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The Measurement of the Gross Domestic Product affected by the shadow economy

Anna Czapkiewicz[1], **Katarzyna Brzozowska-Rup**[2]

Abstract

The article presents a method for balancing Gross Domestic Product (GDP) when the measurements of its components are distorted by the existence of the shadow economy. Our proposal to measure GDP is based on a multiple ultrastructural model (MUM), where the explanatory variables are subject to error. We show that the expected value of GDP can be divided into two parts: the first part concerns data related to registered activities and the second part concerns unobserved data which may be partly related to the shadow economy. The empirical analysis is based on the annual data for individual voivodeships in Poland for the years 2000–2019. The data are obtained from the Local Data Bank of Statistics Poland. Two approaches to measuring GDP are considered: from the expenditure side and from the production side. The results show that the unobservable part of the variables necessary to balance GDP on the production side does not exceed 1% of GDP, and on the expenditure side, it mostly reaches about 3% of GDP.

Key words: measurement error, ultrastructural model, Gross Domestic Product, shadow economy.

1. Introduction

According to Regulation (EU) No 549/2013 of the European Parliament and of the Council of 21 May 2013 on the European system of national and regional accounts in the European Union, one of the main aggregates in the ESA is gross domestic product (GDP). It is a measure of the total economic activity taking place on a given economic territory. In 1995, ESA documents introduced the concept of including shadow economy data in GDP. According to the definition formulated by the European Commission, the shadow economy is an economic sector comprising a group of economic activities that are productive, in line with the SNA/ESA definition of production, legal in terms of compliance with legal norms and regulations, but hidden from public authorities. Methods for measuring the shadow economy can be divided into two groups: direct methods and indirect methods. The first group includes all survey research, while the second is limited to the analysis of "traces" in macroeconomic data.

In Poland, the size of the shadow economy is determined by national statistical offices on the basis of survey data. However, due to the problems of survey research, other methods

[1]Department of Applications of Mathematics in Economics, Faculty of Management, AGH University of Science and Technology, Cracow, Poland & Centre for Non-observed Economy Studies, Statistical Office in Kielce, Kielce, Poland. E-mail: czapkiewicz@agh.edu.pl. ORCID: https://orcid.org/0000-0002-6144-8381.

[2]Faculty of Management and Computer Modelling, Kielce University of Technology, Kielce, Poland. & Centre for Non-observed Economy Studies, Statistical Office in Kielce, Kielce, Poland. E-mail: krup@tu.kielce.pl. ORCID: https://orcid.org/0000-0003-1231-8027.

of estimating the shadow economy are discussed in the literature. Methods for estimating the shadow economy are described by Adair (2017, 2021), Błasiak (2018), Medina and Schneider (2019), Malczewska (2019), among others.

In the literature, special attention is paid to econometric models used to describe the relationships between some macroeconomic data. Such methods can be useful for estimating the shadow economy. One of the most used econometric models for this purpose is the Multiple Indicators Multiple Causes (MIMIC). The properties of the MIMIC model in the context of shadow economy estimation are discussed, for example, in Schneider et al. (2000, 2005a, 2005b, 2018), Trebicka (2014), Breusch (2005), Buszko (2017), Dybka et al. (2017, 2019). However, MIMIC is a confirmatory rather than an exploratory statistical technique. As Kirchgassner (2016) points out, the conclusion that the variable is considered a statistically significant determinant of the shadow economy may not be fully justified. The model, like many other hidden variable models used to measure an unobservable phenomenon, is based on the assumption that some relationship exists (Dybka et al., 2019).

