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APPLICATION OF COHERENT DISTORTION RISK MEASURES

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ABSTRACT

This paper concentrates on solving the portfolio selection problem. It starts with an extension of the well-known optimization framework for Conditional Value-at-Risk (CVaR)-based portfolio selection problems [1, 2] to optimization over a more general class of risk measure - known as the class of Coherent Distortion Risk Measure (CDRM). The CDRM class of risk measures is the intersection of Coherent Risk Measure (CRM) and Distortion Risk Measure (DRM). It concludes with showing that many of the well-known risk measures are of special cases of the CDRM class what may facilitate to deal with the portfolio optimization problem

Key words: coherent risk measure, distortion risk measure, coherent distortion risk measure.

1. Introduction

The problem of optimal portfolio selection is very important issue to investors, hedgers, fund managers and individual investors. The research on optimal portfolio selection has been growing rapidly. Researchers and practitioners are constantly looking for better and more sophisticated risk measures and reward trade-off in constructing optimal portfolios. The classical Markowitz[2] model used variance as the benchmark for risk measurement and this is perceived to be undesirable since it penalizes equally both sides, regardless of downside risk or upside potential. Consequently, other measures of risk have been proposed in connection with portfolio optimization. These include semi-variance[3],

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[2] H. M. Markowitz. Portfolio selection. The Journal of Finance, 7(1): pp. 77–91, 1952.
[3] H. M. Markowitz. Portfolio selection: efficient diversification of investments. New Haven, CT: Cowles Foundation, 94, 1959.

partial moments[1], safety first principle[2], skewness and kurtosis[3], value-at-risk (VaR) and conditional value-at-risk (CVaR)[4].

Let us take some basic notation: let $\rho(\mathbf{x})$ be a function which measures the riskiness of a portfolio **x**. Then, in general, the portfolio selection problem seeks the solution to

$$\min_{\mathbf{x}\in\mathbf{S}} \rho(\mathbf{x}) \tag{1}$$

where the minimization is taken over all feasible portfolios **S**. If ρ corresponds to the variance of the return of the portfolio **x**, then this problem (1) reduces to the standard Markowitz model. We are concerned with the portfolio selection problem involving a particular class of risk measure known as the coherent distortion risk measure (CDRM). CDRM is the intersection of two important families of risk measures: the coherent risk measure (CRM)[5] and the distortion risk measure (DRM)[6]. We study the intersection of both classes: CVaR is an example of CDRM while VaR is neither CRM nor DRM, and hence not CDRM[7].

2. CVaR-based portfolio optimization model

Let $\boldsymbol{l} = f(\mathbf{x},\mathbf{y})$ be the portfolio loss associated with the decision vector **x** , to be chosen from a set $\mathbf{S} \subseteq n$, and the random vector $\mathbf{y} \in m$. The vector **x** represents what we may generally call a portfolio, with **S** capturing the set of all feasible portfolios subject to certain portfolio constraints. For every **x** , the loss $f(\mathbf{x},\mathbf{y})$ is a random variable having a distribution induced by the distribution of $\mathbf{y} \in m$.

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[1] V. S. Bawa and E.B. Lindenberg. Capital market equilibrium in a mean-lower partial moment framework. Journal of Financial Economics, 5(2):189–200, 1977, Trzpiot G. (2005). Partial Moments and Negative Moments in Ordering Asymmetric Distribution, in: Daniel Baier and Klaus-Dieter Wernecke (eds.): Innovations in Classification, Data Science and Information Systems, Proc. 27th Annual GFKL Conference, University of Cottbus, March 11-14 2003. Springer-Verlag, Heidelberg-Berlin, 181-188.

[2] A. D. Roy. Safety first and the holding of assets. Econometrica: Journal of the Econometric Society, pages 431–449, 1952.

[3] C. R. Harvey, J. Liechty, M. Liechty, and P. Müller. Portfolio selection with higher moments. Quantitative Finance, 10(5):469–485, 2010.

[4] R.T. Rockafellar and S. Uryasev. Optimization of conditional value-at-risk. Journal of risk, 2, 21–42, 2000.

[5] P. Artzner, F. Delbaen, J.M. Eber, and D. Heath. Coherent measures of risk. Mathematical finance, 9(3):203–228, 1999.

[6] S. S. Wang. A class of distortion operators for pricing financial and insurance risks. The Journal of Risk and Insurance, 67(1):15–36, 2000.

[7] Trzpiot G. Własności transformujących miar ryzyka [Properties of transforming risk measures], Economic Studies of the University of Economics in Katowice - Faculty Scientific Papers No. 91, 21–36, 2012.

The underlying probability distribution of **y** is assumed to be discrete with probability masses $\boldsymbol{p}$, i.e., P[$\boldsymbol{l}$ = $L(\mathbf{x}, y_i)$] = p_i for i = 1, … ,m.

