STATISTICS IN TRANSITION new series, September 2024
Vol. 25, No. 3, pp. 83–102, https://doi.org/10.59170/stattrans-2024–029
Received – 07.04.2023; accepted – 25.04.2024

Extropy and entropy estimation based on progressive Type-I interval censoring

Huda H. Qubbaj[1], **Husam A. Bayoud**[2], **Hisham M. Hilow**[3]

Abstract

This paper proposes nonparametric estimates for the two information measures extropy and entropy when a progressively Type-I interval censored data is available. Different nonparametric approaches are used for deriving the estimates, including: moments of the empirical cumulative distribution function and linear regression. The performance of the proposed estimates is studied under various censoring schemes via simulation studies. Furthermore, different real data sets are analyzed for illustrative purposes.The estimates based on linear approximation $\hat{J}_2$ and $\hat{H}_2$ outperform the other estimate in the majority of studied cases.

Key words: entropy; extropy; mean square error; nonparametric statistics; Monte Carlo simulation; Type-I interval censoring.

1. Introduction

[Shannon,1948] defined entropy of a random variable (r.v.) X whose probability density function (pdf) $f(x)$ and cumulative distribution function (cdf) $F(x)$ as:

$$H(X) = -\int_{R_X} f(x)log\left(f(x)\right)dx, \tag{1}$$

where R_X denotes the support of the r.v. X.

For more details on entropy the reader can see [Renyi,1961], [Awad,1987], [Tsallis,1988], [Rao et al.,2004] and [Kittaneh et al.,2016].

The differential extropy of X is defined by [Lad et al.,2015] as:

$$J(X) = -\frac{1}{2}\int f^2(x)dx. \tag{2}$$

Important properties of the extropy measure have been discussed in the literature since 2015. See [Qiu,2017] and [Qiu and Jia,2018a], who studied properties of the residual ex-

[1]Department of Mathematics, The University of Jordan, Jordan. E-mail: h.qubaj@ju.edu.jo; hudaqubbaj.ctc@gmail.com. ORCID: https://orcid.org/0009-0004-9364-5082.
[2]College of Sciences and Humanities, Fahad Bin Sultan University, Saudi Arabia. E-mail: hbayoud@fbsu.edu.sa;husam.awni@yahoo.com. ORCID: https://orcid.org/0000-0002-0066-7206.
[3]Department of Mathematics, The University of Jordan, Jordan. E-mail: hhilow@ju.edu.jo. ORCID: https://orcid.org/0000-0001-7291-3579.

tropy and the extropy of ordered statistics as well as the extropy of record values; that is for random variable X its residual extropy is defined as:

$$J_t(X) = -\frac{1}{2}\int_0^{\infty} f_t^2(x)dx = -\frac{1}{2\bar{F}^2(T)}\int_t^{\infty} f^2(x)dx, t \geq 0 \tag{3}$$

where,

$$f_t(X) = \frac{f(x+t)}{\bar{F}(T)}, x \geq 0, t \geq 0 \tag{4}$$

and $\bar{F}$ is the survival function of X.

[Raqab and Qiu,2019] studied properties of the extropy measure under ranked set sampling. The problem of estimating the extropy based on complete sample data has recently been considered by some authors, including: [Qiu and Jia,2018b] and [Noughabi and Jarrahiferiz, 2019]. [Hazeb et al.,2021a] and [Hazeb et al.,2021b] introduced nonparametric estimates for extropy and entropy based on progressively Type-II censored data. However, estimation of the extropy and entropy measures under progressively Type-I interval censored data have not been considered so far in the literature. Accordingly, our main objective in this paper is to develop different methods for estimating the extropy and entropy measures under the progressive Type-I censoring set-up.

Censoring schemes of statistical experiments arise naturally in survival, reliability and medical studies. [Cohen,1963] introduced progressive Type-I censoring as an extension of Type-I censoring, where in a progressively Type-I censored life test on n items, progressive censoring is carried out at the prefixed censoring times $t_1 < t_2 < \ldots < t_k$. That is, at the i^{th} censoring time t_i, R_i items are randomly removed from the experiment, $1 \leq i \leq k-1$, with the restriction $R_1 + \ldots + R_{k-1} \leq n-l$, $l \in \{0,1,\ldots,n\}$. Then, at the k^{th} censoring time t_k, all remaining items are removed from the life test if there are any left. In many practical situations lifetimes of units placed on a test are observed within intervals, where this censoring scheme is called interval censoring.

[Aggarwala,2001] introduced progressive Type-I Interval censoring combining the two concepts: progressive Type-I censoring and Interval censoring, then developed statistical inference for the exponential distribution based on progressively type-I interval censored data. Under progressive type-I interval censoring, observations are only known within two consecutively pre-scheduled times, where items would be allowed to be withdrawn at pre-scheduled time points. [Ng and Wang,2009] introduced the concept of progressive type-I interval censoring using the Weibull distribution and compared many different estimation methods for two parameters of the Weibull distribution via simulation. [Lio et al.,2011] proposed parameter estimation for the generalized Rayleigh distribution under progressively Type-I interval censored data. Parameters of the inverse Weibull distribution under progressive type-I Interval censoring were estimated by [Singh and Tripathi,2016]. Also [Du et al.,2018] proposed Statistical Inference for the Entropy of the Log-Logistic Distribution under Progressive Type-I Interval Censoring Schemes. Recently, [Al otaibi et al.,2021] introduced Bayesian estimation for Dagum distribution based on progressive type -I Interval censoring.

The rest of the paper is organized as follows. Progressive Type-I interval censoring scheme with its properties are discussed in Section 2. Nonparametric estimates for extropy and entropy measures based on progressive Type-I interval censoring are developed in Section 3. Simulation experiments as well as analyses of real life data are performed in Section 4. Finally, we end the the paper in Section 5 with a conclusion.