This article presents a model that includes unobservable variables that is similar to the MIMIC model but has a slightly different structure. The proposed model determines a linear relationship between random variables where the dependent variable and some of the explanatory variables are subject to measurement error. The problems of linear models with errors in the variables have been discussed by Kendall and Stuart (1973), Dolby (1976), Chan and Mak (1984), Gillard (2006), and others. Dolby (1976) considers an ultrastructural relationship with only a single explanatory variable. The model discussed in this paper extends the reasoning of Dolby (1976) for a multivariate case, hereafter called the Multiple Ultrastructural Model (MUM). A maximum likelihood estimation algorithm is also presented, which allows us to effectively estimate the unknown model parameters and data relationships. The structure of the MUM allows to determine the relationships between GDP and its components, among other things. Gross domestic product is measured using three different approaches - output (production), income and expenditure - which are then compiled (properly balanced) to give a final estimate of GDP. Possible differences between these methods may result from the existence of shadow economy. Therefore, the possibility of defining SE as the difference between GDP in terms of production and expenditure has been considered, for example, by Madzarevic-Sujster (2001), Mikulic (2002), Czapkiewicz and Brzozowska-Rup (2021).

The empirical study of this paper examines the relationships between published data on GDP and its components. The data are taken from the Local Data Bank of Statistics Poland and cover the years 2000-2019. These data already include the informal economy. Therefore, the MUM model used to describe these relationships checks the correctness of the balance between different approaches to calculating GDP, taking into account SE.

The purpose of this article is twofold. First, to test whether the MUM model provides a good forecast of GDP, and second, to examine the balance of GDP resulting from the production and expenditure approach. In addition, this methodology allows us to estimate the percentage of GDP accounted for by unobservable data that may not have been included in the GDP calculations.

The paper is organized as follows. Section two defines the model and discusses its methodological features. Section three presents the data and results of the empirical study. A brief summary and conclusions are presented in the last section.

2. Model

Let $(X_t^1,\ldots,X_t^W,Y_t)$ $(t=1,\ldots,T)$ refer to observations that are subject to measurement error. True but unknown values of the explanatory variables are denoted as s_t^w and

$$X_t^w = s_t^w + \varepsilon_t^w \quad w=1,\ldots,W.$$

Let assume that

$$Y_t = \gamma_1 s_t^1 + \ldots + \gamma_W s_t^W + \eta_t$$

where ε_t^w and η_t are normally distributed measurement errors. However, under these assumptions the model is unidentifiable. Therefore, the dependent and explanatory variables are replicated N times to avoid the unidentifiability problem. They are further represented by the vector $(X_{i,t}^1,\ldots,X_{i,t}^W,Y_{i,t})$, where $i=1,\ldots,N$. The true values of the explanatory variables $(s_{i,t}^1,\ldots,s_{i,t}^W)$ satisfy a linear relation

$$Y_{i,t} = \gamma_1 s_{i,t}^1 + \ldots + \gamma_W s_{i,t}^W + \eta_{i,t}. \tag{1}$$

Furthermore, we assume that

$$s_{i,t}^w \sim N(s_i^w,\sigma_{s,w}), \quad X_{i,t}^w \sim N(s_i^w,\sigma_{\varepsilon,w}),$$

$$\varepsilon_{i,t}^w \sim N(0,\sigma_{\varepsilon,w}), \quad \eta_{i,t} \sim N(0,\sigma_\eta).$$

Assuming that $s_{i,t}^w,\varepsilon_{i,t},\eta_{i,t}$ are independent, the vector $(X_{i,t}^1,\ldots,X_{i,t}^W,Y_{i,t})$ is normally distributed with a mean of $(s_t^1,\ldots,s_t^W,\sum_{w=1}^W \gamma_w s_t^w)$ and a covariance matrix

$$\Sigma = \begin{bmatrix} V_{11} & V_{12} \\ V_{21}^T & V_{22} \end{bmatrix}$$

where $V_{11} = [V_1,\ldots,V_W] \times I_W$ and $V_w = \sigma_{s,w}^2 + \sigma_{\varepsilon,w}^2$,
$V_{22} = \sum_{w=1}^W \gamma_w^2 \sigma_{s,w}^2 + \sigma_\eta^2$, $V_{12} = [U_1,\ldots,U_W]^T$ and $U_w = \gamma_w \sigma_{s,w}^2$.

