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On a family that unifies the generalized Marshall-Olkin and Poisson-G family of distributions

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Abstract

The aim of the article is to propose a unification of the generalized Marshall-Olkin (GMO) and Poisson-G (P-G) distributions into a new family of distributions. The density and survival function are expressed as infinite mixtures of an exponentiated-P-G family. The quantile function, asymptotes, shapes, stochastic ordering and Rényi entropy are derived. The paper presents a maximum likelihood estimation with large sample properties. A Monte Carlo simulation is used to examine the pattern of the bias and the mean square error of the maximum likelihood estimators. The utility of the proposed family is illustrated through its comparison with some important models and sub models of the family in terms of modeling real data.

Key words: GMO family, Poisson-G family, stochastic ordering, MLE, AIC.

1. Introduction

Generalized classes of univariate continuous distributions through introduction of additional shape parameter(s) to a baseline distribution have attracted a lot of attention in recent times. With the basic motivation to bring in more flexibility in the modelling different types of data, a preferred area of research in the field of probability distribution is that of generating new distributions starting with a baseline distribution by inducing one or more additional parameters through various methodologies. A number of useful continuous univariate-G families have been added in the literature in recent times. Notable families introduced since 2017 are Poisson-G family (Abouelmagd et al., 2017), Marshall-OlkinKumaraswamy-G family (Handique et al.,

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2017), Generalized Marshall-Olkin Kumaraswamy-G family (Chakraborty and Handique, 2017), Exponentiated generalized-G Poisson family (Gokarna and Haitham, 2017), Beta Kumaraswamy-G family (Handique et al., 2017), Beta generated Kumaraswamy Marshall-Olkin-G family (Handique and Chakraborty, 2017a), Beta generalized Marshall-Olkin Kumaraswamy-G family (Handique and Chakraborty, 2017b), Beta generated Marshall-Olkin Kumaraswamy-G (Chakraborty et al., 2018), Kumaraswamy generalized Marshall-Olkin-G family (Chakraborty and Handique, 2018), Odd modified exponential generalized family (Ahsan et al., 2018), Zografos-Balakrishnan Burr XII family (Emrah et al., 2018), Exponentiated generalized Marshall-Olkin-G family by (Handique et al., 2019), Beta-G Poisson family (Gokarna et al., 2019), Zero truncated Poisson family (Abouelmagd et al., 2019), Extended generalized Gompertz family (Thiago et al., 2019), Generalized modified exponential-G family (Handique et al., 2020), Odd Half-Cauchy family (Chakraborty et al., 2021), Poisson Transmuted-G family (Handique et al., 2021), Beta Poisson-G family (Handique et al., 2022), Kumaraswamy Poisson-G family (Chakraborty et al., 2022), Complementary Geometric-Topp-Leone-G family (Handique et al., 2023), generalized Marshall-Olkin Transmuted-G family (Handique et al., 2024) and Truncated Cauchy Power Kumaraswamy-G family (Ibrahim et al., 2024), among others.

In this article, a new family of continuous probability distribution called the Generalized Marshall-Olkin Poisson-G ($GMOP-G(\theta, \alpha, \lambda)$) is introduced to unify generalized Marshall-Olkin (GMO) of Jayakumar and Mathew, (2008) and the Poisson-G (P-G) family of distribution (Tahir et al., 2016). Now, we briefly describe the GMO and P-G family and then introduce GMOP-G in the next section.

1.1. Generalized Marshall-Olkin (GMO) family

Jayakumar and Mathew (2008) proposed a new generalization of the Marshall-Olkin family (Marshall and Olkin, 1997) of distributions called the generalized Marshall-Olkin (GMO) family of distributions. The survival function (sf) and probability distribution function (pdf) of the GMO distribution are given respectively by

$$\bar{F}^{\mathrm{GMO}}(t;\theta,\alpha)=\left[\frac{\alpha\bar{F}(t)}{1-\bar{\alpha}\bar{F}(t)}\right]^{\theta} \text{and } f^{\mathrm{GMO}}(t;\theta,\alpha)=\frac{\theta\alpha^{\theta}f(t)\bar{F}(t)^{\theta-1}}{[1-\bar{\alpha}\bar{F}(t)]^{\theta+1}}, \quad (1)$$

where $-\infty < t < \infty, \alpha > 0$ ($\bar{\alpha} = 1-\alpha$), $\theta > 0$ is an additional shape parameter and $\bar{F}(t)$ and $f(t)$ is the baseline sf and pdf respectively.

When $\theta = 1$, $\bar{F}^{\mathrm{GMO}}(t;\theta,\alpha) = \bar{F}^{\mathrm{MO}}(t;\alpha)$ and for $\alpha = \theta = 1$, $\bar{F}^{\mathrm{GMO}}(t;\theta,\alpha) = \bar{F}(t)$.

1.2. Poisson-G (P-G) family

The Poisson-G (P-G) family of distributions with survival function and cdf is given by (see Kumaraswamy Poisson-G family, Chakraborty et al., 2022)

$$\bar{G}^{\mathrm{PG}}(t\,;\lambda) = \frac{e^{-\lambda G(t)} - e^{-\lambda}}{1 - e^{-\lambda}}$$

$$and\;\; G^{\mathrm{PG}}(t\,;\lambda) = \frac{1-e^{-\lambda G(t)}}{1-e^{-\lambda}},\; \lambda \in R - \{0\}; n = 1, 2, \dots \tag{2}$$

The corresponding pdf of (2) is given by

$$g^{\mathrm{PG}}(t\,;\lambda) = (1 - e^{-\lambda})^{-1}\lambda g(t)e^{-\lambda G(t)}, \lambda \in R - \{0\}; -\infty < t < \infty. \tag{3}$$

where $G(t)$and$g(t)$ is the baseline distribution.

