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The concept of a behavioral model of decision-making under risk

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Abstract

This study outlines the decision-making process under risk considering the psychological aspects of the decision-maker. The aim is to construct a principle of an optimal decision for an individual decision-maker. The study considers a finite, discrete set of acceptable decisions, a set of possible world states and a system of probabilities of these states (where the probabilities are either known or subjectively estimated by the decision-maker), and a utility matrix of making each decision in particular world states. The proposed process of optimizing the decision addresses not only the rationality of the person making but also emotional aspects of the person making the decision. Rationality is represented by the value of the utility function of the benefits resulting from making a decision in a possible world state. The behavioral part of the model involves two emotions important to decision-making: regret over making a decision that brought less utility than possible in the given conditions and satisfaction with the choice which proved better than the worst option. The first emotion is represented by the regret function and the other by the satisfaction function. New notions are defined: relative utility and expected relative utility of particular decisions used to construct the principle of an optimal decision under risk.

The presented theory thus supplements the prospect theory, as it accounts for regret and satisfaction in the decision-making process under risk. This idea is part of behavioral economics, yet not standing in opposition to classical economics. Its advantage is that it considers both psychological and rational factors in the decision-making process.

Key words: utility function, prospect theory, decisions under risk, regret function, satisfaction function, expected relative utility.

1. Introduction

This paper is inspired by the prospect theory (Kahneman and Tversky, 1979; Jajuga, 2008), which is part of behavioral economics and states that the actions of an individual making economic decisions are guided not only by rationality, but also by psychological factors.

A decision-making model is examined in which one decision-maker desires to make a decision that is optimum under risk; that is, he knows the possible states of the outside world and the distribution of their probabilities (these probabilities can be understood in the classical sense and treated as objective) or does not know the probabilities of the particular world states but estimates them subjectively (these are subjective probabilities in the circumstances (Sadowski, 1981), a concept first introduced by Savage (1954)). Savage pointed

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out that the probability of certain events (e.g., wars, natural disasters, economic crises) is unknown in decision-making situations involving uncertainty. Subjective probability, introduced by Savage, expresses the degree of someone's conviction that a given event may take place (Tyszka et al., 2004). Here, subjective probability should be understood as the extent of belief that a given state will occur; although this degree, for a given state, in certain circumstances, may differ among various decision-makers.

Let the following be given:

- a set of acceptable decisions $M = \{d_1, \ldots, d_k\}$, which, for the sake of simplicity, is discrete and finite,
- a set of external world states $S = \{s_1, \ldots, s_n\}$ with a corresponding system of probabilities $p_1, \ldots, p_n$, where $p_j = P(S = s_j)$ is the probability of the state s_j for $j = 1, \ldots, n$, with $p_j > 0$ and $\sum_{j=1}^{n} p_j = 1$.
- a benefit function
$$k : M \times S \to \mathbf{R},$$
which assigns each possible pair (d_i, s_j) of a decision $d_i \in M$ and external world state $s_j \in S$ to a real number $x_{ij} \in \mathbf{R}$, to be known as the benefit of a decision d_i, $i = 1, \ldots, k$ in the state s_j, $j = 1, \ldots, n$:
$$k(d_i, s_j) = x_{ij}. \tag{1}$$

The values of the benefit function (1) can be represented by a matrix of benefits (Table 1). Let

$$K := k(M \times S)$$

be a set of benefit function values (1).

Table 1. A matrix of benefit function

Probabilities	p_1	p_2	$\ldots$	p_j	$\ldots$	p_n
Decisions/states	s_1	s_2	$\ldots$	s_j	$\ldots$	s_n
d_1	x_{11}	x_{12}	$\ldots$	x_{1j}	$\ldots$	x_{1n}
d_2	x_{21}	x_{22}	$\ldots$	x_{2j}	$\ldots$	x_{2n}
$\vdots$	$\vdots$	$\vdots$	$\ldots$	$\vdots$	$\ldots$	$\vdots$
d_i	x_{i1}	x_{i2}	$\ldots$	x_{ij}	$\ldots$	x_{in}
$\vdots$	$\vdots$	$\vdots$	$\ldots$	$\vdots$	$\ldots$	$\vdots$
d_k	x_{k1}	x_{k2}	$\ldots$	x_{kj}	$\ldots$	x_{kn}

It is the decision-maker's aim to choose a decision that is optimal from his point of view.