2. Progressive Type-I Interval Censoring

Several censoring schemes have been proposed in the literature for saving time and cost of reliability experiments. Type I and type II censoring are probably the most commonly used methods in this regard. In type I censoring, failure times are recorded up to a specified time point where failures occurring afterward this fixed time are not recorded. On the other hand, in type II censoring a life test continues until a pre-specified number of failures have been recorded. These two censoring methods share one common feature that live units can be withdrawn only at the termination point of the test. In progressive censoring, items put on a test can be withdrawn during life experimentation, hence this censoring scheme is more flexible compared to the other two basic schemes. Progressive censoring has been widely studied in lifetime analysis by various researchers. One may refer to [Balakrishnan and Cramer ,2014] for an exhaustive list of references on this topic and also for a detailed discussion on applications of progressive censoring in life testing experiments. Sometimes it is difficult to record exact failure times of items under life testing due to lack of continuous monitoring of subjects or items under study. In such situations, observations are often recorded in intervals and the corresponding censoring is referred to as the interval censoring. [Aggarwala,2001] initially discussed progressive type I interval censoring in the literature and studied an exponential distribution using this censoring. Since then this censoring scheme has attracted attention among researchers. Progressive Type-I interval censoring can be briefly described as follows: Suppose n identical items are placed simultaneously on life testing at time $t_0 = 0$, where inspection is at m pre-fixed censoring times $t_1 < t_2 <.....< t_m$, and where t_m is the scheduled time to terminate the experiment and m is pre-fixed number of stops. For $i = 1,2,...,m$, let k_i be the number of failures in the interval $(t_{i-1},t_i]$. Let S_i be the number of the surviving items at t_i and R_i be the number of removed items at t_i. In this censoring scheme, k_i and S_i are random numbers while R_i is the number of remaining items, which is also a random number. At the 1^{st} inspection time t_1, we observe k_1 failures, then R_1 surviving items are randomly withdrawn from the remaining items $n-k_1$. One can see that after this step, the number of remaining items is $(n-k_1-R_1)$. Now, after time t_1 and at the 2^{nd} inspection time t_2, we observe k_2 failures where R_2 items are randomly removed from $(n-k_1-k_2-R_1)$ items. Lastly, at the m^{th} inspection time (the last inspection time), we observe k_m failures and all remaining $(n-\sum_{i=1}^{m} k_i - \sum_{i=1}^{m-1} R_i)$ items are immediately removed from the experiment. The observed progressive Type-I interval censored data can be represented as: $\{(k_i,R_i,t_i), i = 1,2,...,m\}$. The associated likelihood function under the progressive type-I interval censoring is given by:

$$L(\theta) \propto \prod_{i=1}^{m} \left[F(t_i;\theta) - F(t_{i-1};\theta)\right]^{k_i} \left[1-F(t_i;\theta)\right]^{R_i}. \tag{5}$$

Note that R_i should not be greater than S_i,where the values of R_i for $i = 1,2,...,m$ are determined based on pre-specified removal proportions $q_1, q_2, ..., q_{k-1}$ and $q_m = 1$, such that $R_i = [S_i q_i]$, for $i = 1,2,...,k-1$, where symbol $[b]$ is the greatest integer less than or equal to b. It can be easily seen that $n = \sum_{i=1}^{m}(R_i + k_i)$. If $R_i = 0$, for $i = 1,2,...,m-1$, then Progressive type-I interval censoring reduces to the conventional type-I censoring.

Theorem 2.1. *Let $U_{i:m:n} = F(t_i)$, $i = 1,2,...,m$ denote a progressively Type-I interval censoring sample obtained from the uniform $(0,1)$ distribution, assuming the sample size is n with progressive Type-I interval censored data $\{(k_i, R_i, t_i), i = 1,2,...,m\}$. Let*

$$U_{i:m:n} = 1 - \prod_{j=m-i+1}^{m} V_j,$$

where,

$$V_1 = \frac{1-U_{m:m:n}}{1-U_{m-1:m:n}}, V_2 = \frac{1-U_{m-1:m:n}}{1-U_{m-2:m:n}}, ..., V_m = 1-U_{1:m:n}, \tag{6}$$

are all independent identically distributed (iid) r.v.'s. Then

$$V_i \overset{d}{=} Beta\left(i + \sum_{j=m-i+2}^{m} k_j + \sum_{j=m-i+1}^{m} R_j, k_{m-i+1} + 1\right), \qquad i = 1,2,...,m. \tag{7}$$

Proof. For simplicity, we denote $U_{i:m:n}$ by U_i.

Since U_i follows $U(0,1)$, then the pdf and cdf of U_i are $f_{U_i}(u) = 1$ and $F_{U_i}(u) = u$, for $0 \le u \le 1$, respectively. So, using Eq.(5), the joint pdf of $U_1 \le U_2 \le ... \le U_m$ is obtained as follows:

$$f_{U_{l:m:n}, U_{2:m:n}, ..., U_{m:m:n}}(u_1, u_2, .., u_m) \propto \prod_{i=1}^{m} (u_i - u_{i-1})^{k_i} (1-u_i)^{R_i}. \tag{8}$$

Since $U_i = 1 - \prod_{j=m-i+1}^{m} V_j$, one can see that:

$$\begin{aligned} U_1 &= 1 - V_m \\ U_2 &= 1 - V_{m-1}V_m \\ U_3 &= 1 - V_{m-2}V_{m-1}V_m \\ &\cdot \\ &\cdot \\ &\cdot \\ U_m &= 1 - V_1V_2V_3...V_m \end{aligned} \tag{9}$$

The Jacobian matrix of this transformation is given by the following lower triangular matrix:

$$J=\frac{\partial}{\partial V}U=\begin{pmatrix} 0 & 0 & \cdots & 0 & -1 \\ 0 & 0 & \cdots & -V_m & -V_{m-1} \\ \vdots & \vdots & \ddots & \vdots & \\ -V_2V_3...V_m & -V_1V_3...V_m & \cdots & -V_1V_2...V_{m-2}V_m & -V_1V_2...V_{m-1} \end{pmatrix}$$

Therefore,

$$|J|=\left|\frac{\partial}{\partial V}U\right|=\prod_{j=2}^{m}V_j^{j-1}. \tag{10}$$

Now, assuming $U_0=0$, we have:

$$\begin{aligned} U_1-U_0 &= 1-V_m \\ U_2-U_1 &= V_m\left(1-V_{m-1}\right) \\ U_3-U_2 &= V_{m-1}V_m\left(1-V_{m-2}\right) \\ &. \\ &. \\ &. \\ U_m-U_{m-1} &= V_2V_3...V_m\left(1-V_1\right) \end{aligned} \tag{11}$$

Now, in order to derive the joint pdf of $V_1,V_2,...,V_m$, we simplify the terms of Eq.(8) separately.

Using Eq.(10), the first term in Eq.(8) is simplified to:

$$\prod_{i=1}^{m}(U_i-U_{i-1})^{k_i}=\prod_{i=1}^{m}(1-V_{m-i+1})^{k_i}\prod_{i=1}^{m}V_{m-i+1}^{\sum_{j=i+1}^{m}k_j}. \tag{12}$$

Using Eq.(9), the second term in Eq.(8) is simplified to:

$$\prod_{i=1}^{m}(1-U_i)^{R_i}=\prod_{i=1}^{m}V_{m-i+1}^{\sum_{j=i}^{m}R_j}. \tag{13}$$

Accordingly, the joint pdf of $V_1,...,V_m$ is obtained as follows:

$$f_{V_l,V_2,...,V_m}(v_1,v_2,..,v_m)\propto\prod_{i=1}^{m}(1-V_i)^{k_m+i-1}\prod_{i=2}^{m}V_i^{\sum_{j=m-i+2}^{m}k_j}\prod_{i=1}^{m}V_i^{\sum_{j=m-i+1}^{m}R_j}\prod_{i=2}^{m}V_i^{i-1}. \tag{14}$$

Since $\prod_{j=2}^{m}V_j^{j-1}=\prod_{j=1}^{m}V_j^{j-1}$ and $\prod_{i=2}^{m}V_i^{\sum_{j=m-i+2}^{m}k_j}=\prod_{i=1}^{m}V_i^{\sum_{j=m-i+2}^{m}k_j}$.

