

To the first degree of approximation, the variance of usual unbiased estimator $\overline{y}$ is given by

$$V\left(\overline{y}\right) = \overline{Y}^2 M_{0,0} \tag{12}$$

and the variance of chain regression type estimator $y_{reg.}^{(dc)} = \overline{y} + b_{yx}\left[\overline{x}_1 + b_{xz}\left(\overline{Z} - \overline{z}'\right) - \overline{x}\right]$, where b_{yx} and b_{xz} are the regression coefficient of y on x and x on z respectively, suggested by Kiregyera (1984) is given by

$$V\left(y_{reg.}^{(dc)}\right)_I = \overline{Y}^2\left[M_{0,0} - M_{0,1}C_{yx} + \left(M_{0,2}C_{yx} - 2M_{1,3}C_{yx}C_{yz} + M_{0,3}C_{yz}\rho_{xz}^2\right)\right] \tag{13}$$

From (10), (11), (12) and (13), it is envisaged that the proposed class of estimator $\bar{y}_{RdR}^{(dc)}$ is better than:

i. the usual chain ratio estimator $\bar{y}_{R}^{(dc)}$ in double sampling if

either $1 < \alpha < 2Q/P\,N^{**} - N_1^{*}$

or $2Q/P\,N^{**} - N_1^{*} < \alpha < 1$

where $N_1^{*} = n_1/(n_1 - 2n)$.

ii. the chain dual to ratio estimator $\bar{y}_{k}^{(dc)}$ in double sampling if

either $0 < \alpha < 2Q/P\,N^{**} - 2n_1^{*}$

or $2Q/P\,N^{**} - 2n_1^{*} < \alpha < 0$.

iii. the usual unbiased estimator $\bar{y}$ if

either $2Q/P\,N^{**} - n_1^{*} < \alpha < -n_1^{*}$

or $-n_1^{*} < \alpha < 2Q/P\,N^{**} - n_1^{*}$.

iv. the chain linear regression estimator $y_{reg.}^{(dc)}$ if

either $\left(Q - \sqrt{Q^2 - PR}\right)\Big/P\,N^{**} - n_1^{*} < \alpha < \left(Q + \sqrt{Q^2 - PR}\right)\Big/P\,N^{**} - n_1^{*}$

or $\left(Q + \sqrt{Q^2 - PR}\right)\Big/P\,N^{**} - n_1^{*} < \alpha < \left(Q - \sqrt{Q^2 - PR}\right)\Big/P\,N^{**} - n_1^{*}$

where $R = M_{0,1}C_{yx} - \left(M_{0,2}C_{yx} - 2M_{1,3}C_{yx}C_{yz} + M_{0,3}C_{yz}\rho_{xz}^{2}\right)$.

Thus, it seems from the above results that the proposed estimator $\bar{y}_{RdR}^{(dc)}$ may be made better than other estimators by making a suitable choice of the value of α.

Also from (10), (11), (12) and (13), it is envisaged that the 'AOE' $\left\{\bar{y}_{RdR}^{(dc)}\right\}_{Iopt.}$ is better than:

i. the usual chain ratio estimator $\bar{y}_{R}^{(dc)}$, since

$$V\left\{\bar{y}_{R}^{(dc)}\right\}_{I} - V\left\{\bar{y}_{RdR}^{(dc)}\right\}_{Iopt.} = \frac{\bar{Y}^2}{P}\left(P - Q\right)^2 > 0.$$

ii. the chain dual to ratio estimator $\bar{y}_{k}^{(dc)}$, since

$$V\left\{\bar{y}_{k}^{(dc)}\right\}_{I} - V\left\{\bar{y}_{RdR}^{(dc)}\right\}_{Iopt.} = \frac{\bar{Y}^2}{P}\left(n^{*}P - Q\right)^2 > 0.$$

iii. the usual unbiased estimator $\bar{y}$, since

$$V\left\{\bar{y}\right\} - V\left\{\bar{y}_{RdR}^{(dc)}\right\}_{Iopt.} = \frac{\bar{Y}^2}{P}Q^2 > 0.$$

iv. the chain linear regression estimator $y_{reg.}^{(dc)}$ if

$$M_{0,1}C_{yx} < \left(M_{0,2}C_{yx} - 2M_{1,3}C_{yx}C_{yz} + M_{0,3}C_{yz}\rho_{xz}^{2}\right).$$

Now, we state the following theorem:

Theorem 2.1. To the first degree of approximation, the proposed strategy under optimality condition (8), is always more efficient than $V\left\{\overline{y}_R^{(dc)}\right\}$, $V\left\{\overline{y}_k^{(dc)}\right\}$, $V\left(\overline{y}\right)$ and more efficient than $V\left\{y_{reg.}^{(dc)}\right\}$ if $\left(M_{0,2}C_{yx}-2M_{1,3}C_{yx}C_{yz}+M_{0,3}C_{yz}\rho_{xz}^2\right)>M_{0,1}C_{yx}$.