The unknown parameters of the model are estimated by the maximum likelihood method. To avoid technical problems in determining the estimators of the unknown parameters, $\sigma_{s,w}^2 = 0$ is assumed, then the vector of unknown parameters is

$$\Phi = (\gamma_1,\ldots,\gamma_W,\sigma_\eta,\sigma_{\varepsilon,1},\ldots,\sigma_{\varepsilon,W},s_1,\ldots,s_W).$$

Consider the vector

$$Z_i = (X_{i,1}^1,\ldots,X_{i,T}^1,\ldots,X_{i,T}^W,\ldots,X_{i,T}^W,Y_{i,1},\ldots,Y_{i,T}).$$

Its expectations (μ) and covariance matrix (V) take the form of

$$\mu = (s_1^1, \ldots, s_T^1, \ldots, s_1^W, \ldots, s_T^W, \gamma_1 s_1^1 + \ldots + \gamma_W s_1^W, \ldots, \gamma_1 s_T^1 + \ldots + \gamma_W s_T^W)$$

and

$$V = \Sigma \otimes I,$$

where $\otimes$ is the Kronecker matrix multiplication. The log-likelihood function corresponding to N replications has a form

$$L = C - \frac{N}{2} \ln |V| - \frac{1}{2} \sum_{i=1}^{N} d_i V^{-1} d_i \text{ where } d_i = Z_i - \mu. \tag{2}$$

Let V_ϕ be the matrix of derivatives computed with respect to ϕ ($\phi \in \Phi$), we get

$$\frac{\partial \ln L(\Phi)}{\partial \phi} = N \left(\frac{1}{2} tr(PV_\phi) - d_\phi^i V^{-1} d \right) \tag{3}$$

where $d = \sum_{i=1}^{N} d_i$, $P = V^{-1}(D - V)V^{-1}$ and $D = \frac{1}{N} \sum_{i=1}^{N} d_i d_i^T$.

By performing the appropriate calculations, we obtain estimators for the unknown parameters of the model. Let $\left(X_{.,t}^w\right)$ and $(Y_{.,t})$ denote averages of N replications at each time t and let

$$R_t^w = \left(X_{.,t}^w - s_t^w\right) = \frac{-z_t \gamma_w \sigma_{\varepsilon,w}^2}{\sigma_\eta^2 + \gamma_1^2 \sigma_{\varepsilon,1}^2 + \ldots + \gamma_W^2 \sigma_{\varepsilon,W}^2}, \quad w = 1, \ldots, W, \tag{4}$$

where

$$z_t = Y_{.,t} - \gamma_1 X_{.,t}^1 - \ldots - \gamma_W X_{.,t}^W. \tag{5}$$

The unknown parameters $\gamma_1, \ldots, \gamma_W$, σ_η^2 and $\sigma_{\varepsilon,w}^2$ satisfy the following non-linear equations

$$\sum_{t=1}^{T} (X_{.,t}^w - R_t^w)(\gamma_1 R_t^1 + \ldots + \gamma_W R_t^W + z_t) = 0$$

$$\sigma_\eta^2 = \frac{1}{TN} \sum_{i=1}^{N} \sum_{t=1}^{T} (Y_{i,t} - Y_{.,t})^2 + \frac{1}{T} \sum_{t=1}^{T} (\gamma_1 R_t^1 + \ldots + \gamma_W R_t^W + z_t)^2$$

$$\sigma_{\varepsilon,w}^2 = \frac{1}{TN} \sum_{i=1}^{N} \sum_{t=1}^{T} (X_{i,t}^w - X_{.,t}^w)^2 + \frac{1}{T} \sum_{i=1}^{N} (R_t^w)^2.$$