We can notice that in many cases it is assumed that X - the portfolio loss has a discrete uniform distribution. This is not a very limiting assumption if we restrict ourselves to discrete portfolio loss distributions, which is typically the case if we are obtaining distributional information via scenario generation or from historical data. In addition, given any arbitrary discrete distribution representable with rational numbers, we may always convert it to discrete uniform distribution for some large enough m.

For every portfolio **x** denote the cumulative distribution function (cdf) of the portfolio loss $\boldsymbol{l} = f(\mathbf{x},\mathbf{y})$ as:

$$\Psi(x,\zeta) = \sum_{i=1}^{m} p_i I\{l_i \le \zeta\}$$

Then α −VaR and α −CVaR are defined as follows[1].

Definition 2.1. *Suppose for each* $\mathbf{x} \in \boldsymbol{S}$, *the distribution of the portfolio loss* $\boldsymbol{l} = f(\boldsymbol{x},\boldsymbol{y})$ *is concentrated in* $m < \infty$ *points, and* $\Psi(\boldsymbol{x}, \cdot)$ *is a step function with jumps at these points. Now, fixing* **x** *and* $l_{(1)} < .... < l_{(m)}$ *denoting the corresponding ordered portfolio loss points and* $p_{(i)} > 0$, $i = 1, \ldots, m$, *represent the probability of realizing loss* $l_{(i)}$.
If

$$\min_{\mathbf{x} \in \mathbf{S}} \rho(\ )$$

x denotes the unique index satisfying α then α *-VaR* and α *-CVaR* of the portfolio loss are given, respectively, by $\zeta_\alpha(\mathbf{x}) = l_{(i\alpha)}$ and

$$\phi_\alpha(x) = \frac{1}{1-\alpha}\left[\left(\sum_{i=1}^{i\alpha} p_{(i)} - \alpha\right) l_{i\alpha} + \sum_{i=i\alpha}^{m} p_{(i)} l_{(i)}\right] \tag{2}$$

As pointed out, if ρ is set to VaR, then the resulting portfolio problem (1) is numerically challenging due to its lack of convexity. In contrast, the CVaR-based portfolio optimization problem (1) is a convex program and hence it is computationally amenable. The CVaR-based portfolio model becomes even more popular and more practical when [1] shows that the convex program can in fact be formulated as a liner program. The key to Rockafellar-Uryasev's linear optimization scheme of CVaR-based portfolio selection problem is expressing $\phi_\alpha(\mathbf{x})$ and $\zeta_\alpha(\mathbf{x})$ in terms of the following special function:

$$F_\alpha(x,\zeta) = \zeta + \frac{1}{1-\alpha} E\left[(l(x,y)-\zeta)^+\right] = \zeta + \frac{1}{1-\alpha}\sum_{i=1}^{m} p_i (l_i - \zeta)^+ \tag{3}$$

[1] R.T. Rockafellar and S. Uryasev. Conditional value-at-risk for general loss distributions. *Journal of Banking & Finance*, 26(7):1443–1471, 2002.

If $f(\mathbf{x},\mathbf{y})$ is convex with respect to $\mathbf{x}$, then $\varphi_\alpha(\mathbf{x})$ is convex with respect to $\mathbf{x}$. In this case, $F_\alpha(\mathbf{x}, \zeta)$ is also jointly convex in $(\mathbf{x}, \zeta)$. Armed with these findings, Rockafellar and Uryasev derived the following equivalence formulation[1]:

Theorem 2.1. *Minimizing $\phi_\alpha(\mathbf{x})$ with respect to $\mathbf{x} \in \mathbf{S}$ is equivalent to minimizing $F_\alpha(\mathbf{x}, \zeta)$ over all $(\mathbf{x}, \zeta) \in \mathbf{S}\times$, in the sense that*

$$\min_{x\in S} \phi_\alpha(x) = \min_{(x,\zeta)\in S\times R} F_\alpha(x,\zeta) \tag{4}$$

where moreover

$$(x^*, \zeta^*) \in \min_{(x,\zeta)\in S\times R} F_\alpha(x,\zeta) \Leftrightarrow x^* \in \arg\min_{x\in S} \phi_\alpha(x),\ \zeta^* \in \min_{(x,\zeta)\in S\times R} F_\alpha(x^*,\zeta) \tag{5}$$

The above theorem links the representation (3) explicitly to both VaR and CVaR simultaneously. The theorem asserts that for the purpose of determining an optimal portfolio with respect to CVaR, we can replace $\phi_\alpha(\mathbf{x})$ by $F_\alpha(\mathbf{x}, \zeta)$ in portfolio selection problems. More importantly, by exploiting (3) the general convex programming of CVaR portfolio optimization problem can be linearized into a linear objective function with additional linear auxiliary constraints. With such linear representation we can cast any portfolio selection problem with CVaR objective and linear constraint(s) as a linear program.