Case II: When the second phase sample of size n is drawn independently of the first phase sample of size n_1, see Bose (1943). Under this condition, we get the following results

$$\left.\begin{aligned}&E(e_0)=E(e_1)=E(e_1')=E(e_2')=0\\&E(e_0^2)=M_{0,0},\quad E(e_1^2)=M_{1,0}\\&E(e_1'^2)=M_{2,0},\quad E(e_2'^2)=M_{3,0}\\&E(e_0e_1)=M_{0,1},\quad E(e_1'e_2')=M_{2,3}\\&E(e_0e_1')=E(e_0e_2')=E(e_1e_1')=E(e_1e_2')=0\end{aligned}\right\}\tag{14}$$

Taking the expectation in (3) and using the results in (14), we get the bias of $\overline{y}_{RdR}^{(dc)}$ up to terms of order n^{-1} as

$$B\left\{\overline{y}_{RdR}^{(dc)}\right\}_{II}=\overline{Y}\left\{-\omega M_{0,1}-\left(\alpha N^*-n^*\right)M_{2,3}+\left(\alpha-1\right)n^*M_{2,0}+\alpha\left(M_{1,0}+M_{3,0}\right)\right\}\tag{15}$$

Squaring both sides in (6), then taking the expectation and using the results in (14), we obtain the variance of the estimator $\overline{y}_{RdR}^{(dc)}$ to the terms of order n^{-1}, as

$$V\left\{\overline{y}_{RdR}^{(dc)}\right\}=\overline{Y}^2\left\{M_{0,0}-2\omega M_{0,1}+\omega^2\left(M_{1,0}+M_{2,0}+M_{3,0}-2M_{2,3}\right)\right\}\tag{16}$$

Minimization of (16) with respect to α yields its optimum value as

$$\alpha=M_{0,1}\big/SN^{**}-n_1^*=\alpha_{IIopt}\text{ (say)}\tag{17}$$

where $S=M_{1,0}+M_{2,0}+M_{3,0}-2M_{2,3}$.

Thus, the resulting optimum variance of $\overline{y}_{RdR}^{(dc)}$ is given by

$$V\left\{\overline{y}_{RdR}^{(dc)}\right\}_{IIopt.}=\overline{Y}^2\left\{M_{0,0}-\frac{\left(M_{0,1}\right)^2}{S}\right\}\tag{18}$$

For simplicity we assume that the population size N is large enough as compared to the sample sizes n and n_1 so that the finite population correction (FPC) terms $1/N$ and $2/N$ are ignored.

Ignoring the FPC in (16), the variance of $\bar{y}_{RdR}^{(dc)}$ is given by

$$V\left\{\bar{y}_{RdR}^{(dc)}\right\}_{II} = \bar{Y}^2 \left\{M'_{0,0} - 2\omega M'_{0,1} + \omega^2 \left(M'_{1,0} + M'_{2,0} + M'_{3,0} - 2M'_{2,3}\right)\right\} \tag{19}$$

where

$(1/n)C_y^2 = M'_{0,0}$, $(1/n)C_x^2 = M'_{1,0}$, $(1/n_1)C_x^2 = M'_{2,0}$, $(1/n_1)C_z^2 = M'_{3,0}$, $(1/n)C_{yx}C_x^2 = M'_{0,1}$ and $(1/n_1)C_{xz}C_z^2 = M'_{2,3}$.

Minimization of (19) with respect to α yields its optimum value as

$$\alpha = M'_{0,1}\big/S'N^{**} - n_1^* = \alpha^*_{IIopt.} \text{ (say).} \tag{20}$$

where $S' = M'_{1,0} + M'_{2,0} + M'_{3,0} - 2M'_{2,3}$.