The article is arranged in the following 5 sections. In Section 2, we introduce the proposed family along with its physical basis and list of some important sub models and also define some mathematical properties. In Section 3, a linear representation of the sf and pdf of the proposed family is discussed along with some statistical properties of the proposed family. In Section 4, maximum likelihood methods of estimation of parameters and simulation are presented. The data fitting applications are presented in Section 5. Final conclusion is provided in Section 6.

2. Generalized Marshall-Olkin Poisson-G family

In this section we introduce the $GMOP - G(\theta, \alpha, \lambda)$ family and also provide its special cases and a statistical genesis.

The sf, cdf, pdf and hrf of this distribution are given respectively by:

$$\bar{F}^{\mathrm{GMOPG}}(t\,;\theta\,,\alpha,\lambda) = \left[\frac{\alpha(e^{-\lambda G(t)}-e^{-\lambda})}{1-\alpha e^{-\lambda}-\bar{\alpha}e^{-\lambda G(t)}}\right]^{\theta}, F^{\mathrm{GMOPG}}(t\,;\theta\,,\alpha,\lambda) = 1 - \left[\frac{\alpha(e^{-\lambda G(t)}-e^{-\lambda})}{1-\alpha e^{-\lambda}-\bar{\alpha}e^{-\lambda G(t)}}\right]^{\theta} \tag{4}$$

$$f^{\mathrm{GMOPG}}(t\,;\theta\,,\alpha,\lambda) = \frac{\theta\lambda\alpha^{\theta}(1-e^{-\lambda})g(t)e^{-\lambda G(t)}(e^{-\lambda G(t)}-e^{-\lambda})^{\theta-1}}{(1-\alpha e^{-\lambda}-\bar{\alpha}e^{-\lambda G(t)})^{\theta+1}}, \tag{5}$$

and

$$h^{\mathrm{GMOPG}}(t\,;\theta\,,\alpha,\lambda) = \frac{\theta\lambda(1-e^{-\lambda})g(t)e^{-\lambda G(t)}(e^{-\lambda G(t)}-e^{-\lambda})^{-1}}{1-\alpha e^{-\lambda}-\bar{\alpha}e^{-\lambda G(t)}}. \tag{6}$$

In particular, we get for

(i) $\theta = 1$, the $MOP - G(\alpha, \lambda)$ distribution.

(ii) $\theta = \alpha = 1$, the $P - G(\lambda)$ distribution.

(iii) $\lambda \to 0$, the $GMO(\theta, \alpha)$ distribution.

(iv) $\theta = 1, \lambda \to 0$, the $MO(\alpha)$ distribution.

Proposition 1 Let $T_{i1}, T_{i2}, \ldots, T_{iN}$, $i = 1,2,\cdots,\theta$ be a sequence of θN *i.i.d.* random variables from Poisson-G distribution and $W_i = min(T_{i1}, T_{i2}, \ldots, T_{iN})$ and $V_i = max(T_{i1}, T_{i2}, \ldots, T_{iN})$. Then

(i) $\min_i W_i$ follows $GMOP - G(\theta, \alpha, \lambda)$ if $N \sim Geometric(\alpha)$ and

(ii) $\max_i V_i$ follows $GMOP - G(\theta, \alpha, \lambda)$ if $N \sim Geometric(1/\alpha)$.

Proof: Case (i) When $0 < \alpha \leq 1$, considering N has a geometric distribution with parameter α, we get

$$P[min\{W_1, W_2, \ldots, W_\theta\} > t] = P[W_1 > t]P[W_2 > t]\ldots P[W_\theta > t]$$

$$= \prod_{i=1}^{\theta} P[W_i > t] = [\bar{F}^{MOPG}(t; \alpha, \lambda)]^\theta = \left[\frac{\alpha(e^{-\lambda G(t)} - e^{-\lambda})}{1 - \alpha e^{-\lambda} - \bar{\alpha} e^{-\lambda G(t)}}\right]^\theta.$$

Case (ii) For $\alpha > 1$, considering N has a geometric distribution with parameter $1/\alpha$, we get

$$P[min\{V_1, V_2, \ldots, V_\theta\} > t] = P[V_1 > t]P[V_2 > t]\ldots P[V_\theta > t]$$

$$= \prod_{i=1}^{\theta} P[V_i > t] = [\bar{F}^{MOPG}(t; \alpha, \lambda)]^\theta = \left[\frac{\alpha(e^{-\lambda G(t)} - e^{-\lambda})}{1 - \alpha e^{-\lambda} - \bar{\alpha} e^{-\lambda G(t)}}\right]^\theta.$$

In what follows we investigate some general properties, parameter estimation and real life applications.

2.1. Special model and shape of the density and hazard function

In this section we have plotted the pdf and hrf of the $GMOP - E(\theta, \alpha, \lambda, \beta)$ for some chosen values of the parameters in Figure 1 and Figure 2 respectively to show the variety of shapes assumed by the family.

The pdf and hrf of the $GMOP - E(\theta, \alpha, \lambda, \beta)$ are as follows:

- The GMOP-Exponential (GMOP-E) distribution.

Considering the Exponential distribution with parameters $\beta > 0$ having pdf and cdf $g(t) = \beta e^{-\beta t}$ and $G(t) = 1 - e^{-\beta t}$ respectively we get the pdf and hrf of $GMOP - E(\theta, \alpha, \lambda, \beta)$ distribution as

$$f^{\text{GMOPE}}(t\,; \theta\,, \alpha, \lambda, \beta) = \frac{\theta \lambda \alpha^\theta (1 - e^{-\lambda}) \beta e^{-\beta t} e^{-\lambda(1 - e^{-\beta t})} (e^{-\lambda(1 - e^{-\beta t})} - e^{-\lambda})^{\theta - 1}}{(1 - \alpha e^{-\lambda} - \bar{\alpha} e^{-\lambda(1 - e^{-\beta t})})^{\theta + 1}},$$

and $$h^{\text{GMOPE}}(t\,; \theta\,, \alpha, \lambda, \beta) = \frac{\theta \lambda \alpha^\theta (1 - e^{-\lambda}) \beta e^{-\beta t} e^{-\lambda(1 - e^{-\beta t})} (e^{-\lambda(1 - e^{-\beta t})} - e^{-\lambda})^{-1}}{1 - \alpha e^{-\lambda} - \bar{\alpha} e^{-\lambda(1 - e^{-\beta t})}}.$$