3. Coherent Risk Measure (CRM) and Distortion Risk Measure (DRM)

The uncertainty for future value of an investment position is usually described by a function $X : \Omega \to R$, where Ω is a fixed set of scenarios with a probability space (Ω, F, P). Let $\mathcal{X}$ be a linear space of random variables on Ω, i.e., a set of functions $X : \Omega \to R$. Note that X can be thought of as a loss from an uncertain position.

We can find out some properties of risk measures.

Property 1. Law-invariance

Law-invariance states that a risk measure $\rho(X)$ does not depend on a risk itself but only on its underlying distribution, i.e. $\rho(X) = \rho(F_X)$, where F_X is the distribution function of X. This condition ensures that F_X contains all the information needed to measure the riskiness of X. Law-invariance can be phrased as:

$$F_X = F_Y \Rightarrow \rho(X) = \rho(Y)$$

[1] Ibid.

for every random portfolio returns X and Y with distribution functions FX and FY. In other words, ρ is law-invariant in the sense that ρ(X) = ρ(Y), whenever X and Y have the same distribution with respect to the initial probability measure P. This assumption is essential for a risk measure to be estimated from empirical data, which ensures its applicability in practice.

Property 2. Positive homogeneity

Positive homogeneity (also known as positive scalability) formulates as follows: for each positive λ and random portfolio return $X \in \mathcal{X}$:

$$\rho(\lambda X) = \lambda^k \rho(X).$$

Positive homogeneity signifies that a measure has the same dimension (scalability) as a variable X. When the parameter k = 0, a risk measure does not depend on the scalability.

From a financial perspective, positive homogeneity implies that a linear increase of the return by a positive factor leads to a linear increase in risk by the same factor.

Property 3. Sums of risks

Consider two different financial instruments with random payoffs $X, Y \in \mathcal{X}$: The payoff of a portfolio consisting of these two instruments will equal X + Y .

Property 3.1. Sub-additivity[1]

Sub-additivity states that the risk of the portfolio is not greater than the sum of the risks of the portfolio components. In other words, “a merger does not create extra risk”.

$$\rho(X + Y) \le \rho(X) + \rho(Y)$$

Compliance with this property tends to the diversification effect. Although Artzner treat sub-additivity as a necessary requirement for constructing a risk measure in order for it to be coherent, empirical evidence suggests that sub-additivity does not always hold in reality[2].

Property 3.2. Additivity

The additivity property is expressed in the following form:

$$\rho(X + Y) = \rho(X) + \rho(Y)$$

This property is valid for independent and comonotonic[3] random variables X and Y. The comonotonic random variables with no-hedge condition result in comonotonic additivity.

[1] Artzner et al. (1999).

[2] Critiques of sub-additivity can be found in Dhaene et al. (2003) and Heyde et al. (2006).

[3] Comonotonic or common monotonic random variables (Yaari (1987), Schmeidler (1986), Dhaene et al. (2002a, 2002b)) are such that if the increase of one follows the increase of the other variable: $P[X \le x, Y \le y] = \min\{P[X \le x], P[Y \le y]\}$ for all $x, y \in R$.
Intuitively, such variables have a maximal level of dependency. Comonotonic random variables are necessarily positively correlated. In financial and insurance markets, this property appears quite frequently.

Property 3.3. Super-additivity

Super-additivity states that the portfolio risk estimate could be greater than the sum of the individual risk estimates.

$$\rho(X + Y) \geq \rho(X) + \rho(Y)$$

The super-additivity property is valid for risks which are positive (negative) dependent.

Property 4. Convexity

(1) For all X, Y $\in \mathcal{X}$, $0 \leq \lambda \leq 1$, the following inequality is true:

$$\rho(\lambda X + (1 - \lambda)Y) \leq \lambda\rho(X) + (1 - \lambda)\rho(Y).$$

Convexity ensures the diversification property and relaxes the requirement that a risk measure must be more sensitive to aggregation of large risks.

(2) For any λ, $\mu \geq 0$, $\lambda+\mu = 1$, and distribution functions *F*,*G*, the following inequality holds

$$\rho(\lambda F + \mu G) \leq \lambda\rho(F) + \mu\rho(G).$$

(3) Generalized convexity. For any λ, $\mu \geq 0$, $\lambda+\mu = 1$ and distribution functions U, V, H, such that the following random variables exist X, Y , $\lambda X + \mu Y$, for which $F_X = U$, $F_Y = V$, $F_{\lambda X+\mu Y} = H$, the inequality is true

$$\rho(H) \leq \lambda\rho(U) + \mu\rho(V).$$

Property 5. Monotonicity

For every random portfolio returns X and Y such that $X \geq Y$,

$$\rho(X) \leq \rho(Y).$$

Monotonicity implies that if one financial instrument with the payoff X is not less than the payoff Y of the other instrument, then the risk of the first instrument is not greater than the risk of the second financial instrument. Another presentation of the monotonicity property with a risk-free instrument is as follows:

$$X \geq 0 \Rightarrow\Rightarrow \rho(X) \leq \rho(0)$$

for X $\in \mathcal{X}$,.