Putting (20) in (1) yields the 'AOE' as

$$\left\{\bar{y}_{RdR}^{(dc^*)}\right\}_{IIopt.} = \bar{y}\left[\alpha^*_{IIopt.}\frac{\bar{x}_1}{\bar{x}}\frac{\bar{Z}}{\bar{z}'} + \left(1-\alpha^*_{IIopt.}\right)\frac{\bar{x}^*\bar{z}'}{\bar{x}_1\bar{Z}}\right]$$

Thus, the resulting variance of $\left\{\bar{y}_{RdR}^{(dc^*)}\right\}_{IIopt.}$ is given by

$$V\left\{\bar{y}_{RdR}^{(dc^*)}\right\}_{IIopt.} = \bar{Y}^2\left\{M'_{0,0} - \frac{\left(M'_{0,1}\right)^2}{S'}\right\}. \tag{21}$$

Remark 2.3. For $\alpha = 1$, the estimator $\bar{y}_{RdR}^{(dc)}$ in (1) boils down to the chain ratio estimator $\bar{y}_R^{(dc)}$ (Chand, 1975) in double sampling. Thus, putting $\alpha = 1$ in (19), we get the variance of $\bar{y}_R^{(dc)}$ to the first degree of approximation as

$$V\left\{\bar{y}_R^{(dc)}\right\}_{II} = \bar{Y}^2\left\{M'_{0,0} + M'_{1,0} + M'_{2,0} + M'_{3,0} - 2\left(M'_{0,1} + M'_{2,3}\right)\right\}. \tag{22}$$

Remark 2.4. For $\alpha = 0$, the estimator $\bar{y}_{RdR}^{(dc)}$ in (1) boils down to the chain dual to ratio estimator $\bar{y}_k^{(dc)}$ in double sampling. Setting $\alpha = 0$ in (19), we get the variance of $\bar{y}_k^{(dc)}$ to the first degree of approximation as

$$V\left\{\bar{y}_k^{(dc)}\right\}_{II} = \bar{Y}^2\left(M'_{0,0} - 2n^*M'_{0,1} + n^{*2}S'\right). \tag{23}$$

Ignoring the FPC, the variance of sample mean $\bar{y}$ under SRSWOR is given by

$$V\left(\bar{y}\right)=\bar{Y}^2 M'_{0,0}\,. \tag{24}$$

and chain regression type estimator $y^{(dc)}_{reg.}$ suggested by Kiregyera (1984) in double sampling is

$$V\left(y^{(dc)}_{reg.}\right)_{II}=\bar{Y}^2\left\{M'_{0,0}\left(1-\rho^2_{yx}\right)+M'_{0,2}C_{yx}\left(1-\rho^2_{xz}\right)\right\} \tag{25}$$

where $\left(1/n_1\right)C_{yx}C_x^2=M'_{0,2}$.

From (22), (23), (24) and (25), it is observed that the proposed class of estimator $\bar{y}^{(dc)}_{RdR}$ is better than:

i. the usual chain ratio estimator $\bar{y}^{(dc)}_{R}$ if

either $\left(2M'_{0,1}/S'N^{**}\right)-N_1^*<\alpha<1$

or $1<\alpha<\left(2M'_{0,1}/S'N^{**}\right)-N_1^*$.

ii. the chain dual to ratio estimator $\bar{y}^{(dc)}_{k}$ if

either $0<\alpha<\left(2M'_{0,1}/S'N^{**}\right)-2n_1^*$

or $\left(2M'_{0,1}/S'N^{**}\right)-2n_1^*<\alpha<0$.

iii. the usual unbiased estimator $\bar{y}$ if

either $\left(M'_{0,1}/S'N^{**}\right)-n_1^*<\alpha<-n_1^*$

or $-n_1^*<\alpha<\left(M'_{0,1}/S'N^{**}\right)-n_1^*$.

iv. the chain linear regression estimator $y^{(dc)}_{reg.}$ if

either $-n_1^*+\left(M'_{0,1}-\sqrt{\left(M'_{0,1}\right)^2-S'R'}\right)\Big/S'N^{**}<\alpha<-n_1^*+\left(M'_{0,1}+\sqrt{\left(M'_{0,1}\right)^2-S'R'}\right)\Big/S'N^{**}$

or $-n_1^*+\left(M'_{0,1}+\sqrt{\left(M'_{0,1}\right)^2-S'R'}\right)\Big/S'N^{**}<\alpha<-n_1^*+\left(M'_{0,1}-\sqrt{\left(M'_{0,1}\right)^2-S'R'}\right)\Big/S'N^{**}$

where $R'=M'_{0,0}\rho^2_{yx}-M'_{0,2}C_{yx}\left(1-\rho^2_{xz}\right)$.