Then, Eq.(14), can be simplified to:

$$f_{V_l,V_2,\ldots,V_m}(v_1,v_2,..,v_m) \propto \prod_{i=1}^{m}(1-V_i)^{k_m+i-1}\, V_i^{\sum_{j=m-i+2}^{m}k_j+\sum_{j=m-i+1}^{m}R_j+i-1} \tag{15}$$

where $0<V_i<1$.

By factorization theorem, we see that $V_1,V_2,\ldots,V_m$ are independent and

$$V_i \stackrel{d}{=} Beta\left(i+\sum_{j=m-i+2}^{m}k_j+\sum_{j=m-i+1}^{m}R_j, k_{m-i+1}+1\right), \qquad i=1,2,\ldots,m.$$

□

Corollary 2.2. *As a result of Theorem (2.1), we find*

$$E(U_{i:m:n}) = 1-\prod_{j=m-i+1}^{m}\gamma_j, \tag{16}$$

where,

$$\gamma_i = \frac{i+\sum_{j=m-i+2}^{m}k_j+\sum_{j=m-i+1}^{m}R_j}{1+i+\sum_{j=m-i+1}^{m}k_j+R_j},$$

such that $\gamma_j=\gamma_1$ *if* $j\le 1$ *and* $\gamma_j=\gamma_m$ *if* $j\ge m$ *provided that* $\sum_{j=m+1}^{m}k_j=0.$

3. Nonparametric Extropy and Entropy Estimates

This section develops non-parametric estimates for the extropy and entropy measures based on progressively Type-I interval censored samples. It is of importance here to mention that for a random variable (r.v.) T, extropy and entropy measures $J(T)$ and $H(T)$ are expressed as:

$$J(T) = -\frac{1}{2}\int_0^1\left(\frac{d}{dp}F^{-1}(p)\right)^{-1}dp \tag{17}$$

and

$$H(T) = \int_0^1 log\left(\frac{d}{dp}F^{-1}(p)\right)dp \tag{18}$$

3.1. Moments Approximation Method

The first entropy and extropy estimate will be obtained by using the difference operator that was proposed by Vasicek (1976) for estimating the entropy. This method is based on the following fact:

$$\frac{d}{dp}F^{-1}(p) \approx \frac{T_{i+w:m:n}-T_{i-w:m:n}}{F(T_{i+w:m:n})-F(T_{i-w:m:n})}, \tag{19}$$

where the window size $w \leq m/2$, also $T_{i:m:n} = T_0 = 0$ if $i < 1$, while $T_{i:m:n} = T_{m:m:n}$ if $i > m$. Then $T_0 = 0 \leq T_{1:m:n} \leq T_{2:m:n} \leq \ldots \leq T_{m:m:n}$ are progressively Type-I interval censoring times of size m, which are pre-fixed. In the interval $[T_{i-1}, T_i)$, we observe a random number of failures, say k_i, then R_i surviving units are immediately removed from the remaining $(n - \sum_{j=1}^{i} k_i - \sum_{j=1}^{i-1} R_i)$ items.

Next, it can be seen that (17) is approximately equal to:

$$J(T) \approx -\frac{1}{2m} \sum_{i=1}^{m} \frac{F(T_{i+w:m:n}) - F(T_{i-w:m:n})}{T_{i+w:m:n} - T_{i-w:m:n}}. \tag{20}$$

Notice that $U_{i+w:m:n} = F(T_{i+w:m:n})$ and $U_{i-w:m:n} = F(T_{i-w:m:n})$, the moments-based estimate is proposed by replacing $U_{i+w:m:n}$ and $U_{i-w:m:n}$ by their moments, i.e. their expected values $E(U_{i+w:m:n})$ and $E(U_{i-w:m:n})$, respectively, which are given in (16). Consequently, an estimate of $J(T)$ is

$$\hat{J}_1 = -\frac{1}{2m} \sum_{i=1}^{m} \frac{\prod_{j=m-(i-w)+1}^{m} \gamma_j - \prod_{j=m-(i+w)+1}^{m} \gamma_j}{T_{i+w:m:n} - T_{i-w:m:n}}. \tag{21}$$

On the lines of Vasicek (1976), an estimate of the entropy is

$$\hat{H}_1 = \frac{1}{m} \sum_{i=1}^{m} log \left(\frac{T_{i+w:m:n} - T_{i-w:m:n}}{\prod_{j=m-(i-w)+1}^{m} \gamma_j - \prod_{j=m-(i+w)+1}^{m} \gamma_j} \right). \tag{22}$$

Proposition 3.1.1. *Let $Y = aT + b$, $a > 0$. Then $\hat{J_1}^Y = \frac{1}{a}\hat{J_1}^T$ and $\hat{H_1}^Y = log(a) + \hat{H_1}^T$.*

Proof. The result follows since for all i, $\gamma_i^Y = \gamma_i^T$. □

Proposition 3.1.2. *Estimates $\hat{J}_1$ and $\hat{H}_1$ are consistent estimates for J and H respectively, i.e.*

$$\hat{J}_1 \xrightarrow{p} J,$$

and

$$\hat{H}_1 \xrightarrow{p} H.$$

as $n \to \infty$, $m \to \infty$, and $m/n \to 0$.

Proof. Since when $m \to n$, the progressively Type-I interval censored sample becomes the complete sample. Then the estimates, $\hat{J}_1$ and $\hat{H}_1$ converge to the estimates proposed by [Qiu and Jia,2018b] and [Vasicek,1976], which are consistent estimates of J and H. □

3.2. Linear Approximation Method

The second estimate for extropy $J(T)$ in (2) is proposed following the steps of Correa (1995) by noticing that the quantity:

$$\frac{F(T_{i+w:m:n}) - F(T_{i-w:m:n})}{T_{i+w:m:n} - T_{i-w:m:n}}, \tag{23}$$

represents the slope of a straight line joining the following two points

$$(T_{i-w:m:n}, F(T_{i-w:m:n})) \; and \; (T_{i+w:m:n}, F(T_{i+w:m:n})).$$

This estimation approach is based on estimating the function $F(T_j)$ by a local linear model on $(T_{i-w:m:n}, T_{i+w:m:n})$ by using $2w+1$ ordered pairs:

$$F(T_j) = U_j = a + bT_j + \varepsilon, j = i-w, ..., i+w. \tag{24}$$

On the other hand, slope in (23) can be approximated by b in (22), which can be estimated by the least squares method using $(2w+1)$ ordered pairs as follows:

$$b = \frac{S_{TU}}{S_T^2} = \frac{\sum_{j=i-w}^{i+w}(T_{j:m:n} - \bar{T}_{(i)})(U_{j:m:n} - \bar{U}_{(i)})}{\sum_{j=i-w}^{i+w}(T_{j:m:n} - \bar{T}_{(i)})^2}, \tag{25}$$

where

$$\bar{T}_{(i)} = \frac{1}{2w+1} \sum_{j=i-w}^{i+w} T_{j:m:n}, \text{ and } \bar{U}_{(i)} = \frac{1}{2w+1} \sum_{j=i-w}^{i+w} U_{j:m:n}.$$