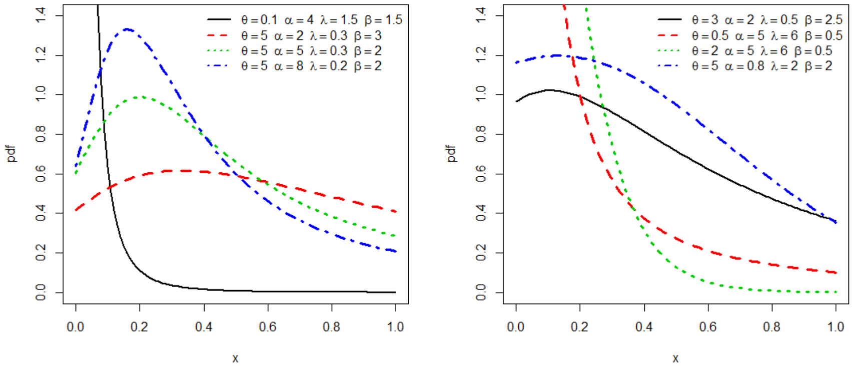


Figure 1: pdf plots of the $GMOP - E(\theta, \alpha, \lambda, \beta)$ distribution

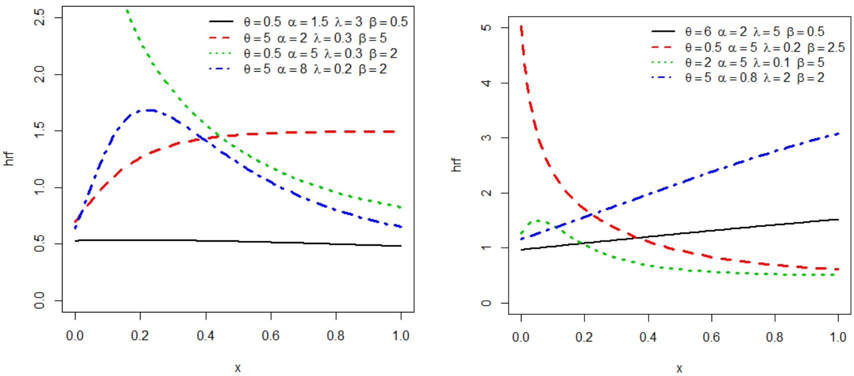


Figure 2: hrf plots of the $GMOP - E(\theta, \alpha, \lambda, \beta)$ distribution

Remark 1. From Figures 1 and 2 it can be seen that the proposed family of distributions is very flexible and can offer different types of shapes for density and hazard like increasing, decreasing and right skewed.

Quantile and related measures

The p^{th} quantile t_p for $GMOP - G(\theta, \alpha, \lambda)$ can be easily obtained by solving the equation $F^{\text{GMOPG}}(t) = p$ as

$$t_p = G^{-1}\left[-\frac{1}{\lambda} log\left[\frac{\alpha e^{-\lambda} + (1 - \alpha e^{-\lambda})(1 - F(t))^{1/\theta}}{\alpha + \bar{\alpha}(1 - F(t))^{1/\theta}}\right]\right].$$

Here the flexibility of skewness and kurtosis of $GMOP - G(\theta, \alpha, \lambda)$ is checked by plotting Galton skewness (S) that measures the degree of the long tail and Moors (1988) kurtosis (K) that measures the degree of tail heaviness in Figure 3 for the $GMOP -$

$E(\theta,\alpha,\lambda,\beta)$ distribution for some values of parameters. These are respectively defined by

$$S = \frac{Q(6/8)-2Q(4/8)+Q(2/8)}{Q(6/8)-Q(2/8)} \text{ and } K = \frac{Q(7/8)-Q(5/8)+Q(3/8)-Q(1/8)}{Q(6/8)-Q(2/8)}.$$

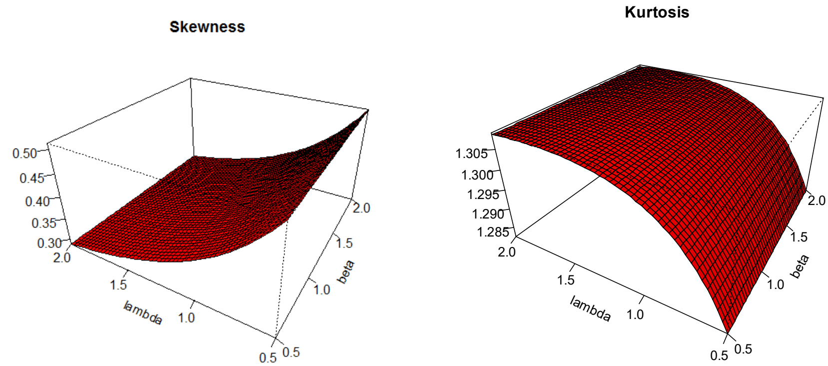


Figure 3: Plots of the Galton skewness S and the Moor kurtosis K for the GMOP-E distribution with parameters $\theta = 3, \alpha = 2, 0.2 < \lambda, \beta < 2$

2.2. Asymptotes and shapes

Two propositions regarding asymptotes of the proposed family are discussed here.
Proposition 2 The asymptotes of pdf, cdf and hrf of $GMOP - G(\theta,\alpha,\lambda)$ as $t \to 0$ are given by

$$f^{\text{GMOPG}}(t\,;\theta\,,\alpha,\lambda) \sim \frac{\theta\lambda g(t)}{\alpha(1-e^{-\lambda})},$$

$$F^{\text{GMOPG}}(t\,;\theta\,,\alpha,\lambda) \sim 0 \text{ and}$$

$$h^{\text{GMOPG}}(t\,;\theta\,,\alpha,\lambda) \sim \frac{\theta\lambda g(t)}{\alpha(1-e^{-\lambda})}.$$