Thus, it seems from the above results that the proposed estimator $\bar{y}^{(dc)}_{RdR}$ may be made better than other estimators by making a suitable choice of the value of α in Case II.

Also from (22), (23), (24) and (25), it is envisaged that the 'AOE' $\left\{\bar{y}^{(dc^*)}_{RdR}\right\}_{IIopt.}$ is better than

i. the usual chain ratio estimator $\bar{y}^{(dc)}_{R}$ if

$$M'_{1,0} + M'_{2,0} + M'_{3,0} > 2M'_{2,3}.$$

ii. the chain dual to ratio estimator $\overline{y}_k^{(dc)}$ if

$$M'_{1,0} + M'_{2,0} + M'_{3,0} > 2M'_{2,3}.$$

iii. the usual unbiased estimator $\overline{y}$ if

$$M'_{1,0} + M'_{2,0} + M'_{3,0} > 2M'_{2,3}.$$

iv. the chain linear regression estimator $y_{reg.}^{(dc)}$ if

$$M'_{1,0} + M'_{2,0} + M'_{3,0} > 2M'_{2,3} \text{ and } M'_{0,0}\rho_{yx}^2 < M'_{0,2}C_{yx}\left(1-\rho_{xz}^2\right).$$

Thus, we consider the following theorem:

Theorem 2.2. To terms of order n^{-1}, the proposed strategy under optimality condition (20) is always more efficient than $V\left\{\overline{y}_R^{(dc)}\right\}$, $V\left\{\overline{y}_k^{(dc)}\right\}$, $V\left(\overline{y}\right)$ if $M'_{1,0} + M'_{2,0} + M'_{3,0} > 2M'_{2,3}$ and more efficient than $V\left\{y_{reg.}^{(dc)}\right\}$ if

$$M'_{1,0} + M'_{2,0} + M'_{3,0} > 2M'_{2,3} \text{ and } M'_{0,0}\rho_{yx}^2 < M'_{0,2}C_{yx}\left(1-\rho_{xz}^2\right).$$

3. Cost aspect

The different estimators reported in this paper have so far been compared with respect to their variances. In practical applications, the cost aspect should also be taken into account. In the literature, therefore, convention is to fixe the total cost of the survey and then to find optimum sizes of preliminary and final samples so that the variance of the estimator is minimized. In most of the practical situations, total cost is a linear function of samples selected at first and second phases.

In this section, we shall consider the cost of the survey and find the optimum sizes of the preliminary and second-phase samples in *Case I* and *Case II* separately.

Case I: When we use one auxiliary variate x then the cost function is given by

$$C = nC_1 + n_1C_2$$

Where C, C_1 and C_2 are the total cost, cost per unit of collecting information on the study variate y and the cost per unit of collecting information on the auxiliary variate x respectively of the survey.

When we used additional auxiliary variate z to estimate $\overline{y}_{RdR}^{(dc)}$, then the cost function is given by

$$C = nC_1 + n_1\left(C_2 + C_3\right) \tag{26}$$

where C_3 is the cost per unit collecting information on auxiliary variate z .

Ignoring FPC, the variance of $\left\{\overline{y}_{RdR}^{(dc)}\right\}$ in (7) can be expressed as

$$V\left\{\overline{y}_{RdR}^{(dc)}\right\}_I = \frac{1}{n}V_1 + \frac{1}{n_1}V_2$$

Where

$$V_1 = \overline{Y}^2\left(M'_{0,0} - 2\omega M'_{0,1} + \omega^2 M'_{1,0}\right),\ V_2 = \overline{Y}^2\left\{M'_{0,3} + 2\omega M'_{0,2} + \omega^2\left(M'_{3,0} - M'_{2,0}\right)\right\}$$

and $\left(1/n_1\right)C_{yz}C_z^2 = M'_{0,3}$.