Now, by replacing $U_{j:m:n}$ by its expected value, we get

$$\hat{\bar{U}}_{(i)} = \frac{1}{2w+1} \sum_{j=i-w}^{i+w} \left(1 - \prod_{k=m-j+1}^{m} \gamma_k \right).$$

Consequently, a second estimate of $J(T)$ using the slope b in Eq.(25) and replacing $F(T_{j:m:n}) = U_{j:m:n}$ by its respective expected value in terms of γ_j in Eq.(21) is as follows:

$$\hat{J}_2 = -\frac{1}{2m} \sum_{i=1}^{m} \frac{\sum_{j=i-w}^{i+w}(T_{j:m:n} - \bar{T}_{(i)})(\frac{\sum_{j=i-w}^{i+w}\prod_{k=m-j+1}^{m}\gamma_k}{2w+1} - \prod_{k=m-j+1}^{m}\gamma_k)}{\sum_{j=i-w}^{i+w}(T_{j:m:n} - \bar{T}_{(i)})^2}. \tag{26}$$

Following the same argument proposed for $\hat{J}_2$ in Eq.(26), we consider the slope of the linear regression of T on F as follows:

$$b_h = \frac{S_{TU}}{S_U^2} = \frac{\sum_{j=i-w}^{i+w}(T_{j:m:n} - \bar{T}_{(i)})(U_{j:m:n} - \bar{U}_{(i)})}{\sum_{j=i-w}^{i+w}(U_{j:m:n} - \bar{U}_{(i)})^2}, \tag{27}$$

Using Eq(18), Eq(19) and Eq(27), a second estimate for $H(T)$ in (1) can be introduced as

$$\hat{H}_2 = \frac{1}{m} \sum_{i=1}^{m} log \left[\frac{\sum_{j=i-w}^{i+w}(T_{j:m:n} - \bar{T}_{(i)})(\frac{\sum_{j=i-w}^{i+w}\prod_{k=m-j+1}^{m}\gamma_k}{2w+1} - \prod_{k=m-j+1}^{m}\gamma_k)}{\sum_{j=i-w}^{i+w}(\frac{\sum_{j=i-w}^{i+w}\prod_{k=m-j+1}^{m}\gamma_k}{2w+1} - \prod_{k=m-j+1}^{m}\gamma_k)^2} \right]. \tag{28}$$

Proposition 3.2.1. *Let $Y = aT + b$, $a > 0$. Then $\hat{J}_2^{\,Y} = \frac{1}{a}\hat{J}_2^{\,T}$ and $\hat{H}_2^{\,Y} = log(a) + \hat{H}_2^{\,T}$.*

Proof. The result follows since for all i, $\gamma_i^Y = \gamma_i^T$. □

Proposition 3.2.2. *Estimates $\hat{J}_2$ and $\hat{H}_2$ are consistent estimates for J and H respectively, i.e.*

$$\hat{J}_2 \xrightarrow{p} J,$$

and

$$\hat{H}_2 \xrightarrow{p} H.$$

as $n \to \infty$, $m \to \infty$ and $m/n \to 0$.

Proof. Proof is obvious by [Correa,1995] and [Qiu and Jia,2018b] and so it is omitted. □

4. Simulation study and data analysis

Here, we perform an extensive simulation study to compare the act of the proposed extropy and entropy estimates. Moreover, a real data set is discussed for illustrative purposes.

4.1. Simulation study

In this subsection, we carry out a Monte Carlo simulation to analyze the behavior of our proposed estimators of entropy and extropy. In order to perform the simulation process, we consider different sample sizes, i.e. $n = 10, 20, 30, 50, 100$ with different inspection times, i.e. $m = 3, 4, 5$. Next, we generate 1000 progressive type-I interval censoring data sets in each experimentation case. Also, we work with different withdrawal (removal) schemes as detailed in Table 1. We consider the uniform distribution $U(0,\theta)$ with $\theta = 1$, the exponential distribution $Exp(\beta)$ with $\beta = 1$ and normal distribution $N(\mu,\sigma)$ with $\mu = 0, \sigma = 1$, which are commonly used.

Table 1: Progressive interval censoring schemes used in the Monte Carlo simulation study

Scheme No.	m	$(t_1,\ldots,t_m)$	$(q_1,\ldots,q_m)$
1	3	$0.1, 0.5, 0.9$	$0.25, 0.1, 1$
2	3	$0.1, 0.5, 0.9$	$0.5, 0.1, 1$
3	3	$0.1, 0.5, 0.9$	$0, 0.25, 1$
4	3	$0.1, 0.5, 0.9$	$0.1, 0.1, 1$
5	4	$0.1, 0.5, 0.7, 0.9$	$0, 0, 0.25, 1$
6	4	$0.1, 0.5, 0.7, 0.9$	$0, 0.25, 0.25, 1$
7	4	$0.1, 0.5, 0.7, 0.9$	$0.25, 0, 0, 1$
8	4	$0.1, 0.5, 0.7, 0.9$	$0.2, 0.2, 0.2, 1$
9	5	$0.1, 0.3, 0.5, 0.7, 0.9$	$0.25, 0, 0, 0, 1$
10	5	$0.1, 0.3, 0.5, 0.7, 0.9$	$0, 0, 0, 0, 1$
11	5	$0.1, 0.3, 0.5, 0.7, 0.9$	$0.25, 0.25, 0, 0, 1$
12	5	$0.1, 0.3, 0.5, 0.7, 0.9$	$0.1, 0.1, 0.2, 0.2, 1$

For each generated data set, we compute the average estimator and the corresponding mean squared error (MSE) of the proposed entropy and extropy estimators over 1000 simulations. Simulation results are shown in Tables (2-10), where the bold type in these tables indicates the estimator achieving the minimal MSE. Comments on simulation results are provided after these tables. Here, we will detail the withdrawal schemes used through Tables 2 to 10.

For schemes 1, 2, 3 and 4, we specify three inspection (stopping) times at which surviving units are progressively removed at different proportions as detailed in Tables 2, 5 and 8. In scheme 1, censoring by removal is heavier at the end of the first interval and lighter at the end of the second interval. In scheme 2, the same censoring approach is applied but with different proportions. In scheme 3, the reverse situation is applied, i.e. the censoring by removal is lighter at the end of the first interval and heavier at the end of the second interval. In scheme 4, equal (uniform) proportions are used.

Similarly, for schemes 5, 6, 7 and 8, we specify the proportions of the surviving units to be removed at four inspection times as presented in Tables 3, 6 and 9. In scheme 5, censoring by removal is lighter at the end of the first and second intervals and heavier at the end of the third interval. In scheme 6, censoring by removal is lighter at the end of the first interval and heavier at the end of the second and third intervals. In scheme 7, only left-most and right-most removal is applied. In scheme 8, equal proportions (uniform) are used.