Property 6. Translation invariance

Property 6.1. For the non-negative number $\alpha \geq 0$ and C $\in$ R, the property has the following form:

$$\rho(X + C) = \rho(X) - \alpha C.$$

This property states that if the payoff increases by a known constant, the risk correspondingly decreases. In practice, $\alpha = 0$ or $\alpha = 1$ are often used.

Property 6.2. When $\alpha = 0$, it implies that the addition of a certain wealth does not increase risk. This property is also known as the Gaivoronsky-Pflug (G-P) translation invariance[1] .

Property 6.3. The case when $\alpha = 1$ implies that by adding a certain payoff, the risk decreases by the same amount.

$$\rho(X + C) = \rho(X) - C.$$

Property 6.4. When a constant wealth has a positive value, i.e., $C \geq 0$, one gets

$$\rho(X + C) \leq \rho(X).$$

This result is in agreement with the monotonicity property of $X + C \geq X$.

Property 6.5. In particular, translation invariance involves

$$\rho(X + \rho(X)) = \rho(X) - \rho(X) = 0,$$

obtaining a risk-neutral position by adding $\rho(X)$ to the initial position X.

Property 7. Consistency

Property 7.1. Consistency with respect to n-order stochastic dominance has the following general form:

$$X \geq_n Y,\ \rho(X) \geq \rho\ (Y\ ).$$

In practice, the maximal value of n = 2; n = 0 just stands for a monotonicity property.

Property 7.2. Monotonic dominance of n-order

$$X \geq_{M(n)} Y, \text{ if } E[u(X)] \geq E[u(Y\ )]$$

for any monotonic of order n functions, that is $u^{(n)}(t) \geq 0$.

It is known, that $X \geq_1 Y$ is equivalent to $X \leq_{M(1)} Y$. $X \leq_{M(2)} Y$ is also called the Bishop-de Leeuw ordering or Lorenz dominance.

Property 7.3. First-order stochastic dominance (FSD)

$$\text{For } X \geq_1 Y,\ F_X(x) \leq F_Y\ (x)$$

If an investor prefers X to Y, then FSD will indicate that the risk of X is less than the risk of Y. In terms of utility function u, the following holds

$$\text{If } X \geq_1 Y, \text{ then } E[u(X)] \geq E[u(Y\ )]$$

[1] Gaivoronski A. and Pflug G., 2001, Value at risk in portfolio optimization: properties and computational approach, Technical Report, University of Vienna.

for all increasing utility functions u. FSD characterizes the preferences of risk-loving investors. Ortobelli et al. (2006) classified risk measures consistent with respect to FSD as *safety-risk measures*[1].

Property 7.4. Rothschild-Stiglitz stochastic order dominance (RSD)

RSD has the form[2]:

$$\text{If } X \leq_{RS} Y, \text{ then } E[u(X)] \geq E[u(Y)]$$

for any concave, not necessarily decreasing, utility function u. RSD describes preferences of risk-averse investors. *Dispersion measures* are normally consistent with RSD.

Property 7.5. Second-order stochastic dominance (SSD)

SSD has the following form:

$$\text{For } X \geq_2 Y,\ E[u(X)] \geq E[u(Y)]$$

for all increasing, concave utility functions u. SSD characterizes non-satiable risk averse investors[3].

Property 7.6. Stochastic order - stop-loss Y dominates X ($Y \geq_{SL} X$) in stop-loss order, if for any number α the following inequality is true:

$$E[(Y - \alpha)^+] \geq E[(X - \alpha)^+].$$

Here, $\alpha^+ = \max\{0, \alpha\}$. Such order is essential in the insurance industry. If the insurer takes the responsibility for the claims greater than α (deductible), then the expected claim Y is not smaller than X.

Property 7.7. Convex order Y dominates X with respect to convex order ($Y \geq_{CX} X$) if the relation $Y \geq_{SL} X$ is true and when $\alpha = -\infty$ in stop-loss order, i.e. $E[X] = E[Y]$. Convex ordering is related to the notion of risk aversion[4].

Consistency with the stochastic dominance is a necessary property for a risk measure, because it enables one to characterize the set of all optimal portfolio choices when either wealth distributions or expected utility functions depend on a finite number of parameters [5].

Property 8. Non-negativity

Property 8.1. $\rho(X) \geq 0$, while $\rho(X) > 0$ for all non-constant risk.

Property 8.2. If $X \geq 0$, then $\rho(X) \leq 0$; if $X \leq 0$, then $\rho(X) \geq 0$.