The principle of maximizing the expected value (benefit), formulated by Pascal, is the oldest method for selecting an optimum decision under risk. For any decision d_i, $i = 1, \ldots, k$, the expected value of benefits needs to be calculated, considering all the known states of the external world and their probabilities:

$$\mathrm{E}X_i = \sum_{j=1}^{n} x_{ij} p_j, \tag{2}$$

The decision is optimal for which the expected value of benefits formulated with (2) is maximum:

$$\max_i \mathrm{E}X_i. \tag{3}$$

Although the theory of maximizing expected value was the starting point for Markowitz's portfolio theory (Markowitz, 1952), it has faced criticism. Its weakness consists in the fact that it fails to address the risk or the decision-maker's subjective approach, for instance. It was already noted by Daniel Bernoulli, in 1738, in the so-called St. Petersburg paradox, in which the value of the expected payoff is infinite (the game involves flipping a coin and ends when tails are up, and the payoff is 2^n, where n is the number of tosses). Thus, someone guided by the principle of maximizing expected profit should pay an infinite sum to take part in the game, which sounds like a paradox. Bernoulli pointed out that people, in practice, are driven in their decisions not by the rule of maximizing the expected payoff, but by maximizing the expected utility, which he understood as the psychological value of money. Based on this observation, Bernoulli suggested a novel approach to evaluating bets. As Daniel Kahneman (2012) writes: "His idea was simple: human choices are based not on the monetary values of possible choices but on their psychological value, or utility. The psychological value of a bet is therefore not a weighted average of possible financial results, but an average of their utilities, where the utility of each result is weighted based on its probability":

$$\mathrm{E}u(X_i) = \sum_{j=1}^{n} u(x_{ij}) p_j, \tag{4}$$

where $u(x_{ij})$ is the utility of benefits from making the decision d_i given the world's state s_j. The optimum decision is the one for which the expected value of utility formulated as (4) is maximized:

$$\max_i \mathrm{E}u(X_i). \tag{5}$$

The utility of a certain amount of money is a subjective value that can vary among individuals. In order to arrive at a reliable decision-making criterion, therefore, one should be guided not only by the rule (5), but also take into account the value of standard deviation, for example, a measure of risk, or the coefficient of variation. The classic version of the theory of maximizing the expected utility is available in the monograph by von Neumann and Morgenstern (1947). New theories emerged over successive years of selecting optimum decisions under risk that began to consider psychological factors, for instance, Savage's theory of subjective expected utility (Savage, 1954) and the prospect theory (Kahneman and Tversky, 1979) – for which Daniel Kahneman was awarded the Nobel Prize in 2002 – are based on the assumption that human choices are unstable and context-dependent, chiefly on whether a decision is made when losses are sustained or profits are generated. The theory of regret, introduced independently by Bell (1982) and Loomes and Sugden (1982) and developed by Quiggin (1994), continues the concept of decision-making by addressing behavioral factors. Regret is expressed as the difference between the utility of a given decision's result and the utility of the best result of all decisions in a given situation (Zatoń, 2010).