At last, for schemes 9, 10, 11 and 12, we specify the proportions of the surviving units to be removed at five inspection times as shown in Tables 4, 7 and 10. In scheme 9, only left-most and right-most removal is applied. In scheme 10, a conventional interval censoring scheme is employed, i.e. all removal is carried out following the final time interval, immediately prior to experiment termination. In scheme 11, censoring by removal is heavier at the end of the first and second intervals and lighter at the end of the third and fourth intervals. In scheme 12, censoring by removal is lighter at the end of the first and second intervals and heavier at the end of the third and fourth intervals.

Table 2: MSEs of the estimates of extropy $J(Y)$ and entropy $H(Y)$ for uniform $(0,1)$ assuming $m=3$

Scheme Number	n	$\hat{J}_1$	$\hat{J}_2$	$\hat{H}_1$	$\hat{H}_2$
1	10	**0.0015**	0.0098	**0.0098**	0.0160
	20	**0.0003**	0.0066	**0.0018**	0.0050
	30	**0.0003**	0.0054	**0.0013**	0.0040
	50	**0.0002**	0.0046	**0.0005**	0.0025
	100	**0.0001**	0.0038	**0.0003**	0.0015
2	10	**0.0021**	0.0087	**0.0125**	0.0214
	20	**0.0009**	0.0050	**0.0044**	0.0088
	30	**0.0007**	0.0036	**0.0028**	0.0051
	50	**0.0006**	0.0025	**0.0023**	0.0033
	100	**0.0006**	0.0015	**0.0021**	0.0013
3	10	**0.0012**	0.0100	**0.0077**	0.0144
	20	**0.0006**	0.0074	**0.0033**	0.0084
	30	**0.0003**	0.0062	**0.0015**	0.0055
	50	**0.0001**	0.0053	**0.0006**	0.0039
	100	**0.0001**	0.0047	**0.0003**	0.0031
4	10	**0.0009**	0.0115	**0.0068**	0.0099
	20	**0.0005**	0.0088	**0.0031**	0.0066
	30	**0.0002**	0.0074	**0.0014**	0.0039
	50	**0.0001**	0.0066	**0.0008**	0.0031
	100	**0.0000**	0.0057	**0.0003**	0.0020

Table 3: MSEs of the estimates of extropy $J(Y)$ and entropy $H(Y)$ for uniform $(0,1)$ distribution assuming $m=4$

Scheme Number	n	$\hat{J}_1$	$\hat{J}_2$	$\hat{H}_1$	$\hat{H}_2$
5	10	**0.0037**	0.0076	**0.0246**	0.0281
	20	**0.0030**	0.0065	**0.0206**	0.0235
	30	**0.0019**	0.0048	**0.0120**	0.0143
	50	**0.0013**	0.0038	**0.0081**	0.0098
	100	**0.0006**	0.0027	**0.0035**	0.0048
6	10	**0.0041**	0.0077	**0.0278**	0.0311
	20	**0.0038**	0.0068	**0.0277**	0.0302
	30	**0.0028**	0.0054	**0.0207**	0.0226
	50	**0.0020**	0.0042	**0.0141**	0.0154
	100	**0.0010**	0.0028	**0.0064**	0.0075
7	10	**0.0036**	0.0070	**0.0233**	0.0271
	20	**0.0027**	0.0051	**0.0186**	0.0210
	30	**0.0020**	0.0038	**0.0132**	0.0151
	50	**0.0011**	0.0023	**0.0060**	0.0071
	100	**0.0006**	0.0012	**0.0024**	0.0027
8	10	**0.0039**	0.0072	**0.0257**	0.0295
	20	**0.0031**	0.0057	**0.0206**	0.0234
	30	**0.0030**	0.0052	**0.0200**	0.0221
	50	**0.0019**	0.0034	**0.0110**	0.0120
	100	**0.0009**	0.0020	**0.0053**	0.0059

Table 4: MSEs of the estimates of extropy $J(Y)$ and entropy $H(Y)$ for uniform $(0,1)$ distribution assuming $m=5$

Scheme Number	n	$\hat{J}_1$	$\hat{J}_2$	$\hat{H}_1$	$\hat{H}_2$
9	10	0.0039	**0.0038**	**0.0145**	0.0158
	20	0.0038	**0.0031**	**0.0136**	0.0142
	30	0.0037	**0.0025**	**0.0123**	**0.0123**
	50	0.0030	**0.0014**	0.0063	**0.0056**
	100	0.0026	**0.0007**	0.0042	**0.0026**
10	10	**0.0038**	**0.0038**	**0.0155**	0.0163
	20	0.0030	**0.0025**	**0.0101**	0.0104
	30	0.0028	**0.0018**	**0.0075**	**0.0075**
	50	0.0023	**0.0009**	0.0035	**0.0030**
	100	0.0021	**0.0003**	0.0023	**0.0014**
11	10	**0.0045**	0.0048	**0.0178**	0.0201
	20	0.0042	**0.0035**	**0.0155**	0.0162
	30	0.0040	**0.0030**	0.0145	**0.0143**
	50	0.0036	**0.0021**	0.0107	**0.0096**
	100	0.0030	**0.0011**	0.0075	**0.0049**
12	10	**0.0043**	0.0045	**0.0183**	0.0199
	20	**0.0036**	0.0039	**0.0175**	0.0184
	30	**0.0029**	0.0031	**0.0134**	0.0138
	50	**0.0024**	0.0025	0.0118	**0.0117**
	100	0.0015	**0.0013**	0.0057	**0.0050**

Table 5: MSEs of the estimates of extropy $J(Y)$ and entropy $H(Y)$ for exponential distribution with $\theta=1$ assuming $m=3$

Scheme Number	n	$\hat{J}_1$	$\hat{J}_2$	$\hat{H}_1$	$\hat{H}_2$
1	10	0.0247	**0.0095**	0.5494	**0.4789**
	20	0.0220	**0.0082**	0.5288	**0.4469**
	30	0.0204	**0.0073**	0.5154	**0.4305**
	50	0.0181	**0.0059**	0.4977	**0.4079**
	100	0.0163	**0.0047**	0.4853	**0.3922**
2	10	0.0262	**0.0113**	0.5742	**0.4888**
	20	0.0241	**0.0107**	0.5411	**0.4580**
	30	0.0221	**0.0096**	0.5206	**0.4328**
	50	0.0195	**0.0080**	0.4973	**0.4076**
	100	0.0178	**0.0066**	0.4892	**0.3949**
3	10	0.0209	**0.0080**	0.4945	**0.4214**
	20	0.0187	**0.0067**	0.4817	**0.4020**
	30	0.0176	**0.0057**	0.4805	**0.3958**
	50	0.0166	**0.0050**	0.4744	**0.3866**
	100	0.0146	**0.0037**	0.4614	**0.3708**
4	10	0.0228	**0.0078**	0.5300	**0.4692**
	20	0.0193	**0.0063**	0.4963	**0.4214**
	30	0.0189	**0.0060**	0.4995	**0.4205**
	50	0.0176	**0.0051**	0.4928	**0.4094**
	100	0.0160	**0.0041**	0.4841	**0.3967**