Proposition 3 The asymptotes of pdf, cdf and hrf of $GMOP - G(\theta,\alpha,\lambda)$ as $t \to \infty$ are given by

$$f^{\text{GMOPG}}(t\,;\theta\,,\alpha,\lambda) \sim \theta\,\lambda\alpha^{\theta}e^{-\lambda}g(t)(e^{-\lambda G(t)} - e^{-\lambda})^{\theta-1}/(1-e^{-\lambda})^{\theta},$$

$$F^{\text{GMOPG}}(t\,;\theta\,,\alpha,\lambda) \sim 1 - \alpha^{\theta}(e^{-\lambda G(t)} - e^{-\lambda})^{\theta}/(1-e^{-\lambda})^{\theta} \text{ and}$$

$$h^{\text{GMOPG}}(t\,;\theta\,,\alpha,\lambda) \sim \theta\,\lambda e^{-\lambda}g(t)(e^{-\lambda G(t)} - e^{-\lambda})^{-1}.$$

Analytically the shapes of the pdf and hazard rate function can be stated through critical points. The critical points of the pdf are the roots of the equation

$$\frac{g'(t)}{g(t)} - \lambda g(t) - (\theta-1)\frac{\lambda e^{-\lambda G(t)}g(t)}{e^{-\lambda G(t)}-e^{-\lambda}} - (\theta+1)\frac{\bar{\alpha}\lambda e^{-\lambda G(t)}g(t)}{1-\alpha e^{-\lambda}-\bar{\alpha}e^{-\lambda G(t)}} = 0. \quad (7)$$

The critical point of $GMOP-G(\theta,\alpha,\lambda)$ family hazard rate are the roots of the equation

$$\frac{g'(t)}{g(t)}-\lambda g(t)+\frac{\lambda e^{-\lambda G(t)}g(t)}{e^{-\lambda G(t)}-e^{-\lambda}}-\frac{\bar{\alpha}\lambda e^{-\lambda G(t)}g(t)}{1-\alpha e^{-\lambda}-\bar{\alpha}e^{-\lambda G(t)}}=0. \tag{8}$$

Equations (7) and (8) may have multiple solutions. If $t=t_0$ is a root then it is a local maximum, a local minimum or a point of inflexion if $\psi(t_0)<0,\psi(t_0)>0 or \psi(t_0)=0$ and for (8) if $\omega(t_0)<0,\omega(t_0)>0 or \omega(t_0)=0$ where $\psi(t)=(d^2/dt^2)\,log[f(t)]$ and $\omega(t)=(d^2/dt^2)\,log[h(t)]$

2.3. Stochastic orderings

Let X and Y be two random variables with cdfs F and G, respectively, corresponding pdf's f and g. Then X is said to be smaller than Y in the likelihood ratio order ($X\leq_{lr}Y$) if $f(t)/g(t)$ is decreasing in $t\geq 0$. Here we present a result of likelihood ratio ordering.

Theorem 1 Let $X\sim GMOPG \rightleftarrows (\theta,\alpha_1,\lambda)$ and $Y\sim GMOPG(\theta,\alpha_2,\lambda)$. If $\alpha_1<\alpha_2$, then $X\leq_{lr}Y$

Proof: $\frac{f(t)}{g(t)}=\left(\frac{\alpha_1}{\alpha_2}\right)^{\theta}\left[\frac{1-\alpha_2 e^{-\lambda}-\bar{\alpha}_2 e^{-\lambda G(t)}}{1-\alpha_1 e^{-\lambda}-\bar{\alpha}_1 e^{-\lambda G(t)}}\right]^{\theta+1}$

$$\frac{d}{dt}(f(t)/g(t))=(\theta+1)(\frac{\alpha_1}{\alpha_2})^{\theta}(\alpha_1-\alpha_2)\frac{[1-\alpha_2e^{-\lambda}-\bar{\alpha}_2e^{-\lambda G(t)}]^{\theta}\lambda e^{-\lambda G(t)}g(t)(1-e^{-\lambda})}{[1-\alpha_1e^{-\lambda}-\bar{\alpha}_1e^{-\lambda G(t)}]^{\theta+2}}.$$

Now this is always less than 0, since $\alpha_1<\alpha_2$. Hence, $f(t)/g(t)$ is decreasing in *t*. That is $X\leq_{lr}Y$.

3. Linear representation

Linear representation of sf and pdf, etc. in terms of corresponding functions of known distributions is an important tool for further mathematical properties. In this section we present some important results for the proposed family.

3.1. Expansions of the survival and density functions as infinite linear mixture

Here the sf and pdf of the $GMOP-G(\theta,\alpha,\lambda)$ are expressed as linear mixture of the corresponding functions of exponentiated-$P-G(\lambda)$ distribution.

Consider the series representation

$$(1-z)^{-k}=\sum_{j=0}^{\infty}\frac{\Gamma(k+j)}{\Gamma(k)j!}z^j=\sum_{j=0}^{\infty}\frac{(j+k-1)!}{(k-1)!j!}z^j, |z|<1 \text{ and } k>0, \tag{9}$$

where $\Gamma(.)$is the gamma function.