2. The behavioral model of decision-making under risk considering regret and satisfaction

In this study, a method for choosing the optimum decision under risk is proposed. This method is an extension of the methods described in Chapter 1. It refers back to the theory of regret (Bell, 1982; Loomes and Sugden, 1982) by considering the role of this emotion in the decision-making process. It is also an expansion: an emotion contrary to regret – joy, satisfaction – is taken into account. In addition, the model uses the utility function that meets the assumptions of the prospect theory (Kahneman and Tversky, 1979). However, as Daniel Kahneman (2012) states: "The prospect theory and the theory of utility are unable to explain the phenomenon of regret. They both assume that all available options are evaluated separately and independently when making a decision; then, the option with the maximum value is selected. (...) such an assumption is certainly wrong."
Besides rationality, the considered model takes behavioral factors into account. Two strong emotions are analyzed from the viewpoint of decision-making: regret at making a decision that brings lesser utility than is possible in a given world state (this section refers back to the regret theory, Quiggin (1994))–and an opposite emotion–satisfaction with making a choice that is better than the worst in a given world state. Each emotion is represented by their respective functions of regret and satisfaction.

Let $u : K \to \mathbf{R}$ be the function of utility that, for each benefit $x_{ij} = k(d_i, s_j) \in K \subset \mathbf{R}$ of making decision d_i, $i = 1, \dots, k$ in a world state s_j, $j = 1, \dots, n$ (described by the function of benefit (1) and present in the matrix of benefits in Table 1), ascribes a certain subjective value, which is individual for each decision-maker:

$$x_{ij} \mapsto u(x_{ij}). \tag{6}$$

Since the function of utility depends on decision-makers' individual preferences, it may vary for different persons. It is assumed, nonetheless, that the function has certain characteristics shared by all decision-makers. Namely, the utility curve $u : K \to \mathbf{R}$ is assumed to meet the assumptions of the value function from the prospect theory (Kahneman and Tversky, 1979); that is, its shape is different for gains and for losses.

Assuming $0 \in K$ is the reference point for the decision-maker's gains and losses, the utility function can be described as follows:

1. $u \in C^2(K)$, i.e., it is continuous and has first and second order derivatives,
2. $\forall_{x \in K, x>0}\ u(-x) < 0 < u(x) \wedge u(0) = 0$ - the utility of losses is negative, that of gains is positive, and that of zero benefit is zero,
3. $\forall_{x_1, x_2 \in K}(x_1 < x_2 \to u(x_1) < u(x_2))$ - the utility is an increasing function,
4. $\forall_{x \in K, x>0}\ |u(-x)| > u(x)$ - the function is steeper for losses than for gains (a loss is more painful than a gain is enjoyable),
5. $\forall_{x \in K, x \le 0}\ \frac{\partial u}{\partial x} > 0 \wedge \frac{\partial^2 u}{\partial x^2} > 0$ - the utility function for losses rises in a convex manner,

6. $\forall_{x\in K, x\geq 0}\ \frac{\partial u}{\partial x} > 0 \wedge \frac{\partial^2 u}{\partial x^2} < 0$ - the utility function for gains rises in a concave manner.

Where the sets of decisions M and external world states S are finite, the values of the utility functions (6) can be illustrated with a matrix of utility (Table 2), as in the case of the benefit function (1):

Table 2. Utility matrix

Probabilities	p_1	p_2	$\dots$	p_j	$\dots$	p_n
Decisions/states	s_1	s_2	$\dots$	s_j	$\dots$	s_n
d_1	$u(x_{11})$	$u(x_{12})$	$\dots$	$u(x_{1j})$	$\dots$	$u(x_{1n})$
d_2	$u(x_{21})$	$u(x_{22})$	$\dots$	$u(x_{2j})$	$\dots$	$u(x_{2n})$
$\vdots$	$\vdots$	$\vdots$	$\dots$	$\vdots$	$\dots$	$\vdots$
d_i	$u(x_{i1})$	$u(x_{i2})$	$\dots$	$u(x_{ij})$	$\dots$	$u(x_{in})$
$\vdots$	$\vdots$	$\vdots$	$\dots$	$\vdots$	$\dots$	$\vdots$
d_k	$u(x_{k1})$	$u(x_{k2})$	$\dots$	$u(x_{kj})$	$\dots$	$u(x_{kn})$