The next step is to estimate the unobservable parts of the explanatory variables. We found that the expected value $E(Y_{i,t})$ can be represented by two components $E(X_{i,t})$ and SE

$$E(Y_{i,t}) = \gamma_1 s_t^1 + \ldots + \gamma_W s_t^W = \gamma_1 (X_{.,t}^1 - R_t^1) + \ldots + \gamma_W (X_{.,t}^W - R_t^W) =$$

$$\gamma_1 X_{.,t}^1 + \ldots + \gamma_W X_{.,t}^W - (\gamma_1 R_t^1 + \ldots + \gamma_W R_t^W) = E(X_{i,t}) - SE.$$

The first component refers to the observed data. The second component refers to the unobservable part of the explanatory variables, which can be estimated using the formula

$$| E(Y_{i,t}) - E(X_{i,t}) | = | \hat{\gamma}_1 \hat{R}_t^1 + \ldots + \hat{\gamma}_W \hat{R}_t^W | = | SE | . \tag{6}$$

3. Empirical study

Gross Domestic Product is the primary indicator of economic activity, which can be estimated in three independent and theoretically equivalent ways. The results of these methods may differ, so a balancing process is carried out to obtain the final GDP (SNA93, ESA95). The article discusses two approaches: from the production side and from the expenditure side.

In the production approach, GDP is the gross value added of institutional sectors or industries (GP) minus the value of intermediate consumption (IC). In the expenditure approach, GDP is equal to the sum of the final uses of goods and services (all uses except intermediate consumption) plus exports and minus imports of goods and services. It corresponds to the expenditure of all purchasers of final goods produced during the year, including consumption (Cn), investment (In), government expenditure (Gov) and net exports (Exports (Ex)-Imports (Im)). In empirical study the MUM model is used to check the correctness of the balance between two approaches to calculating GDP when its components include unknown values related to the shadow economy.

3.1. Data

The analysis considers voivodeship data (there are 16 voivodeships in Poland) from 2000-2019. The voivodeship observations for each year are assumed to be replications in the MUM model. These data are transformed to ensure the desired statistical properties. Taking into account the differences in the logarithms of the observations, it is possible to compare values while avoiding large differences between voivodships.

The dependent variable is defined as $Y_{i,t} = \log\left(\frac{GDP_{i,t}}{GDP_{i,t-1}}\right)$. In the first model (the production approach) the explanatory variables are $P_{i,t} = \log\left(\frac{GP_{i,t}}{GP_{i,t-1}}\right)$ and $Z_{i,t} = \log\left(\frac{IC_{i,t}}{IC_{i,t-1}}\right)$, whereas in the second model (the expenditure approach) the explanatory variables are: $C_{i,t} = \log\left(\frac{Cn_t}{Cn_{t-1}}\right)$, $I_{i,t} = \log\left(\frac{In_t}{In_{t-1}}\right)$, $G_{i,t} = \log\left(\frac{Gov_{i,t}}{Gov_{i,t-1}}\right)$, $E_{i,t} = \log\left(\frac{Ex_{i,t}}{Ex_{i,t-1}}\right) - \log\left(\frac{Im_{i,t}}{Im_{i,t-1}}\right)$ and $i = 1\ldots,16$. The consumption and investment are noted by voivodeship, whereas government expenditures and net export are aggregated data for Poland. Hence, the model includes also data related to voivodeship budget expenditures, $B_{i,t}$.

Figure 1 shows the data considered in the empirical study.