Using equation (9) in equation (4), for $\alpha \in (0,1)$ we obtain

$$\bar{F}^{GMOPG}(t\,;\theta\,,\alpha,\lambda) = \alpha^{\theta}\{\bar{G}^{\mathrm{PG}}(t\,;\lambda)\}^{\theta}\sum_{j=0}^{\infty}\frac{(j+\theta-1)!}{(\theta-1)!\,j!}(1-\alpha)^{j}\{\bar{G}^{\mathrm{PG}}(t\,;\lambda)\}^{j}$$

$$= \sum_{j=0}^{\infty}\eta'_{j}\,[\bar{G}^{\mathrm{PG}}(t\,;\lambda)]^{j+\theta}\,. \tag{10}$$

Differentiating in equation (10) with respect to 't' we get

$$f^{GMOPG}(t\,;\theta\,,\alpha,\lambda) = g^{PG}(t\,;\lambda)\sum_{j=0}^{\infty}\eta_{j}[\bar{G}^{PG}(t\,;\lambda)]^{j+\theta-1} \tag{11}$$

$$= -\sum_{j=0}^{\infty}\eta'_{j}\frac{d}{dt}[\bar{G}^{\mathrm{PG}}(t\,;\lambda)]^{j+\theta} \tag{12}$$

where $\eta'_{j} = \eta'_{j}(\alpha) = \binom{j+\theta-1}{j}(1-\alpha)^{j}\alpha^{\theta}, \quad \eta_{j} = \eta_{j}(\alpha) = (j+\theta)\eta'_{j}$.

We have presented a numerical evaluation result of mean, variance, skewness and kurtosis of the $GMOP-E(\theta,\alpha,\lambda,\beta)$ distribution for some selected parameter values in Table 1.

Table 1: Mean, variance, skewness and kurtosis of the $GMOP-E(\theta,\alpha,\lambda,\beta)$ distribution with different values of θ, α, λ and β

θ	α	λ	β	Mean	Variance	Skewness	Kurtosis
10	10	2	2	0.1572	0.0163	1.2096	4.6452
10	10	1	2	0.2224	0.0306	1.0761	4.0834
10	10	0.5	2	0.2648	0.0408	0.9749	3.7373
10	10	0.1	2	0.3033	0.0502	0.8840	3.4698
10	10	2	1	0.3145	0.0652	1.2096	4.6452
10	10	2	0.5	0.6291	0.2610	1.2096	4.6452
10	10	0.5	0.5	1.0592	0.6528	0.9749	3.7373
10	10	0.1	0.1	6.0675	20.1022	0.8840	3.4698
10	5	2	2	0.0918	0.0066	1.5144	6.0241
10	2	2	2	0.0419	0.0016	1.9171	8.4744
10	0.5	2	2	0.0115	0.0001	2.4163	12.8206
10	0.5	0.5	0.5	0.0834	0.0077	2.3537	12.1190
5	10	2	2	0.2692	0.0424	1.0978	4.3497
5	5	2	2	0.1676	0.0206	1.4448	5.8193
2	5	2	2	0.3615	0.0914	1.5105	6.3958
2	2	2	2	0.2049	0.0427	2.1968	10.7924
1	2	2	2	0.4180	0.19284	2.3136	11.2913
5	0.1	0.1	0.1	0.2309	0.0804	3.6320	30.5202
5	5	5	3	0.0489	0.0016	1.4027	5.7915
5	5	3	5	0.04882	0.0017	1.5071	6.2776
5	5	5	8	0.0183	0.0002	1.3967	5.7764
5	5	10	10	0.0069	0.00001	1.1168	2.1041

3.2. Rényi entropy

Entropy of a random variable is a measure of uncertainty and has been used in various situations in science and engineering. The Rényi entropy (see details, Song, 2001) is defined by

$$I_R(\delta) = (1-\delta)^{-1}\, log\left(\int_{-\infty}^{\infty} f(t)^{\delta} dt\right), \text{ where } \delta > 0 \text{ and } \delta \neq 1.$$

Thus the Rényi entropy of $GMOP - G(\theta, \alpha, \lambda)$ distribution can be obtained as

$$I_R(\delta) = (1-\delta)^{-1}\, log\left(\sum_{j=0}^{\infty} \mu_j \int_{-\infty}^{\infty} [g^{\mathrm{PG}}(t\,;\lambda)\bar{G}^{\mathrm{PG}}(t\,;\lambda)^{\theta-1}]^{\delta}[\bar{G}^{\mathrm{PG}}(t\,;\lambda)]^{j} dt\right),$$

where $\mu_j = \mu_j(\alpha) = \{\theta^{\delta}\alpha^{\delta\theta}(1-\alpha)^j \Gamma[\delta(\theta+1)+j]\}/\{\Gamma[\delta(\theta+1)]j!\}$.

Table 2 shows the values of numerical values of Rényi entropy $GMOP - E(\theta, \alpha, \lambda, \beta)$ distribution for some selected parameter values. As expected, the Rényi entropy turns out to be non-increasing with δ.

Table 2: Rényi entropy $GMOP - E(\theta, \alpha, \lambda, \beta)$ distribution with different values of θ, α, λ and β

Parameter				δ					
θ	α	λ	β	**0.2**	**0.5**	**1.5**	**2**	**3**	**5**
10	10	**2**	**2**	-0.2550	-0.6403	-0.9916	-1.0647	-1.1549	-1.2490
5	5	0.5	0.5	1.6816	1.3053	0.9722	0.9032	0.8176	0.7280
5	5	2	0.5	1.3469	0.8661	0.4519	0.3687	0.2669	0.1621
3	3	2	0.5	1.6090	1.0220	0.5252	0.42839	0.3113	0.1924
1.5	1.5	2	0.5	2.1288	1.3773	0.6945	0.5640	0.4097	0.2571
2	0.5	0.5	0.5	1.7182	0.8594	0.4390	-0.1133	-0.2982	-0.4789

4. Estimation

Here, we consider the parameter estimation of $GMOP - G(\theta, \alpha, \lambda)$ via the maximum likelihood (ML) method.