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[1] In the portfolio selection literature, two disjoint categories of risk measures are defined: dispersion measures and safety-first risk measures. (Ortobelli S., Rachev S., Shalit H. and Fabozzi F., 2006).

[2] Rothschild M. and Stiglitz J., 1970.

[3] Hadar J. and Russell W., 1969.

[4] See also Kaas et al. (1994, 2001).

[5] Ortobelli S., 2001.

Property 9. Continuity

Property 9.1. Probability convergence continuity: If $X_n \xrightarrow{P} X$, then $\rho(X_n)$ converges and has the limit $\rho(X)$.

Property 9.2. Weak topology continuity: If $F_{X_n} \xrightarrow{w} F_X$, then $\rho(F_{X_n})$ converges and has the limit $\rho(F_X)$.

Property 9.3. Horizontal shift continuity: $\lim_{\delta\to 0} \rho(X + \delta) = \rho(X)$.

Property 9.4. Opportunity of arbitrary risk approximation with the finite carrier is expressed by the equality[1]:

$$\lim_{\sigma\to+\infty} \rho(\min\{X,\delta\}) = \lim_{\sigma\to-\infty} \rho(\min\{X,\delta\}) = \rho(X)$$

Property 9.5. Lower semi-continuity: For any $C \in R$, the set $\{X \in \mathit{X}: \rho(X) \leq C\}$ is $\sigma(L^\infty,L^1)$ - closed.

Property 9.6. Fatough property[2]

For any bounded sequence (X_n) for which $X_n \xrightarrow{P} X$, the following holds:

$$\rho(X) \leq \lim_{n\to\infty} \inf \rho(X_n).$$

These properties are cardinally important. Nonfulfilment of the continuity property implies that even a small inaccuracy in a forecast can lead to the poor performance of a risk measure.

Property 10. Strictly expectation-boundedness

The risk of a portfolio is always greater than the negative of the expected portfolio return.

$$\rho(X) \geq -E[X], \text{ while } \rho(X) > -E[X] \text{ for all non-constant } X,$$

where E[X] is the mathematical expectation of X.

Property 11. Lower-range dominated

Deviation measures possess lower-range dominated property of the following form:

$$D(X) \leq E(X)$$

for a non-negative random variable. From property 10 and property 11 one can derive:

$D(X) = \rho(X - EX)$, $\rho(X) = D(X) - E(X)$

Property 12. Risk with risk-free return C

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[1] Wang et al. (1997).

[2] In some contexts, it is equivalent to the upper semi-continuity condition with respect to $\sigma(L^\infty,L^1)$.

Property 12.1. ρ (C) = −C, it follows from the invariance property 6.3. If C > 0, then the situation is stable, risk is negative. The opposite situation occurs with C < 0.
Property 12.2. ρ (C) = 0, risk does not deviate with the zero certain return.

Property 13. Symmetric property

(1) ρ(−X) = −ρ(X), which corresponds to property 8.1.
(2) ρ(−X) = ρ(X), this property makes sense for the measures with possible negative values (property 8.2 fulfilled).

Property 14. Allocation

A risk measure need not be defined on the whole set of values of a random variable. Formally, in a given set U, from the condition $F_X = F_Y$, when $x \notin U$, it follows that ρ(X) = ρ(Y). Apparently, this property holds only for law-invariant measures. Most often, some threshold value T is assigned, and the set U takes values $U = (-\infty, T]$ or $U = [T,\infty)$.

Property 15. Static and dynamic natures

It is useful to use a dynamic and multi-period framework to answer the following question: how should an institution proceed with new information in each period and how should it reconsider the risk of the new position? Riedel (2004) introduced the specific axioms such as predictable translation invariance and dynamic consistency for a risk measure to capture the dynamic nature of financial markets.

CRM satisfies properties of *monotonicity, translation invariance, positive homogeneity*, and *subaddivity*[1].

DRM satisfies properties of conditional *state independence, monotonicity, comonotonic additivity* and *continuity*[2].

The notion of comonotonicity is central in risk measures[3]. Imposed axiom comonotonic additivity based on the argument that the comonotonic random variables do not hedge against each other, leads to additivity of risks.

It was also proved[4] that for all the *Bernoulli*(*n, p*) random variables, if $0 \le p \le 1$, then a DRM ρ satisfies $\rho(1) = 1$ if and only if ρ has a Choquet integral representation with respect to a distorted probability; i.e.