Definition 2.1 ***The general model for decision-making under risk (GMR)*** *is called the system (M,S,P,U), consisting of:*

- *the set $M = \{d_1,\dots,d_k\}$ of acceptable decisions, discrete and finite for the sake of simplicity,*
- *the set $S = \{s_1,\dots,s_n\}$ of external world states,*
- *the probability distribution $P = (p_1,\dots,p_n)$ of the external world states, where $p_j = P(s_j)$ for $j = 1,\dots,n$, with $\forall_{j=1,\dots,n} p_j > 0$ and $\sum_{j=1}^{n} p_j = 1$ (as part of the model, the probabilities p_j may be either objective or subjectively estimated by the decision-maker),*
- *the utility matrix $U = [u(x_{ij})]_{i=1,\dots,k,\ j=1,\dots,n}$, where $u(x_{ij})$ is the utility of making decision d_i, $i = 1,\dots,k$ in a world state s_j, $j = 1,\dots,n$, i.e. the value of utility function u that satisfies the above assumptions 1.-6. of the prospect theory.*

Remark 1 *Insofar as, there is precisely one benefit function (1) for all decision-makers for a given decision-making system (M,S,P,U), there can be a different utility function (6) for every decision-maker (the function of utility depends on the individual preferences of persons making decisions).*

It is the decision-maker's intention to choose an optimum decision in probable world states $s_1,\dots,s_n$.

Remark 2 *Note that the rows of the benefit matrix (Table 1) can be considered as the values of random variables $X_1,\dots,X_k$, where X_i is the benefit of making a decision d_i, $i = 1,\dots,k$. Therefore, for every $i = 1,\dots,k$, the distribution of random variable X_i is known (Table 3).*

Table 3. Benefits distribution

x_{ij}	x_{i1}	x_{i2}	$\dots$	x_{in}
$P(X_i = x_{ij})$	p_1	p_2	$\dots$	p_n

Since the utility function $u : K \to \mathbf{R}$ is continuous, from the theory of probability (Evans and Rosenthal, 2009), it is known that $u(X_i)$ is also a random variable for every $i = 1, \ldots, k$. What is more, since u is a monotonous function, the probability of the particular benefits' utility is equal to the probabilities of these benefits, i.e.

$$P(u(X_i) = u(x_{ij})) = P(X_i = x_{ij}) = p_j \text{ for } i = 1, \ldots, k, \; j = 1, \ldots, n. \tag{7}$$

Therefore, the probability distribution of the random variable $u(X_i)$ denoting the utility of making i-th decision for every $i = 1, \ldots, k$ is the same as that for the random variable X_i (Table 4).

Table 4. Utility distribution

$u(x_{ij})$	$u(x_{i1})$	$u(x_{i2})$	$\ldots$	$u(x_{in})$
$P(u(X_i) = u(x_{ij}))$	p_1	p_2	$\ldots$	p_n

The present model of decision-making under risk is designed to consider not only the rationality of a decision-maker but also psychological factors. The rational determinants are the values of the utility function (from the utility matrix, Table 2) for the specific gains in the subsequent world states. The behavioral determinants are two strong emotions associated with decision-making: regret at selecting a suboptimal result possible in a given world state, and satisfaction with choosing a result better than the worst in a given world state. Two functions are defined for describing emotions: one of regret and one of satisfaction. Let U be the set of values of the utility function $u : K \to \mathbf{R}$, i.e.

$$U := u(K). \tag{8}$$

Definition 2.2 *Let*

$$r : U \to \mathbf{R}_{-} \cup \{0\} \tag{9}$$

be a continuous function that attributes to the utility $u(x_{ij})$ of making i-th decision in j-th world state the difference between this utility and the maximum utility that can be derived from any decision in a known j-th world state:

$$r(u(x_{ij})) = u(x_{ij}) - \max_{1 \le i \le k} u(x_{ij}) := r_{ij}. \tag{10}$$