4.1 Maximum likelihood method

Let $T = (t_1,\ t_2, \ldots,\ t_n)$ be a random sample of size n from $GMOP - G(\theta,\ \alpha,\ \lambda)$ with parameter vector $\rho = (\theta,\ \alpha,\ \lambda,\ \xi)$, where $\xi = (\xi_1,\ \xi_2, \ldots,\ \xi_q)$ is the parameter vector of G. Then, the log-likelihood function for ρ is given by

$$\ell = \ell(\boldsymbol{\rho}) = n\log(\theta\lambda\alpha^{\theta}) - n\log(1-e^{-\lambda}) + \sum_{i=1}^{n}\log[g(t_i, \boldsymbol{\xi})] - \lambda\sum_{i=1}^{n}[G(t_i, \boldsymbol{\xi})]$$

$$+(\theta-1)\sum_{i=1}^{n}\log(e^{-\lambda G(t_i, \boldsymbol{\xi})} - e^{-\lambda}) - (\theta+1)\sum_{i=1}^{n}\log(1-\alpha e^{-\lambda} - \overline{\alpha} e^{-\lambda G(t_i, \boldsymbol{\xi})}) .$$

Due to its complex form, this function cannot be solved precisely, but it can be numerically maximized by using optimization methods available with the software R.

We obtain the components of the score vector $U_{\boldsymbol{\rho}} = (U_\theta, U_\alpha, U_\lambda, U_{\boldsymbol{\xi}})$ by taking the partial derivatives of the log-likelihood function with respect to θ, α, λ and $\boldsymbol{\xi}$.

The asymptotic variance-covariance matrix of the MLEs of parameters is obtained by inverting the Fisher information matrix $I(\rho)$ derived using the second partial derivatives of the log-likelihood function with respect to each parameter. The ij^{th} elements of $I_n(\rho)$ are given by

$$I_{ij} = -E[\partial^2 l(\rho)/\partial \rho_i \partial \rho_j], \; i,j = 1,2,\cdots,3+q.$$

In practice one can estimate $I_n(\rho)$ by the observed Fisher's information matrix $\hat{I}_n(\hat{\rho}) = (\hat{I}_{ij})$ defined as

$$\hat{I}_{ij} \approx \left(-\partial^2 l(\rho)/\partial \rho_i \partial \rho_j\right)_{\boldsymbol{\eta}=\hat{\boldsymbol{\eta}}}, \; i,j = 1,2,\cdots,3+q.$$

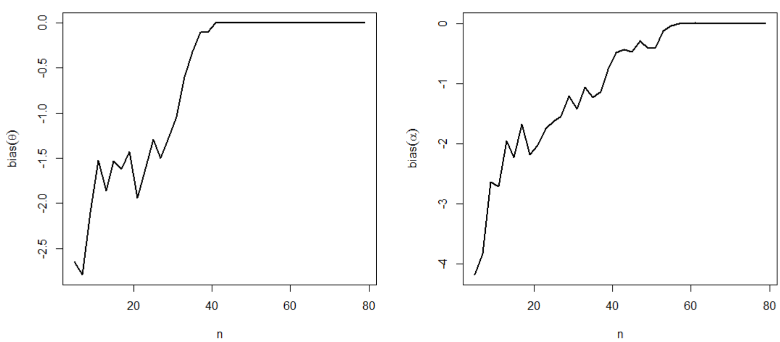
From the asymptotic theory of MLEs under some regularity conditions on the parameters as $n \to \infty$ the asymptotic distribution of $\sqrt{n}(\hat{\rho} - \rho)$ is $N_k(0, V_n)$ where $V_n = (v_{jj}) = I_n^{-1}(\rho)$. This holds even if V_n is replaced by $\hat{V}_n = \hat{I}^{-1}(\hat{\rho})$. Using this result large sample standard errors of j^{th} parameter ρ_j is given by $\sqrt{\hat{v}_{jj}}$.

4.2 Simulation

Here, a Monte Carlo simulation study is conducted to compare the performance of the different estimators of the unknown parameters for the $GMOP - E(\theta, \alpha, \lambda, \beta)$ distribution using R program. We generate $N = 3000$ samples of size $n = 5\text{to}80\text{from}$ GMOP-E distribution with true parameter values $\theta = 2, \alpha = 8, \lambda = 5, \beta = 0.5$, and calculate the bias and mean square error (MSE) of the MLEs empirically by

$$Bias_h = \frac{1}{N}\sum_{i=1}^{N}(\hat{h}_i - h) \text{ and } MSE_h = \frac{1}{N}\sum_{i=1}^{N}(\hat{h}_i - h)^2 \text{respectively (for} h = \theta, \alpha, \lambda, \beta).$$

Results of this simulation study are presented graphically in Figures 4 and 5 and tell us that as the sample sizes increases the biases and MSE's approach to 0 in all caess, which is consistent with the theoretical properties of the MLE and hence appropriate for estimating the $GMOP - E(\theta, \alpha, \lambda, \beta)$ distribution parameters.



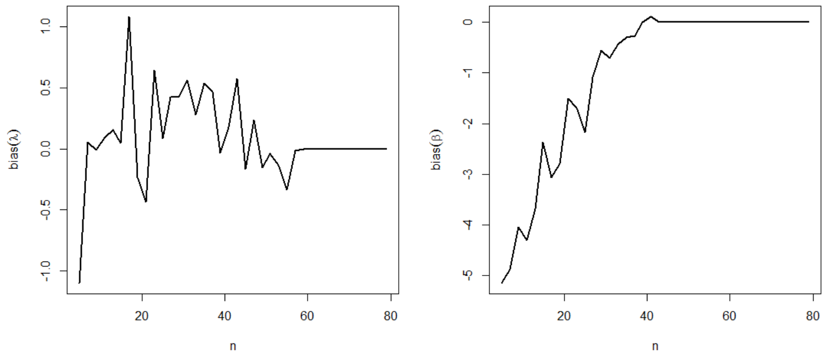


Figure 4: The Biases for the parameter values $\theta = 2, \alpha = 8, \lambda = 5,\ \beta = 0.5$ for $GMOP - E(\theta, \alpha, \lambda, \beta)$ distribution