[1] P. Artzner, F. Delbaen, J.M. Eber, and D. Heath. Coherent measures of risk. *Mathematical finance*, 9(3):203–228, 1999.
[2] Trzpiot G., O własnościach transformujących miar ryzyka, Studia Ekonomiczne UE w Katowicach nr 91, 21–36, 2012.
[3] J. Dhaene, S.S. Wang, V.R. Young, and M.J. Goovaerts.
[4] Wang, 2000.

$$\rho_g(X) = \int X d(g \circ P) = \int_{-\infty}^{0} [g(P(X > x)) - 1] dx + \int_{0}^{\infty} [g(P(X > x)) dx \tag{6}$$

where $g(\cdot)$ is known as the distortion function which is nondecreasing with $g(0) = 0$,
and
$g(1) = 1$ and $(g \circ P)(A) := g(P(A))$ is called the distorted probability. The Choquet integral representation of DRM can be used to explore its mathematical properties. Furthermore, calculations of DRMs can easily be done by taking the expected value of X under probability measure $P^* := g \circ P$.

For discretely distributed portfolio losses random variable $\boldsymbol{l}$ □$(l_1, \ldots, l_m)$ with probability masses $P[\boldsymbol{l} = l_i] = p_i$ dla $i = 1, \ldots, m$, with cdf done as

$$F_l(l) = \sum_{i=1}^{m} p_i \mathbf{1}_{\{l_i \le l\}}$$

and the survival function $S_l(l) = 1 - F_l(l)$, (6) becomes

$$E_{P^*}(X) = \int_0^\infty S^*(x) dx - \int_0^\infty F^*(-x) dx = \int_0^\infty g[S(x)] dx - \int_0^\infty \tilde{g}[F(-x)] dx$$

$$= \int_0^\infty g[S(x)] dx - \int_0^\infty \{1 - g[S(-x)]\} dx = \rho_g(X).$$

Here we list some commonly used distortion functions:

− CVaR distortion:

$$g_{CVaR}(x, \alpha) = \min\{x/(1-\alpha), 1\} \text{ dla } \alpha \in [0,1) \tag{7}$$

− Wang Transform (WT) distortion:

$$g_{WT}(x, \beta) = \Phi[\Phi^{-1}(x) - \Phi^{-1}(\beta)] \text{ dla } \beta \in [0,1) \tag{8}$$

- the dual-power g_{DP} distortion[1]:

$$g_{DP}(x, \nu) = 1 - (1 - x)^{\frac{1}{\nu}}, \; x \in [0, 1], \; \nu \le 1 \tag{9}$$

– Proportional hazard (PH) distortion:

$$g_{PH}(x, \gamma) = x^{\gamma}, \; x \in [0, 1], \; \gamma \le 1. \tag{10}$$

[1] Wirch J, Hardy MR (2001), Trzpiot (2004a, 2006).

4. Coherent Distortion Risk Measure (CDRM)-based portfolio selection

Recall that CDRM is the intersection of CRM and DRM. There are two ways to define CDRM.

Definition 4.1. *We say ρ is a coherent distortion risk measure (CDRM) if:*

a) $\rho\, g$ is a distortion risk measure (DRM) with a concave distortion function g

or equivalently

b) ρ is a coherent risk measure (CRM) that is also comonotonic and law-invariant.

The following representation theorem for CDRM is the key result that enables us to use a convex optimization framework for any CDRM portfolio selection problem.

Theorem 4.1.[1] *For any random variable X and a given concave distortion function g, risk measure ρ_g is a CDRM if and only if there exists a function $w:[0,1] \to [0,1]$, satisfying $\int_{\alpha=0}^{1} w(\alpha)d\alpha = 1$ such that*

$$\rho_\alpha(X) = \int_{\alpha=0}^{1} w(\alpha)\phi_\alpha(X)d\alpha \tag{11}$$

where $(X)\ \alpha\,\phi$ is the α-*CVaR* of X.

This representation theorem says that any CDRM can be represented as a convex combination of $CVaR_\alpha\ (X)$, $\alpha \in [0, 1]$ and we can construct any CDRM based on a convex combination of $CVaR_\alpha\ (X)$. Such result was proved for continuous portfolio loss distributions[2]. Proved and strengthened was the representation theorem that any CDRM can be represented as a convex combination of finite number of $CVaR\alpha\ (X)$ under the assumption that the portfolio loss has a discrete uniform distribution[3].

For solving convex programming formulation CDRM portfolio selection problem, we need a theorem for CDRM to general discrete loss distributions. We notice the following definition[4]:

[1] S. Kusuoka, On law invariant coherent risk measures. Advances in mathematical economics, 3:83–95, 2001.
[2] Ibid.
[3] D. Bertsimas and D.B. Brown. Constructing uncertainty sets for robust linear optimization. Operations research, 57(6):1483–1495, 2009.
[4] Feng M. B., Tan K. S. (2012).