It is assumed that $C_1 > C_2 > C_3$. The optimum values of n and n_1 for fixed cost $C = C_0$ which minimizes the variance of $\overline{y}_{RdR}^{(dc)}$ at (7) under cost function are given by

$$n_{opt.} = \frac{C_0\sqrt{V_1/C_1}}{\sqrt{V_1C_1} + \sqrt{V_2\left(C_2+C_3\right)}} \text{ and } n_{1opt.} = \frac{C_0\sqrt{V_2/\left(C_2+C_3\right)}}{\sqrt{V_1C_1} + \sqrt{V_2\left(C_2+C_3\right)}}$$

Thus, the resulting variance of $\overline{y}_{RdR}^{(dc)}$ is given by

$$V_{opt.}\left\{\overline{y}_{RdR}^{(dc)}\right\}_I = \frac{1}{C_0}\left\{\sqrt{V_1C_1} + \sqrt{V_2\left(C_2+C_3\right)}\right\}^2 \tag{27}$$

If all the resources were diverted towards the study variate y only, then we would have optimum sample size as below

$$n^{**} = C/C_1$$

Thus, the variance of sample mean $\overline{y}$ for a given fixed cost $C = C_0$ in case of large population is given by

$$V_{opt.}\left(\overline{y}\right) = \frac{C_1}{C_0}S_y^2. \tag{28}$$

From (27) and (28), the proposed sampling strategy would be profitable as long as

$$V_{opt.}\left\{\overline{y}_{RdR}^{(dc)}\right\}_I < V_{opt.}\left(\overline{y}\right).$$

or equivalently,

$$\frac{C_2+C_3}{C_1} < \left[\frac{S_y - \sqrt{V_1}}{\sqrt{V_2}}\right]^2.$$

Case II: we assume that y measured on n units, x and z are measured on n_1 units. We consider a simple cost function

$$C = nC_1 + n_1\left(C'_2 + C'_3\right) \tag{29}$$

where C'_2 and C'_3 denote costs per unit of observing x and z values respectively.

The variance of $\left\{\bar{y}_{RdR}^{(dc)}\right\}$ at (19) can be written as

$$V\left\{\bar{y}_{RdR}^{(dc)}\right\}_{II} = \frac{1}{n}V_1 + \frac{1}{n_1}V_3 \tag{30}$$

where $V_3 = \bar{Y}^2\omega^2\left(M'_{3,0} + M'_{2,0} - 2M'_{2,3}\right)$.

To obtain the optimum allocation of sample between phases for a fixed cost $C = C_0$, we minimized (30) with condition (29). It is easily found that this minimum is attained for

$$n_{opt.} = \frac{C_0\sqrt{V_1/C_1}}{\sqrt{V_1C_1} + \sqrt{V_3\left(C'_2 + C'_3\right)}} \text{ and } n_{1opt.} = \frac{C_0\sqrt{V_3/\left(C'_2 + C'_3\right)}}{\sqrt{V_1C_1} + \sqrt{V_3\left(C'_2 + C'_3\right)}}.$$

Thus, the optimum variance corresponding to these optimum values of n and n_1 are given by

$$V_{opt.}\left\{\bar{y}_{RdR}^{(dc)}\right\}_{II} = \frac{1}{C_0}\left\{\sqrt{V_1C_1} + \sqrt{V_3\left(C'_2 + C'_3\right)}\right\}^2 \tag{31}$$

From (28) and (31) it is obtained that the proposed estimator $\bar{y}_{RdR}^{(dc)}$ yields less variance than that of sample mean $\bar{y}$ for the same fixed cost if

$$\frac{C'_2 + C'_3}{C_1} < \left[\frac{S_y - \sqrt{V_1}}{\sqrt{V_3}}\right]^2.$$

4. Empirical study

To examine the merits of the proposed estimator, we have considered the six natural population data sets. The description of the populations are given as follows.

<u>*Population I*</u>- Source: Cochran (1977)

Y : Number of 'placebo' children, X : Number of paralytic polio cases in the placebo group, Z : Number of paralytic polio cases in the 'not inoculated' group.

$N = 34,\ n = 10,\ n_1 = 15,\ \bar{Y} = 4.92,\ \bar{X} = 2.59,\ \bar{Z} = 2.91,\ \rho_{yx} = 0.7326,\ \rho_{yz} = 0.6430,$
$\rho_{xz} = 0.6837,\ C_y^2 = 1.0248,\ C_x^2 = 1.5175,\ C_z^2 = 1.1492.$

<u>*Population II*</u>- Source: Sukhatme and Chand (1977)

Y : Apple trees of bearing age in 1964, X : Bushels of apples harvested in 1964, Z : Bushels of apples harvested in 1959.