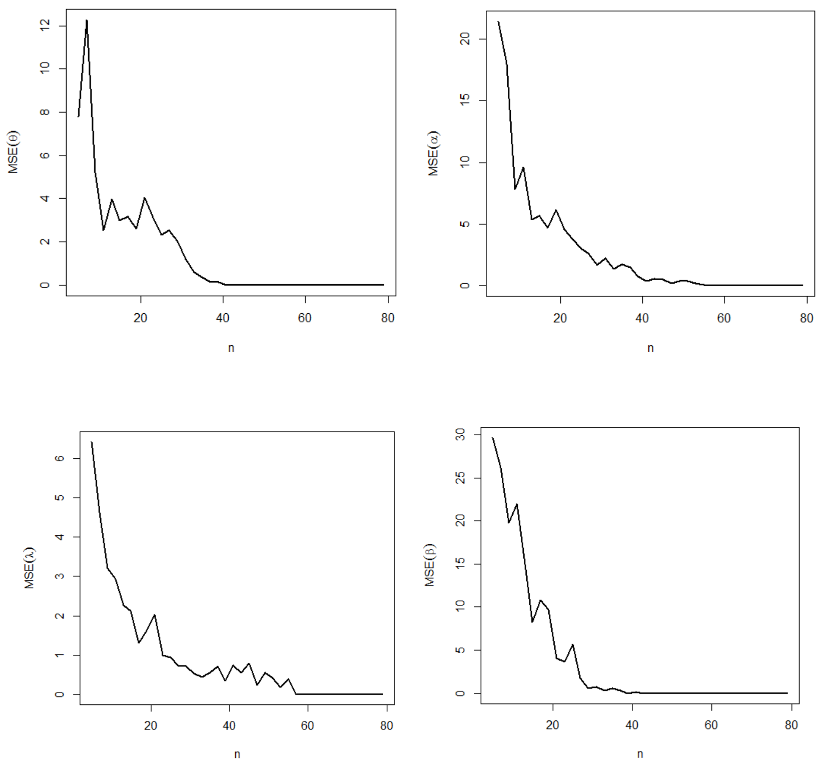


Figure 5: The MSEs for the parameter values $\theta = 2, \alpha = 8, \lambda = 5, \beta = 0.5$ for $GMOP - E(\theta, \alpha, \lambda, \beta)$ distribution

5. A Real Data Application

Here, we consider modelling of the one failure time data set to illustrate the suitability of the $GMOP-E(\theta,\alpha,\lambda,\beta)$ distribution in comparison to some existing distributions by estimating the parameters by numerical maximization of log-likelihood functions. The data set consists of survival time of 72 guinea pigs infected with virulent tubercle bacilli, reported by Bjerkedal (1960). The descriptive statistics about the data set shown in Table 3 reveal that the data set is positively skewed as expected from the nature of life time data and has higher kurtosis.

Table 3: Descriptive Statistics for the guinea pigs survival time's data set

Data Set	n	Min.	Mean	Median	s.d.	Skewness	Kurtosis	1st Qu.	3rd Qu.	Max.
I	72	0.100	1.851	1.560	1.200	1.788	4.157	1.080	2.303	7.000

We have compared the $GMOP-E(\theta,\alpha,\lambda,\beta)$ distribution with exponential (Exp), moment exponential (ME), transmuted exponential (T-E), Marshall-Olkin exponential (MO-E) (Marshall and Olkin, 1997), generalized Marshall-Olkin exponential (GMO-E) (Jayakumar and Mathew, 2008) and Marshall-Olkin transmuted exponential (MOT-E), Kumaraswamy exponential (Kw-E) (Cordeiro and de Castro, 2011), Beta exponential (BE) (Eugene et al., 2002), Marshall-Olkin Kumaraswamy exponential (MOKw-E) (Handique et al., 2017), Kumaraswamy Marshall-Olkin exponential (KwMO-E) (Alizadeh et al., 2015), beta Poisson exponential (BP-E) (Handique et al., 2022) and Kumaraswamy Poisson exponential (KwP-E) (Chakraborty et al., 2022) distributions for the failure time data set.

Model with the lowest AIC (Akaike Information Criterion), BIC (Bayesian Information Criterion), CAIC (Consistent Akaike Information Criterion), and HQIC (Hannan-Quinn Information Criterion) is chosen as the best. Also, to verify which distribution fits better these data goodness-of-fit tests, Anderson-Darling (A), Cram′er-von Mises (W) and Kolmogorov-Smirnov (K-S) statistics are applied. Asymptotic standard errors of the MLEs for each competing model are also provided. The best fitted density and the fitted cdf are plotted with the corresponding observed histograms and ogives in Figure 7, which indicates that the proposed distributions provide a close fit to this data set.

To check the shape of the observed hazard function the total time on test (TTT) plot Aarset, (1987) is used. A straight diagonal line indicates constant hazard for the data set, whereas a convex (concave) shape implies decreasing (increasing) hazard. The TTT plots for the data set Figure 6 indicate that the data set has increasing hazard rate. We also provide the box plot of the data to summerize the minimum, first quartile, median, third quartile, and maximum, where a box is shown from the first quartile to the third quartile with a vertical line going through the box at the median.

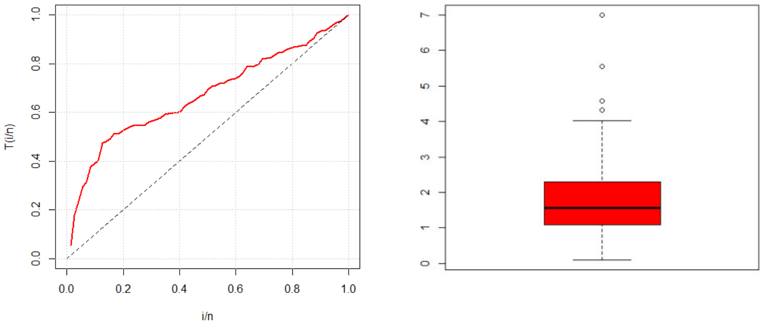


Figure 6: TTT and Box plot for the failure time data set

Table 4: MLEs, standard errors (in parentheses) values for the guinea pigs survival time's data set