Definition 4.2. *For a given loss observation* $\boldsymbol{l} = (l_1, \ldots, l_m)$ *and its ordered losses* $l_{(1)} < l_{(2)} < \ldots < l_{(m)}$, $p_{(i)}$ *be the probability of realizing* $l(i)$, $i = 1, m$ *and* $S_l\left(l_{(i)}\right) = 1 - \sum_{j=1}^{i} p_{(i)}$. *Define a CVaR*$_a$ *matrix* $\mathbf{Q} \in \mathrm{R}^m \times \mathrm{R}^m$ *with columns* m $\mathbf{Q}_i \in \mathrm{R}^m$, $i = 1, m$ *as*

$$\mathbf{Q} = [\mathbf{Q}_1, \mathbf{Q}_2, \ldots, \mathbf{Q}_m] = \begin{bmatrix} p(1) & 0 & 0 & \ldots & 0 \\ p(2) & \dfrac{p(2)}{1 - S_l\left(l_{(1)}\right)} & 0 & \ldots & 0 \\ p(3) & \dfrac{p(3)}{1 - S_l\left(l_{(1)}\right)} & \dfrac{p(3)}{1 - S_l\left(l_{(2)}\right)} & \ldots & 0 \\ .. & \ldots & \ldots & \ldots & \ldots \\ p(m) & \dfrac{p(m)}{1 - S_l\left(l_{(1)}\right)} & \dfrac{p(m)}{1 - S_l\left(l_{(2)}\right)} & \ldots & \dfrac{p(m)}{1 - S_l\left(l_{(m-1)}\right)} \end{bmatrix}$$

Since portfolio losses are discretely distributed at m points, there are m jumps in the cumulative function of $\boldsymbol{l}$.

By defining

$$\alpha_i = \begin{cases} 0, & dla \quad i = 1 \\ \sum_{j=1}^{i-1} p_{(j)} & dla \quad i = 2, \ldots m \end{cases} \tag{12}$$

at these m jumps, the m CVaRs at these probability levels are given by

$$\phi_{\alpha_i}(l) = \frac{1}{1 - \alpha_i} \sum_{j=i}^{m} p(j) l(j) = \sum_{j=i}^{m} \frac{p(j)}{1 - S_l\left(l_{(m-1)}\right)} l(j) = \sum_{j=i}^{m} Q_{ij} l(j) \tag{13}$$

for $i = 1, m$ and Q_{ij} is the (i, j) -th entry of $\mathbf{Q}$. Note that column $\mathbf{Q}_i$ is essential to the calculation of $CVaR_{(i-1)/m}(\boldsymbol{l})$ and hence it explains the name of the matrix.

We consider the following special function for some $w(\alpha) \geq 0$ and $\int_{\alpha=0}^{1} w(\alpha) d\alpha = 1$,

$$M_g(x, \zeta) = \int_{\alpha=0}^{1} w(\alpha) F(x, \zeta_\alpha) d\alpha . \tag{14}$$

Theorem 4.1. of CDRM ensures the existence of $w(\alpha)$, $\alpha \in [0,1]$ and defines CDRM for a given set of weights. For each α there is a corresponding auxiliary variable ζ_α . Taking partial derivatives w.r.t. all ζ_α and setting them equal to zeros give the extremal properties of $Mg\,(\mathbf{x},\boldsymbol{\zeta})$. This provides more insights about the connection between a particular CDRM, $\rho_g(\mathbf{x})$, and its convex representation $Mg\,(\mathbf{x},\boldsymbol{\zeta})$[1] . Yet $\boldsymbol{\zeta}$ may have infinitely many entries ζ_α.

Taking partial derivative w.r.t. all $\zeta\alpha$ for $\alpha \in [0,1]$ requires calculus of variations, which is outside the scope of this thesis. We alleviate such difficulty by applying properties of Choquet integrals because CDRM is a subclass of DRM.

We conclude by presenting this generalized CVaR-based portfolio model of [1] to the more general class of CDRM-based portfolio model:

Theorem 4.2 *Let $\rho_g(\mathbf{x})$ be a CDRM with a corresponding distortion function g. Minimizing $\rho_g(\mathbf{x})$ with respect to $\mathbf{x}\in\mathbf{S}$ is equivalent to minimizing $Mg\,(\mathbf{x},\boldsymbol{\zeta})$ over all $(\mathbf{x}, \boldsymbol{\zeta}) \in \mathbf{S}\times |\boldsymbol{\zeta}|$, in the sense that*

$$\min_{x\in S} \rho_g(x) = \min_{(x,\zeta)\in S\times R^{\varsigma}} M_g(x,\zeta) \tag{15}$$

where moreover

$$\left(x^*,\zeta^*\right)\in \underset{(x,\zeta)\in S\times R^{\varsigma}}{\arg\min}\; M_g(x,\zeta) \Leftrightarrow x^* \in \underset{x\in S}{\arg\min}\,\rho_g(x),\; \zeta^* \in \underset{\zeta\in R^{\varsigma}}{\arg\min}\, M_g\left(x^*,\zeta\right) \tag{16}$$

As some remarks we can notice that:

- It is of interest to note that the PH portfolio optimization is almost equivalent to optimization over two extreme CVaR-based portfolios: one with $\alpha = 0.99$ and the other with $\alpha = 0$. Recall that minimizing CVaR with high value of α implies that you are someone who is very risk averse and hence is interested in risk minimization. In contrast, minimizing CVaR with α close to 0 implies an investor is a risk seeker and is only interested in maximizing expected return.
- Consistent with the classical trade-off theory on risk and reward, a more risk averse investor seeks an optimal portfolio with lower risk (as measured by the respective CDRM) but at the expense of lower expected return. Hence, the expected return of the optimal portfolio decreases with α for CVaR, decreases with β for WT, increases with γ for PH, and increases with δ for LB.[2]

[1] Ibid.