Table 6: MSEs of the estimates of extropy $J(Y)$ and entropy $H(Y)$ for exponential distribution with $\theta = 1$ assuming $m = 4$

Scheme Number	n		$\hat{J}_1$	$\hat{J}_2$	$\hat{H}_1$	$\hat{H}_2$
5	10	0.0169	**0.0101**	0.4120	**0.3790**	
	20	0.0122	**0.0076**	0.3306	**0.3036**	
	30	0.0096	**0.0065**	0.2832	**0.2602**	
	50	0.0071	**0.0048**	0.2488	**0.2273**	
	100	0.0040	**0.0031**	0.1767	**0.1592**	
6	10	0.0211	**0.0127**	0.4892	**0.4473**	
	20	0.0139	**0.0090**	0.3453	**0.3147**	
	30	0.0120	**0.0083**	0.3106	**0.2842**	
	50	0.0083	**0.0057**	0.2547	**0.2321**	
	100	0.0049	**0.0040**	0.1823	**0.1653**	
7	10	0.0194	**0.0119**	0.4544	**0.4148**	
	20	0.0144	**0.0092**	0.3654	**0.3298**	
	30	0.0122	**0.0080**	0.3303	**0.2973**	
	50	0.0093	**0.0059**	0.3006	**0.2681**	
	100	0.0064	**0.0037**	0.2589	**0.2262**	
8	10	0.0191	**0.0115**	0.4568	**0.4149**	
	20	0.0147	**0.0095**	0.3602	**0.3261**	
	30	0.0125	**0.0081**	0.3317	**0.2999**	
	50	0.0096	**0.0065**	0.2888	**0.2612**	
	100	0.0059	**0.0041**	0.2182	**0.1953**	

Table 7: MSEs of the estimates of extropy $J(Y)$ and entropy $H(Y)$ for exponential distribution with $\theta = 1$ assuming $m = 5$

Scheme Number	n	$\hat{J}_1$	$\hat{J}_2$	$\hat{H}_1$	$\hat{H}_2$
9	10	0.0283	**0.0180**	0.5597	**0.5208**
	20	0.0259	**0.0170**	0.5053	**0.4699**
	30	0.0215	**0.0139**	0.4658	**0.4303**
	50	0.0164	**0.0101**	0.4143	**0.3791**
	100	0.0127	**0.0073**	0.3744	**0.3391**
10	10	0.0300	**0.0189**	0.5563	**0.5235**
	20	0.0232	**0.0144**	0.4737	**0.4436**
	30	0.0194	**0.0117**	0.4436	**0.4132**
	50	0.0154	**0.0090**	0.3993	**0.3695**
	100	0.0114	**0.0058**	0.3599	**0.3289**
11	10	0.0359	**0.0238**	0.6342	**0.5922**
	20	0.0254	**0.0171**	0.4983	**0.4576**
	30	0.0236	**0.0158**	0.4919	**0.4500**
	50	0.0191	**0.0126**	0.4443	**0.4039**
	100	0.0131	**0.0081**	0.3733	**0.3347**
12	10	0.0251	**0.0159**	0.4953	**0.4621**
	20	0.0212	**0.0137**	0.4481	**0.4146**
	30	0.0186	**0.0121**	0.4180	**0.3857**
	50	0.0133	**0.0088**	0.3463	**0.3189**
	100	0.0079	**0.0048**	0.2723	**0.2477**

Table 8: MSEs of the estimates of extropy $J(Y)$ and entropy $H(Y)$ for normal distribution with $\mu = 0, \sigma = 1$ assuming $m = 3$

Scheme Number	n	$\hat{J}_1$	$\hat{J}_2$	$\hat{H}_1$	$\hat{H}_2$
1	10	0.0845	**0.0520**	1.5360	**1.4058**
	20	0.0821	**0.0515**	1.5007	**1.3609**
	30	0.0796	**0.0499**	1.4879	**1.3401**
	50	0.0771	**0.0487**	1.4700	**1.3076**
	100	0.0754	**0.0472**	1.4700	**1.3060**
2	10	0.0867	**0.0551**	1.5636	**1.4262**
	20	0.0832	**0.0541**	1.5046	**1.3688**
	30	0.0815	**0.0535**	1.4869	**1.3474**
	50	0.0801	**0.0526**	1.4803	**1.3360**
	100	0.0762	**0.0498**	1.4576	**1.3037**
3	10	0.0800	**0.0486**	1.4779	**1.3399**
	20	0.0768	**0.0478**	1.4434	**1.2884**
	30	0.0755	**0.0471**	1.4392	**1.2781**
	50	0.0735	**0.0457**	1.4330	**1.2654**
	100	0.0717	**0.0443**	1.4324	**1.2590**
4	10	0.0825	**0.0508**	1.5122	**1.3729**
	20	0.0804	**0.0491**	1.4912	**1.3513**
	30	0.0786	**0.0481**	1.4801	**1.3315**
	50	0.0763	**0.0467**	1.4733	**1.3145**
	100	0.0742	**0.0454**	1.4634	**1.2965**

Table 9: MSEs of the estimates of extropy $J(Y)$ and entropy $H(Y)$ for normal distribution with $\mu = 0, \sigma = 1$ assuming $m = 4$

Scheme Number	n	$\hat{J}_1$	$\hat{J}_2$	$\hat{H}_1$	$\hat{H}_2$
5	10	0.0735	**0.0555**	1.4046	**1.3288**
	20	0.0638	**0.0482**	1.2529	**1.1836**
	30	0.0553	**0.0411**	1.1298	**1.0654**
	50	0.0484	**0.0352**	1.0534	**0.9925**
	100	0.0406	**0.0284**	0.9559	**0.9005**
6	10	0.0768	**0.0586**	1.4402	**1.3612**
	20	0.0665	**0.0507**	1.2799	**1.2066**
	30	0.0590	**0.0445**	1.1722	**1.1058**
	50	0.0486	**0.0358**	1.0230	**0.9669**
	100	0.0427	**0.0304**	0.9581	**0.9077**
7	10	0.0799	**0.0615**	1.4823	**1.4038**
	20	0.0677	**0.0521**	1.3037	**1.2296**
	30	0.0622	**0.0477**	1.2309	**1.1578**
	50	0.0568	**0.0431**	1.1638	**1.0907**
	100	0.0511	**0.0380**	1.1216	**1.0472**
8	10	0.0770	**0.0589**	1.4454	**1.3674**
	20	0.0682	**0.0525**	1.3116	**1.2354**
	30	0.0629	**0.0484**	1.2217	**1.1512**
	50	0.0533	**0.0402**	1.0890	**1.0263**
	100	0.0462	**0.0337**	1.0106	**0.9537**

Table 10: MSEs of the estimates of extropy $J(Y)$ and entropy $H(Y)$ for normal distribution with $\mu = 0, \sigma = 1$ assuming $m = 5$