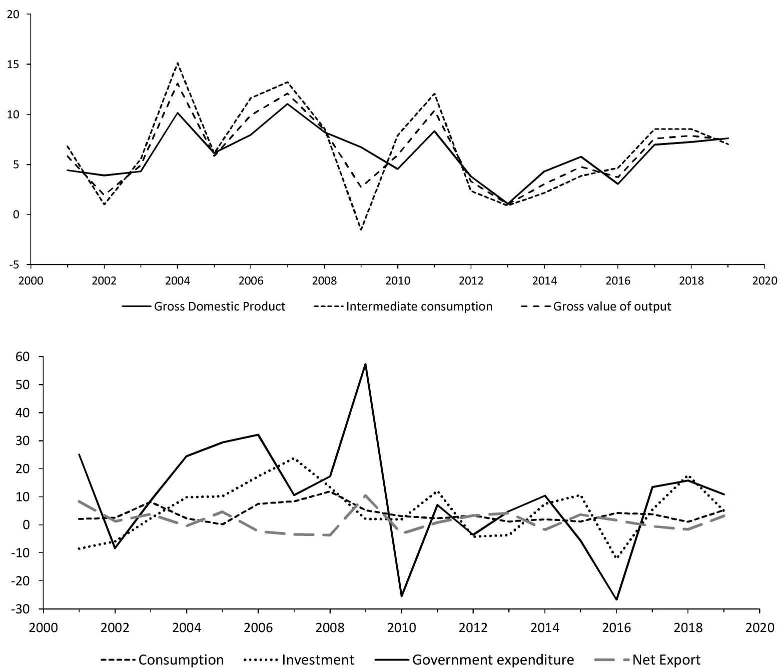

Figure 1: The average values (calculated by voivodeship) of transformed data related to GDP, intermediate consumption and gross value of output ($Y_{i,t}$, $P_{i,t}$, $Z_{i,t}$) (first panel); and the average values of transformed data related to consumption, investment, governed expenditure and net export ($C_{i,t}$, $I_{i,t}$, $B_{i,t}$, $E_{i,t}$) (second panel).

In order to illustrate the lack of meaningful differences in the replication by voivodeships taken into account in the MUM model, Figure 2 is presented. It shows four voivodeships with the lowest GDP growth (first panel) and four voivodeships with the highest GDP growth (second panel). The highest growth in the period under review was observed in the Małopolskie voivodeship and the lowest in the Zachodniopomorskie voivodeship. It should be noted, however, that the increases in GDP within each voivodeship are relatively small, so the data from the voivodeships can be used as replications in the MUM model.

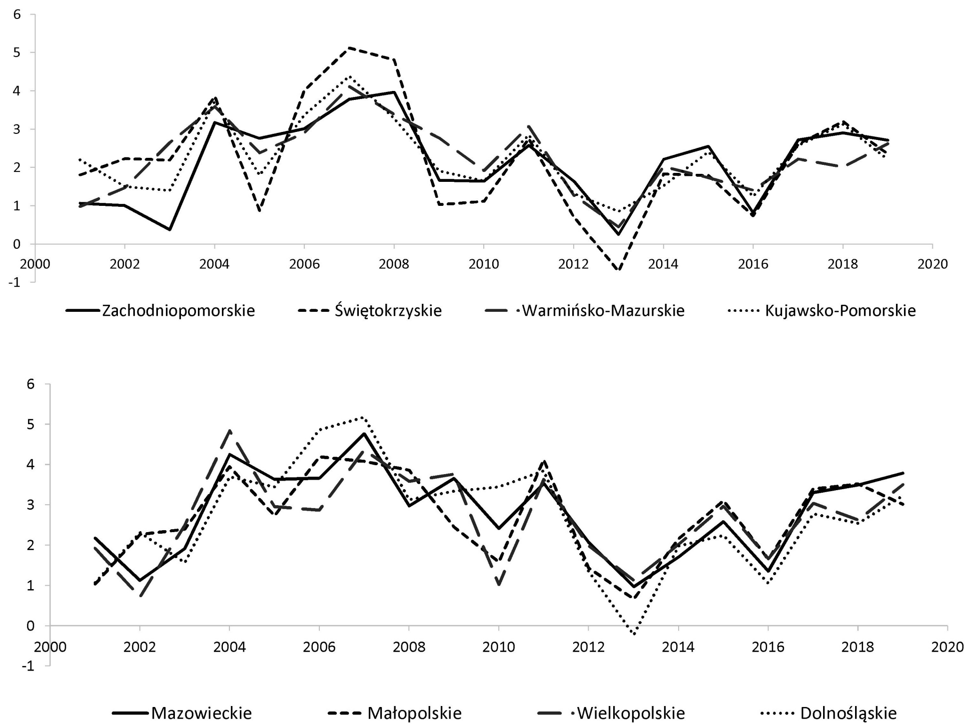

Figure 2: GDP growth in percent for voivodeships: voivodeships with the lowest GDP (first panel), voivodeships with the highest GDP (second panel).