[2] Trzpiot G. (2010). Pessimistic portfolio optimization, In: Modelling of preferences and risk '09, Scientific Papers, University of Economics in Katowice, 121–128.

5. Conclusion

In this paper we present extension of the well-known linear optimization framework for CVaR to a general class of risk measure known as the CDRM. We generalized the finite generation theorem for CDRM and showed that any CDRM can be defined as a convex combination of ordered portfolio losses and equivalently a convex combination of CVaRs. We make use of the latter to develop a CDRM-based portfolio optimization framework.

REFERENCES

ARTZNER, P., DELBAEN F., EBER J. M., HEATH, D., (1999). Coherent measures of risk. Mathematical finance, 9(3):203–228.

BAWA, V. S., LINDENBERG E. B., (1977). Capital market equilibrium in a mean-lower partial moment framework. Journal of Financial Economics, 5(2): 189–200.

BERTSIMAS, D. D. B., (2009). Brown, Constructing uncertainty sets for robust linear optimization. Operations research, 57(6):1483–1495.

DENNEBERG, D., (1994). Non-additive measure and integral. Kluwer, Dordrecht.

DHAENE, J., WANG, S. S., YOUNG, V. R., GOOVAERTS, M. J., (2000). Comonotonicity and maximal stop-loss premiums. Bulletin of the Swiss Association of Actuaries, 2:99–113.

FENG, M. B., TAN, K. S., (2012). Coherent Distortion Risk Measures in Portfolio Selection, System Engineering Procedia 4, 25–34.

HADAR, J., RUSSELL, W., (1969). Rules for ordering uncertain prospects, American Economic Review 59, 25–34.

HARVEY, C. R. J., LIECHTY, M., LIECHTY, P., (2010). Müller. Portfolio selection with higher moments. Quantitative Finance, 10(5):469–485.

KUSUOKA, S., (2001). On law invariant coherent risk measures. Advances in mathematical economics, 3:83–95.

MARKOWITZ, H. M., (1952). Portfolio selection. The Journal of Finance, 7(1):77–91.

MARKOWITZ, H. M., (1959). Portfolio selection: efficient diversification of investments. New Haven, CT: Cowles Foundation, 94.

ORTOBELLI, S., (2001). The classification of parametric choices under uncertainty: analysis of the portfolio choice problem, Theory and Decision 51, 297–327.

ORTOBELLI, S., RACHEV, S., SHALIT, H., FABOZZI, F., (2006). Risk probability function- analysis and probability metrics applied to portfolio theory, http://www.statistik.unikarlsruhe.de.

ROCKAFELLAR, R. T., URYASEV, S., (2002). Conditional value-at-risk for general loss distributions. Journal of Banking & Finance, 26(7):1443–1471.

ROCKAFELLAR, R. T., URYASEV, S., (2000). Optimization of conditional value-at-risk. Journal of risk, 2:21–42.

ROTHSCHILD, M., STIGLITZ, J., (1970). Increasing risk. I.a. definition, Journal of Economic Theory 2, 225–243.

ROY, A. D., (1952). Safety first and the holding of assets. Econometrica: Journal of the Econometric Society, pages 431–449.

TRZPIOT, G., (2005). Partial Moments and Negative Moments in Ordering Asymmetric Distribution, in: Daniel Baier and Klaus-Dieter Wernecke (eds.): Innovations in Classification, Data Science and Information Systems, Proc. 27th Annual GFKL Conference, University of Cottbus, March 11-14 2003. Springer-Verlag, Heidelberg-Berlin, 181–188.

TRZPIOT, G., (2010). Pesymistyczna optymalizacja portfelowa [Pessimistic portfolio optimization], In: Modelling of preferences and risk '09, Scientific Papers, University of Economics in Katowice, 121–128

TRZPIOT, G., (2012). Własności transformujących miar ryzyka [Properties of transforming risk measures], Economic Studies of the University of Economics in Katowice - Faculty Scientific Papers No. 91, Katowice, 21–36

WANG, S. S., (2000). A class of distortion operators for pricing financial and insurance risks. The Journal of Risk and Insurance, 67(1):15–36.

WIRCH, J., HARDY, M. R., (2001). Distortion risk measures: coherence and stochastic dominance. Working paper.