Models	$\hat{\theta}$	$\hat{\alpha}$	$\hat{a}$	$\hat{b}$	$\hat{\lambda}$	$\hat{\beta}$
Exp	---	---	---	---	---	0.540
(β)						(0.063)
ME	---	---	---	---	---	0.925
(β)						(0.077)
T-E	---	---	---	---	-0.812	1.041
(λ, β)					(0.038)	(0.105)
MO-E	---	8.778	---	---	---	1.379
(α, β)		(3.555)				(0.193)
GMO-E	0.179	47.635	---	---	---	4.465
(θ, α, β)	(0.070)	(44.901)				(1.327)
MOT-E	---	3.245	---	---	-0.696	1.354
(α, λ, β)		(1.863)			(0.137)	(0.125)
Kw-E	---	---	3.304	1.100	---	1.037
(a, b, β)			(1.106)	(0.764)		(0.614)
B-E	---	---	0.807	3.461	---	1.331
(a, b, β)			(0.696)	(1.003)		(0.855)
MOKw-E	---	0.008	2.716	1.986	---	0.099
(α, a, b, β)		(0.002)	(1.316)	(0.784)		(0.048)
KwMO-E	---	0.373	3.478	3.306	---	0.299
(α, a, b, β)		(0.136)	(0.861)	(0.779)		(1.112)
BP-E	---	---	3.595	0.724	0.014	1.482
(a, b, λ, β)			(1.031)	(1.590)	(0.010)	(0.516)
KwP-E	---	---	3.265	2.658	4.001	0.177
(a, b, λ, β)			(0.991)	(1.984)	(5.670)	(0.226)
GMOP-E	0.333	12.584	---	---	0.054	2.858
$(\theta, \alpha, \lambda, \beta)$	(0.151)	(7.696)			(1.376)	(0.959)

Table 5: Log-likelihood, AIC, BIC, CAIC, HQIC, A, W and KS (p-value) values for the guinea pigs survival times data set

Models	AIC	BIC	CAIC	HQIC	A	W	KS (p-value)
Exp	234.63	236.91	234.68	235.54	6.53	1.25	0.27
(β)							(0.06)
ME	210.40	212.68	210.45	211.30	1.52	0.25	0.14
(β)							(0.13)
T-E	209.94	214.50	210.11	211.74	0.98	0.19	0.10
(λ, β)							(0.17)
MO-E	210.36	214.92	210.53	212.16	1.18	0.17	0.10
(α, β)							(0.43)
GMO-E	210.54	217.38	210.89	213.24	1.02	0.16	0.09
(θ, α, β)							(0.51)
MOT-E	208.26	215.10	208.61	210.96	0.86	0.15	0.10
(α, λ, β)							(0.47)
Kw-E	209.42	216.24	209.77	212.12	0.74	0.11	0.08
(a, b, β)							(0.50)
B-E	207.38	214.22	207.73	210.08	0.98	0.15	0.11
(a, b, β)							(0.34)
MOKw-E	209.44	218.56	210.04	213.04	0.79	0.12	0.10
(α, a, b, β)							(0.44)
KwMO-E	207.82	216.94	208.42	211.42	0.61	0.11	0.08
(α, a, b, β)							(0.73)
BP-E	205.42	214.50	206.02	209.02	0.55	0.08	0.09
(a, b, λ, β)							(0.81)
KwP-E	206.63	215.74	207.23	210.26	0.48	0.07	0.09
(a, b, λ, β)							(0.79)
GMOP-E	204.24	213.36	204.83	207.84	0.44	0.04	0.07
$(\theta, \alpha, \lambda, \beta)$							(0.83)

MLEs of parameters with standard errors for all the fitted models and AIC, BIC, CAIC, HQIC, A, W and K-S statistic with p-value for the failure time data set are presented respectively in Tables 4 and 5. It is obvious from these results that the $GMOP - E$distribution is not only a better model than the entire sub models but is also better than most of the recently introduced three or four parameters models. The plots in Figure 7 also indicate that the proposed distribution provides a close fit to the data set considered here.

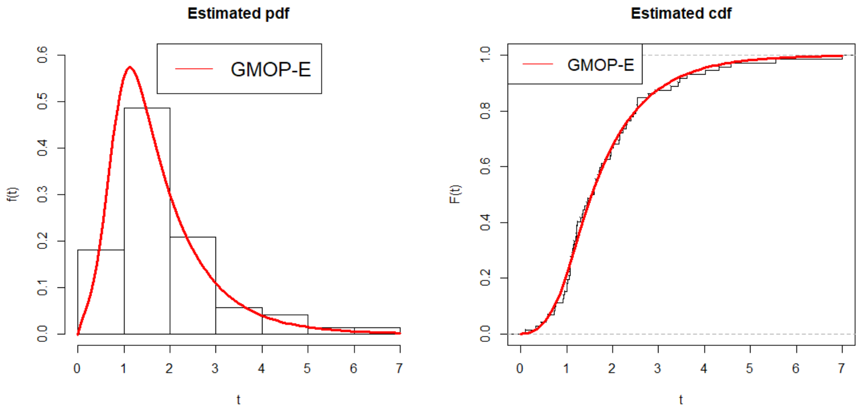


Figure 7: Plots of the observed histogram and estimated pdf on left and on right the observed ogive and estimated cdf for failure time data set for the GMOP-E model

6. Conclusions

In this work, we propose a new family of continuous distributions called the *Generalized Marshall-Olkin Poisson -G* family of distributions. Several new models can be generated by considering special distributions for G. We demonstrate that the pdf of any GMOPG distribution can be expressed as a linear combination of exponentiated-G density functions, which allowed us to derive some of its mathematical and statistical properties. The estimations of the model parameters are obtained by the maximum likelihood method. One application of the proposed family empirically proves its flexibility to model real data sets. In particular, we verified that a special case of the GMOPG family can provide better fits than its sub models and other models generated from well-known families.

Conflicts of Interest

Authors have no conflict of interest.

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