Scheme Number	n	$\hat{J}_1$	$\hat{J}_2$	$\hat{H}_1$	$\hat{H}_2$
9	10	0.1021	**0.0810**	1.6604	**1.5913**
	20	0.0925	**0.0740**	1.5344	**1.4681**
	30	0.0848	**0.0678**	1.4609	**1.3933**
	50	0.0787	**0.0627**	1.4157	**1.3462**
	100	0.0716	**0.0565**	1.3670	**1.2956**
10	10	0.1002	**0.0788**	1.6234	**1.5584**
	20	0.0863	**0.0678**	1.4751	**1.4082**
	30	0.0806	**0.0632**	1.4172	**1.3496**
	50	0.0743	**0.0578**	1.3672	**1.2976**
	100	0.0686	**0.0528**	1.3462	**1.2732**
11	10	0.1047	**0.0836**	1.6928	**1.6201**
	20	0.0944	**0.0761**	1.5693	**1.4960**
	30	0.0871	**0.0704**	1.4807	**1.4087**
	50	0.0791	**0.0637**	1.3985	**1.3251**
	100	0.0690	**0.0549**	1.3175	**1.2431**
12	10	0.0990	**0.0779**	1.6263	**1.5586**
	20	0.0833	**0.0659**	1.4533	**1.3839**
	30	0.0773	**0.0611**	1.3866	**1.3168**
	50	0.0656	**0.0512**	1.2499	**1.1834**
	100	0.0554	**0.0424**	1.1382	**1.0757**

It can be seen from Tables 2-10 that the performance of the proposed estimates is affected by the distribution of the sample under study. For uniform distribution, the obtained simulation results are given in Tables 2-4. We observe that extropy estimates $\hat{J}_1$ and $\hat{J}_2$ perform satisfactorily, as shown in Tables 2 and 3. Moreover, $\hat{J}_1$ dominates $\hat{J}_2$ under all censoring schemes for all sample sizes. It is also obvious from Tables 2 and 3 that both entropy estimates $\hat{H}_1$ and $\hat{H}_2$ perform well. Moreover, $\hat{H}_1$ dominates $\hat{H}_2$ under all censoring schemes for all sample sizes. Table 4 shows that for scheme 9, $\hat{J}_1$ and $\hat{J}_2$ perform well and $\hat{J}_2$ dominates $\hat{J}_1$ for all sample sizes; while for scheme 10 we see that the MSEs of $\hat{J}_2$ are always smaller than those of $\hat{J}_1$ except when $n = 10$, where $\hat{J}_1$ and $\hat{J}_2$ are equal. As for scheme 11, we see that MSEs of $\hat{J}_2$ are always smaller than those of $\hat{J}_1$ except when $n = 10$. On the other hand, in scheme 12, we see that MSEs of$\hat{J}_1$ are always smaller than those of $\hat{J}_2$ except when $n = 100$. Furthermore, from Table 4, we observe that for schemes 9 and 10, estimates of $\hat{H}_1$ and $\hat{H}_2$ perform satisfactorily and we also see that MSEs of $\hat{H}_1$ are always smaller than those of $\hat{H}_2$ except when $n = 50$ and $n = 100$ and they are equal when $n = 30$. As for scheme 11, we see that MSEs of $\hat{H}_2$ are always smaller than those of $\hat{H}_1$ except when $n = 10$ and $n = 20$. On the other hand, in scheme 12 we see that MSEs of $\hat{H}_1$ are always smaller than those of $\hat{H}_2$ except when $n = 50$ and $n = 100$.

For exponential distribution, results obtained are displayed in Tables 5-7. It is clear from Tables 5, 6 and 7 that extropy estimates $\hat{J}_1$ and $\hat{J}_2$ perform satisfactorily where $\hat{J}_2$ dominates $\hat{J}_1$ under all censoring schemes for all sample sizes. Moreover, it is clear from Tables 9, 11 and 13, entropy estimates $\hat{H}_1$ and $\hat{H}_2$ perform well where $\hat{H}_2$ dominates $\hat{H}_1$ under all censoring schemes for all sample sizes.

For normal distribution, the obtained results are arranged in Tables 8-10. We observe that $\hat{J}_1$ and $\hat{J}_2$ perform satisfactorily, as shown in Tables 8, 9 and 10, where $\hat{J}_2$ dominates $\hat{J}_1$ under all censoring schemes for all sample sizes. It is also obvious from Tables 8, 9 and 10 that both $\hat{H}_1$ and $\hat{H}_2$ estimators perform well, where $\hat{H}_2$ dominates $\hat{H}_1$ under all censoring schemes for all sample sizes.

As expected, MSE decreases as the sample size n increases. It is worthwhile to point that the extropy and entropy estimates are affected by the sample size, censoring schemes and the type of distribution of data. For example, we notice that the MSEs generated from uniform distribution seems to outperform their parallel MSEs generated from Exponential distribution under all censoring schemes and for all sample sizes.

4.2. Real data analysis

In this subsection, we present an example to show the behavior of the proposed entropy and extropy estimates in real case. Here we consider the insulating fluid example from [Nelson, 1982, P.105].

Example : The following data, represents failure times (in minutes) for an insulating fluid between two electrodes, subject to a voltage of 34 kV.:

```
0.19, 0.78, 0.96, 1.31, 2.78, 3.16, 4.15, 4.67, 4.85, 6.50,
7.35, 8.01, 8.27, 12.06, 31.75, 32.52, 33.91, 36.71, 72.89.
```

It is well known that the exponential distribution, $Exp(\theta)$ with pdf $f(x) = \theta \exp(-\theta x)$is appropriate fitting model for this dataset. The MLE of θ based on the complete sample is equal to $1/\bar{X}$ which equals 0.0696. Therefore, assuming $\theta = 0.0696$, the extropy and entropy of X in this case are obtained to be

$$J(X) = -\frac{\theta}{4} = -0.0174, \text{ and } H(X) = 1 - \log\, \theta = 3.6644,$$

respectively.

Next, we study the behavior of the proposed estimators based on the following progressive Type-I Interval censoring schemes that are presented in Table 11. In this study, we consider $m = 4$ and $w = 2$.

Table 11: Progressive censoring schemes used in this real data example

Censoring scheme No.	(T_1, T_2, T_3, T_4)	(q_1, q_2, q_3, q_4)
1	$1, 5, 15, 25$	$0.25, 0, 0.25, 1$
2	$2.5, 5, 10, 35$	$0.1, 0.2, 0.3, 1$
3	$1, 6, 18, 30$	$0.05, 0.05, 0.05, 1$
4	$0.5, 4, 10, 30$	$0, 0, 0, 1$
5	$0.9, 4, 12, 40$	$0.3, 0.2, 0.1, 1$

Continuing with the exploration of progressive Type-I Interval censoring under this lifetime model, the following censored data are observed according to the applied censoring scheme on the insulation data. The generated censored data are summarized in Table 12.