$N = 200,\ n = 20,\ n_1 = 30,\ \bar{Y} = 0.103182\times10^4,\ \bar{X} = 0.293458\times10^4,\ \bar{Z} = 0.365149\times10^4,$
$\rho_{yx} = 0.93,\ \rho_{yz} = 0.77,\ \rho_{xz} = 0.84,\ C_y^2 = 2.55280,\ C_x^2 = 4.02504,\ C_z^2 = 2.09379.$

Population III- Source: Murthy (1967)

Y : Area under wheat in 1964, X : Area under wheat in 1963, Z : Cultivated area in 1961.

$N = 34,\ \ n = 7,\ \ n_1 = 10,\ \ \bar{Y} = 199.44\ \ acre,\ \ \bar{X} = 208.89\ \ acre,\ \ \bar{Z} = 747.59\ \ acre,$

$\rho_{yx} = 0.9801,\ \ \rho_{yz} = 0.9043,\ \ \rho_{xz} = 0.9097,\ \ C_y^2 = 0.5673,\ \ C_x^2 = 0.5191, C_z^2 = 0.3527.$

Population IV- Source: Srivastava *et al.* (1989, Page 3922)

Y : The measurement of weight of children, X : Mid arm circumference of children

Z : Skull circumference of children.

$N = 82,\ \ n = 25,\ \ n_1 = 43,\ \ \bar{Y} = 5.60\ \ kg,\ \ \bar{X} = 11.90\ \ cm,\ \ \bar{Z} = 39.80\ \ cm,\ \ \rho_{yx} = 0.09,$

$\rho_{yz} = 0.12,\ \ \rho_{xz} = 0.86,\ \ C_y^2 = 0.0107,\ \ C_x^2 = 0.0052,\ \ C_z^2 = 0.0008.$

Population V- Source: Srivastava *et al.* (1989, Page 3922)

Y : The measurement of weight of children, X : Mid arm circumference of children

Z : Skull circumference of children.

$N = 55,\ \ n = 18,\ \ n_1 = 30,\ \ \bar{Y} = 17.08\ \ kg,\ \ \bar{X} = 16.92\ \ cm,\ \ \bar{Z} = 50.44\ \ cm,\ \ \rho_{yx} = 0.54,$

$\rho_{yz} = 0.51,\ \ \rho_{xz} = -0.08,\ \ C_y^2 = 0.0161,\ \ C_x^2 = 0.0049,\ \ C_z^2 = 0.0007.$

Population VI- Source: Srivastava *et al.* (1990)

$N = 120,\ \ n = 30,\ \ n_1 = 60,\ \ \bar{Y} = 2934.58,\ \ \bar{X} = 1031.82,\ \ \bar{Z} = 3651.49,\ \ \rho_{yx} = 0.93,$

$\rho_{yz} = 0.84,\ \ \rho_{xz} = 0.77,\ \ C_y^2 = 4.02004,\ \ C_x^2 = 2.55280,\ \ C_z^2 = 2.09379.$

To reflect the gain in the efficiency of the proposed estimator $\bar{y}_{RdR}^{(dc)}$ over the estimators $\bar{y}$,

$\bar{y}_R^{(dc)}$ $\bar{y}_k^{(dc)}$ and $\bar{y}_{reg.}^{(dc)}$, the effective ranges and optimum value of α are shown in Table 1 with respect to the above population data sets.

To observe the relative performance of different estimators of $\bar{Y}$, we have computed the percent relative efficiencies (PRE) of the proposed estimator $\bar{y}_{RdR}^{(dc)}$, usual ratio estimator $\bar{y}_R^{(dc)}$, chain dual to ratio estimator $\bar{y}_k^{(dc)}$ and linear regression estimator $\bar{y}_{reg.}^{(dc)}$ with respect to usual unbiased estimator $\bar{y}$ in *Case I* and *Case II* and the findings are presented in Table 2.