3.2. Results

3.2.1 GDP - production approach

In this subsection GDP is calculated using global production and intermediate consumption. In this case, the MUM model has a form

$$P_{i,t} = s_t^P + \varepsilon_{i,t}^P$$

$$Z_{i,t} = s_t^Z + \varepsilon_{i,t}^Z$$

$$Y_{i,t} = \gamma_1 s_t^P + \gamma_2 s_t^Z + \eta_{i,t} \tag{7}$$

where

$$\varepsilon_{i,t}^P \sim N(0, \sigma_{\varepsilon,p}),\ \varepsilon_{i,t}^Z \sim N(0, \sigma_{\varepsilon,z}),\ \ \eta_{i,t} \sim N(0, \sigma_\eta),\quad i = 1, \ldots, N,\ \ t = 1, \ldots, T.$$

and s_t^P, s_t^Z denote true but unknown (unobserved) values of explanatory variables.

Table 1 shows the ML estimates of the parameters γ_1 and $\gamma 2$. For comparison, these parameters are also estimated from the Ordinary Least Square (OLS) regression model. To evaluate the goodness of fit of two models, the mean squared error (MSE) is calculated:

$$MSE = \sqrt{\frac{1}{T}\sum_{i=1}^{T}(Y_t - \hat{Y}_t)^2},$$

where Y_t denotes annual GDP published by Statistics Poland and $\hat{Y}_t$ are estimates derived from the given model. The last column of Table 1 shows the *MSE* results for both models.

Table 1: Estimates of the model parameters of the relationship between GDP and its components: global production and intermediate consumption, obtained from the OLS and MUM model.

Method	γ_1	γ_2	*MSE*
OLS	1.948^{***}	-0.962^{***}	0.449
MUM	2.036^{***}	-1.041^{***}	0.053

Note that the MUM model's mean square error is much smaller than the OLS model's (0.053 and 0.449, respectively).

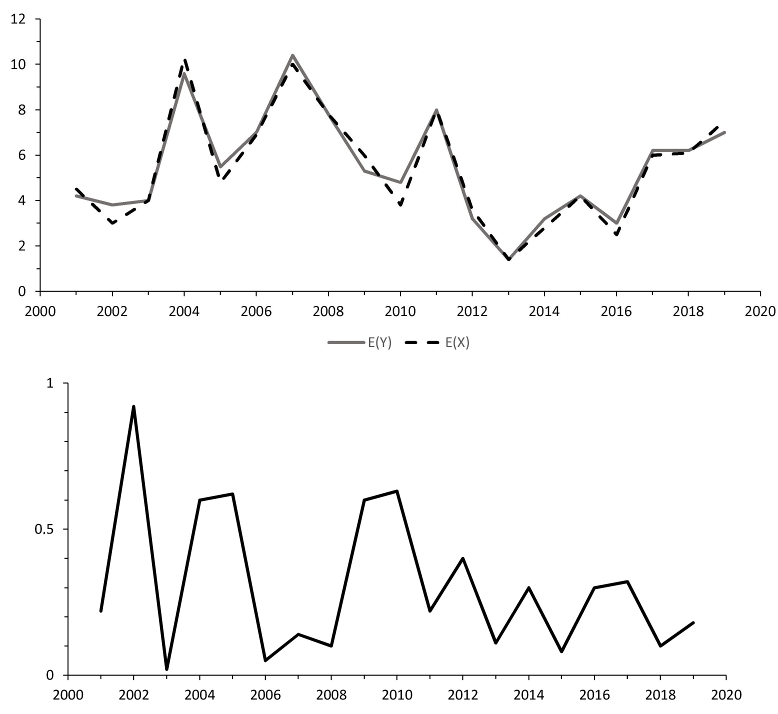

Figure 3: The expected value $E(Y_{i,t})$ and $E(X_t)$ - part of expected values, which is explained by observed data (first panel) and $\mid SE \mid$ (second panel).