Table 12: The observed censored data from this real data example

Scheme No.	$\{k_1,k_2,k_3,k_4\}$	$\{R_1,R_2,R_3,R_4\}$
Scheme 1	$\{3,4,4,0\}$	$\{4,0,1,3\}$
Scheme 2	$\{4,4,3,2\}$	$\{2,2,1,1\}$
Scheme 3	$\{3,6,4,0\}$	$\{1,0,0,5\}$
Scheme 4	$\{1,5,7,1\}$	$\{0,0,0,5\}$
Scheme 5	$\{2,3,5,1\}$	$\{5,2,0,1\}$

Results of estimation for the extropy and entropy measures using the proposed estimates under the above censoring schemes are shown in Table 13.

Table 13: Extropy and entropy estimates for this real data example

Scheme Number	$\hat{J}_1$	$\hat{J}_2$	$\hat{H}_1$	$\hat{H}_2$
1	-0.0158	-0.0141	3.5088	3.5066
2	-0.0192	-0.0155	3.4825	3.5296
3	-0.0125	-0.0112	3.7624	3.7314
4	-0.0175	-0.0155	3.4866	3.4543
5	-0.0157	-0.0137	3.6187	3.6018

From Table 13, we can clearly see, for the real data set, that all estimates $\hat{J}_1$, $\hat{J}_2$, $\hat{H}_1$ and $\hat{H}_2$ provide close results to those obtained using the complete sample with slight differences. The proposed extropy estimates $\hat{J}_1$ and $\hat{J}_2$ perform satisfactorily when comparing their values to $J(X) = -0.0174$. On the other hand, the estimates $\hat{H}_1$ and $\hat{H}_2$ perform well under all suggested censoring schemes when comparing their values to $H(X) = 3.6644$. This confirms that these results are in agreement with what have been concluded from the simulation studies.

5. Conclusions

In this paper, we have considered the estimation problem of the extropy and entropy measures based on progressive Type-I interval censoring samples. Nonparametric-based methods involving moments approximation and linear approximation have been proposed for estimating the extropy and entropy measures. The performance of the proposed estimates have been studied via simulation studies and real data sets considering various censoring schemes and three probability distributions, namely uniform, exponential and normal distributions. It has been observed that the proposed estimates of the extropy and entropy measures are affected by the sample size, censoring schemes and the type of distribution of data. The Monte Carlo simulations show that both estimates perform well in terms of the MSE. Yet, the estimates based on linear approximation $\hat{J}_2$ and $\hat{H}_2$ outperform the other estimate in the majority of studied cases.

Acknowledgements

The authors would like to thank the associate editor and anonymous referees for their constructive comments which improved the quality and presentation of our results.

References

Aggarwala, R., (2001). Progressive interval censoring: some mathematical results with applications to inference. *Commun Stat: Theory Methods*, 30, pp. 1921–1935.

Alotaibi, R., Rezk, H., Dey, S., and Okasha, H., (2021). Bayesian estimation for Dagum distribution based on progressive type I interval censoring. *PLOS ONE*, 16(6), DOI: 10.1371/journal.pone.0252556

Awad, A. M. and Alawneh, A., (1987). Application of entropy of a life time model. *IMA J. Math. Control Inf.*, 4, pp. 143–147.

Balakrishnan, N., Cramer, E., (2014). *The Art of Progressive Censoring: Applications to Reliability and Quality*. Boston.

Cohen, A. C., (1963). Progressively censored sample in life testing. *Technometrics*, 5, pp. 327–339.

Correa, J. C., (1995). A new estimator of entropy. *Communications in Statistics Theory and Methods*, 24, pp. 2439–2449.

Du Y., Guo Y. and Gui W., (2018). Statistical Inference for the Information Entropy of the Log-Logistic Distribution under Progressive Type-I Interval Censoring Schemes. *Symmetry*, 10, p. 445.

Hazeb, R., Raqab, M. and Bayoud, H., (2021a). Non-parametric estimation of the extropy and the entropy measures based on progressive type-II censored data with testing uniformity. *Journal of Statistical Computation and Simulation*, 91 (11), pp. 2178–2210.

Hazeb, R., Bayoud, H. and Raqab, M., (2021b). Kernel and CDF-Based Estimation of Extropy and Entropy from Progressively Type-II Censoring with Application for Goodness of Fit Problems. *Stochastic and Quality Control*, 36 (1), pp. 73–83.

Kittaneh, O. A., Khan, M. A., Akbar, M., and Bayoud, H. A. (2016). Average entropy: a new uncertainty measure with application to image segmentation. *The American Statistician*, 70 (1), pp. 18–24.

Lad, F., Sanfilippo, G. and Agro, G., (2015). Extropy: Complementary Dual of Entropy. *Statistical Science*, 30, pp. 40–58.

Lio, YL., Chen, D. G. and Tasi, T. R., (2011). Parameter estimations for generalized exponential distribution under progressive type-I interval censoring.*Comput. Stat. Data Anal.*, 54, pp. 1581–1591.

Nelson, W., (1982). *Applied Life Data Analysis*, Wiley, New York.

Ng, H. K. T., Wang, Z., (2009). Statistical estimation for the parameters of Weibulldistribution based on progressively Type-I interval censored sample.*J.Stat.Comput.Simulat*, 79(2), pp. 145–159.

Noughabi, H. A., Jarrahiferiz, J., (2019). On the estimation of extropy. *Journal of Nonparametric Statistics*, 31(1), pp. 88–99, DOI: 10.1080/10485252.2018.1533133.

Qiu, G., (2017). The Extropy of Order Statistics and Record values. *Statistics and Probability Letters*, 120, pp. 52–60.

Qiu, G. and Jia, K., (2018a). The Residual Extropy of Order statistics. *Statistics and Probability Letters*, 133, pp. 15–22.

Qiu, G. and Jia, K., (2018b). Extropy Estimators with Applications in Testing Uniformity. *Journal of Nonparametric Statistics*, 30(1), pp. 182–196, DOI: 10.1080/1048252.2017.1404063.

Rao, M., Chen, Y., Vemuri, B. C., and Wang, F., (2004). Cumulative residual entropy: a new measure of information. *IEEE transactions on Information Theory*, 50 (6), pp. 1220–1228.

Raqab, M. Z. and Qiu, G., (2019). On extropy properties of ranked set sampling. *Statistics*, 53(1), pp. 210–226, DOI: 10.1080/02331888.2018.1533963.

Renyi, A., (1961). On measures of entropy and information. *Stat. and Prob.*, 1, pp. 547–561.

Shannon, C. E., (1948). A mathematical theory of communications. *Bell System Tech. J.*, 27(3), pp. 379–423.

Singh, S, Tripathi, Y. M., (2016). Estimating the parameters of an inverse Weibull distribution under progressive type-I interval censoring. *Stat. Pap.*, 59, pp. 21–56.

Tsallis, C., (1988). Possible generalization of Boltzmann Gibbs statistics. *J. Stat. Phys.*, 52, pp. 470–487.

Vasicek, O., (1976). A test for normality based on sample entropy. *Journal of the Royal Statistical Society B*, 38, pp. 54–59.