Table 1. Effective ranges and optimum values of α of $\overline{y}_{RdR}^{(dc)}$

Population	Ranges of α in which $\overline{y}_{RdR}^{(dc)}$ is better than				Optimum value of α
	$\overline{y}_{R}^{(dc)}$	$\overline{y}_{k}^{(dc)}$	$\overline{y}$	$y_{reg.}^{(dc)}$	
Case I					α_{Iopt}
I	(1.00, 1.79)	(0.00, 2.79)	(0.79, 2.00)	*	1.3956
II	(1.00, 1.42)	(0.00, 2.42)	(0.42, 2.00)	*	1.2079
III	(0.87, 1.00)	(0.00, 1.87)	(0.12, 1.75)	*	0.9331
IV	(1.00, 5.33)	(0.00, 6.33)	(2.76, 3.57)	*	3.1660
V	(0.56, 1.00)	(0.00, 1.56)	(-1.44, 3.00)	*	0.7819
VI	(1.00, 101.02)	(0.00, 102.02)	(-300.00, 402.02)	*	51.0103
Case II					α_{IIopt}^{*}
I	(1.00, 2.13)	(0.00, 3.13)	(1.13, 2.00)	*	1.5632
II	(1.00, 1.77)	(0.00, 2.77)	(0.77, 2.00)	*	1.3857
III	(1.00, 0.07)	(0.00, 1.07)	(-0.68, 1.75)	*	0.5366
IV	(1.00, 6.06)	(0.00, 7.06)	(3.49, 3.57)	*	3.5322
V	(1.00, 4.43)	(0.00, 5.43)	(2.43, 3.00)	*	2.7158
VI	(-600.99, 1.00)	(-599.99, 0.00)	(-300.00, -299.99)	*	-299.9968

* *Data is not applicable.*

Table 2. Percent relative efficiencies of different estimators with respect to $\overline{y}$

Population	$\overline{y}$	Efficiencies of the proposed estimator $\overline{y}_{RdR}^{(dc)}$ belonging to the class							
		$\overline{y}_{R}^{(dc)}$	$\overline{y}_{R}^{(dc)}$	$\overline{y}_{k}^{(dc)}$	$\overline{y}_{k}^{(dc)}$	$\overline{y}_{reg.}^{(dc)}$	$\overline{y}_{reg.}^{(dc)}$	$\left\{\overline{y}_{RdR}^{(dc)}\right\}$ or $\left\{\overline{y}_{RdR}^{(dc)}\right\}_{I}$	$\left\{\overline{y}_{RdR}^{(dc)}\right\}$ or $\left\{\overline{y}_{RdR}^{(dc^{*})}\right\}$
		Case I	Case II	Case I	Case II	Case I	Case II	Case I	Case II
I	100.00	136.91	79.50	32.86	17.87	185.53	152.94	189.27	163.78
II	100.00	279.93	176.85	52.21	25.43	326.33	328.03	322.94	353.81
III	100.00	730.78	644.74	79.06	44.77	778.93	643.67	763.27	681.36
IV	100.00	81.92	66.85	67.54	49.40	101.00	100.69	100.81	100.64
V	100.00	131.91	107.73	127.21	77.79	118.31	113.35	132.32	120.39
VI	100.00	488.98	347.59	487.39	347.81	517.80	321.69	531.85	348.90

5. Conclusions

From Table 1, it is clear that the proposed estimator $\overline{y}_{RdR}^{(dc)}$ is more efficient than the conventional estimators $\overline{y}$, $\overline{y}_{R}^{(dc)}$, $\overline{y}_{k}^{(dc)}$ and $\overline{y}_{reg.}^{(dc)}$ for both the cases under the effective ranges of α as far as the variance criterion is considered. We have also computed the percent relative efficiencies of different estimators with respect to $\overline{y}$ as shown in Table 2.

Table 2 clearly indicates that there is considerable gain in efficiency by using the estimators $\overline{y}_{RdR}^{(dc)}$ or $\left[\left\{\overline{y}_{RdR}^{(dc)}\right\}_{Iopt.} \text{and} \left\{\overline{y}_{RdR}^{(dc^*)}\right\}_{IIopt.}\right]$ over the estimators $\overline{y}$, $\overline{y}_{R}^{(dc)}$, $\overline{y}_{k}^{(dc)}$ and $\overline{y}_{reg.}^{(dc)}$ in both the cases except for the data set of population II and III for linear regression estimator $\overline{y}_{reg.}^{(dc)}$ in *Case I*. It is due to poor correlation between y and x, y and z, x and z. It is also observed that overall performances of the proposed estimator $\overline{y}_{RdR}^{(dc)}$ in *Case I* is better than *Case II*. Thus, it is preferred to use the proposed estimators $\overline{y}_{RdR}^{(dc)}$ or $\left[\left\{\overline{y}_{RdR}^{(dc)}\right\}_{Iopt.} \text{and} \left\{\overline{y}_{RdR}^{(dc^*)}\right\}_{IIopt.}\right]$ in practice.

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