From the formula (6) it is possible to determine the unobservable part of the explanatory variables:

$$\mid SE \mid = \mid \gamma_1 s_t^P + \gamma_2 s_t^Z - (\gamma_1 P_{.t} + \gamma_2 Z_{.t}) \mid = \mid \gamma_1 R_t^1 + \gamma_2 R_t^2 \mid .$$

The analysis indicates that the absolute value of the difference (denoted as $\mid SE \mid$) between the expected value $E(Y_{i,t})$ (obtained from the MUM model) and its part explained by the observed components $E(X_{i,t})$ reaches about 0.5% of GDP. The largest difference (1% of GDP) is observed only in the first half of the period. Thus, it can be concluded that in the production approach the data are correctly balanced with respect to the existence of the shadow economy.

The first panel of Figure 3 shows the expected GDP growth as the average of the growth of the voivodeships (denoted as $E(Y)$) and its part explained by the known and observed production components, denoted as $E(X)$. The second panel of Figure 3 shows the absolute value of their difference, $\mid SE \mid$, which may correspond to the existence of a shadow economy.

3.2.2 GDP - expenditure approach

The second approach to calculating GDP is based on expenditures. It takes into account household consumption, investment, government expenditures, and net exports. In addition, the model also takes into account the expenditure of the voivodeship budget. Hence, the MUM takes the form of

$$C_{i,t} = s_t^C + \varepsilon_{it}^C$$

$$I_{i,t} = s_t^I + \varepsilon_{it}^I$$

$$B_{i,t} = s_{i,t}^B + \varepsilon_{it}^B$$

and

$$Y_{i,t} = \gamma_1 s_{i,t}^C + \gamma_2 s_{i,t}^I + \gamma_3 s_{i,t}^B + \gamma_4 G_t + \gamma_5 E_t + \eta_{i,t}. \tag{8}$$

For comparison purposes, the ML estimates of the MUM are also compared with the OLS estimates. The results are shown in Table 2.

Table 2: Estimates of the model parameters of the relationship between GDP and its expenditure components, obtained from the OLS and MUM model

Method	γ_1	γ_2	γ_3	γ_4	γ_5	MSE
OLS	0.164	0.223^{**}	-0.050	0.635^{***}	0.208^{*}	1.738
MUM	0.126^{*}	0.605^{***}	-0.150	0.406^{***}	0.739^{**}	0.126

The MSE comparison of the two models shows that the MUM model is much more effective in the explanation of GDP than the model that does not take into account observation errors (OLS model). For the MUM the mean square error is 0.126, while for the OLS model it is as much as 1.738. Following a similar line of reasoning as in the previous section, the expected value of $Y_{i,t}$ is divided into two parts

$$E(Y_{i,t}) = \gamma_1 s_t^C + \gamma_2 s_t^I + \gamma_3 s_t^B + \gamma_4 G_{.,t} + \gamma_5 E_{.,t} =$$

$$\left(\gamma_1 C_{.,t} + \gamma_2 I_{.,t}^I + \gamma_3 B_{.,t}^B + \gamma_4 G_{.,t} + \gamma_5 E_{.,t}\right) - \left(\gamma_1 R_t^1 + \gamma_2 R_t^2 + \gamma_3 R_t^3\right)$$

In this case the level of the shadow economy $\mid SE \mid$ related to consumption, investment and expenditures of voivodeships, is estimated as:

$$\mid SE \mid = \mid \gamma_1 R_t^1 + \gamma_2 R_t^2 + \gamma_3 R_t^3 \mid .$$

It should be noted that the SE related to export and import is still unknown (no data to replicate). The first panel of Figure 4 shows the expected GDP growth as the average of the growth of the voivodeships (denoted as $E(Y)$) and its part explained by the expected value (denoted as $E(X_t)$) of the observed and known expenditure components. The second panel of Figure 4 shows the absolute value of their difference, denoted as $\mid SE \mid$, and it may also correspond to the existence of the shadow economy.

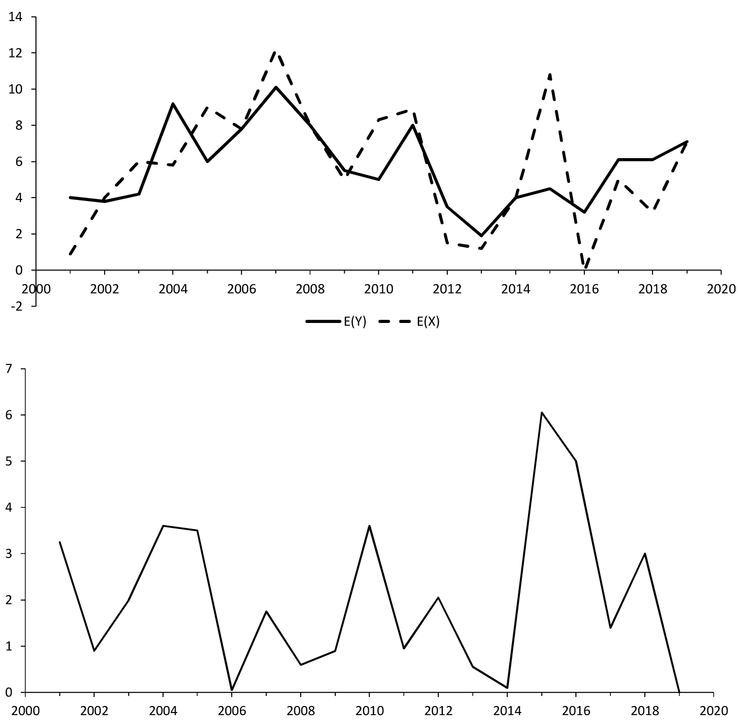

Figure 4: The expected value $E(Y_t)$ and $E(X_t)$ - part of expected values, which is explained by observed data (first panel) and $\mid SE \mid$ (second panel).

Analyzing Figure 4, we notice that the difference between the estimated expected value $E(Y_{i,t})$ and its part which is related to the observed data is relatively large. The average $\mid SE \mid$ estimate is about 3% of GDP. The highest $\mid SE \mid$ value (6% of GDP) is observed in 2016.

4. Conclusion

In the article, we presented an ultrastructural linear model (MUM) in which both the dependent and independent variables are subject to measurement error. We also developed an algorithm to estimate the parameters of this model based on the maximum likelihood method.

The MUM was used to estimate the relationship between GDP and its components, taking into account the existence of the shadow economy. In order to obtain consistent estimates of the model parameters, we adopted the voivodeship data as replications required in the MUM. However, the macroeconomic data used in the study are from Statistics Poland, which already takes into account the shadow economy. Therefore, the proposed method was considered as a way to validate or support the method of GDP adjustment in case of underestimation of the SE.

The results show that the MUM model is much more effective in explaining GDP than a model that does not take observation errors into account (OLS model). In addition, MUM provides estimates of the difference between published GDP and its components on the production side (about 0.5% of GDP) and on the expenditure side (about 3% of GDP).

The proposed methodology has several limitations. The main limitation of this method is the proper selection of variables and the method of creating their replication. In addition, the use of already aggregated data in the study leads to the loss of some information. Original survey data will be more appropriate not only to confirm the balance of GDP calculation methods, but also to estimate the size of the shadow economy. Therefore, the use of survey data in the MUM model to estimate the shadow economy (assuming that production and expenditure are balanced) will be the subject of further research.

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