*The function r defined with (10) will be called **the regret function**.*

Note that the values of the regret function are not positive; they are negative if, in a given world state, the utility of gains from a decision under consideration is lower than the greatest utility of another decision. The value of this negative number serves as a measure of regret. They are zero if the utility of benefits from the considered decision is maximum in a given world state. In the case of a finite, discrete set of decisions and a finite, discrete set of world states, the values of the regret function for the particular decisions can be grouped into a matrix that will be referred to as ***the matrix of regret***:

$$M_r = [r_{ij}]_{i=1,\ldots,k,\; j=1,\ldots,n}. \tag{11}$$

Example 1 *Let a set of acceptable decisions* $M = \{d_1, d_2, d_3, d_4\}$*, a set of external world states* $S = \{s_1, s_2, s_3\}$ *and their corresponding probabilities* $p_1 = 0.8, p_2 = 0.15, p_3 = 0.05$*, and a matrix of utility of making the particular decisions in the subsequent world states be given:*

Table 5. *Utility matrix - example 1,2*

Probabilities	0.8	0.15	0.05
Decisions/states	s_1	s_2	s_3
d_1	*8*	*14*	*20*
d_2	*10*	*10*	*10*
d_3	*15*	*0*	*-40*
d_4	*-10*	*80*	*100*

Considering the first world state s_1, it is clear that the maximum utility of 15 can be attained in the case of the third decision d_3, namely

$$\max_{1 \le i \le 4} u(x_{i1}) = 15.$$

Thus, relying on equation (10), the first column of the regret matrix, or the value of regret in the first world state s_1, can be calculated as:

$$\begin{aligned} r_{11} &= u(x_{11}) - \max_{1 \le i \le 4} u(x_{i1}) = 8 - 15 = -7, \\ r_{21} &= u(x_{21}) - \max_{1 \le i \le 4} u(x_{i1}) = 10 - 15 = -5, \\ r_{31} &= u(x_{31}) - \max_{1 \le i \le 4} u(x_{i1}) = 15 - 15 = 0, \\ r_{41} &= u(x_{41}) - \max_{1 \le i \le 4} u(x_{i1}) = -10 - 15 = -25. \end{aligned} \tag{12}$$

Proceeding likewise for the second state s_2, with its maximum utility of 80 ($\max_{1 \le i \le 4} u(x_{i2}) = 80$) that can be reached for the decision d_4, the second column of the regret matrix is generated:

$$\begin{aligned} r_{12} &= u(x_{12}) - \max_{1 \le i \le 4} u(x_{i2}) = 14 - 80 = -66, \\ r_{22} &= u(x_{22}) - \max_{1 \le i \le 4} u(x_{i2}) = 10 - 80 = -70, \\ r_{32} &= u(x_{32}) - \max_{1 \le i \le 4} u(x_{i2}) = 0 - 80 = -80, \\ r_{42} &= u(x_{42}) - \max_{1 \le i \le 4} u(x_{i2}) = 80 - 80 = 0 \end{aligned} \tag{13}$$

and for the third state s_3, where the maximum utility of 100 ($\max_{1 \le i \le 4} u(x_{i3}) = 100$) can be reached for decision d_4, we generate the third column of the regret matrix:

$$\begin{aligned} r_{13} &= u(x_{13}) - \max_{1\le i\le 4} u(x_{i3}) = 20 - 100 = -80, \\ r_{23} &= u(x_{23}) - \max_{1\le i\le 4} u(x_{i3}) = 10 - 100 = -90, \\ r_{33} &= u(x_{33}) - \max_{1\le i\le 4} u(x_{i3}) = -40 - 100 = -140, \\ r_{43} &= u(x_{43}) - \max_{1\le i\le 4} u(x_{i3}) = 100 - 100 = 0. \end{aligned} \tag{14}$$

Substituting the values (12), (13), and (14) of the regret function into (11), the following regret matrix results:

$$M_r = \begin{bmatrix} -7 & -66 & -80 \\ -5 & -70 & -90 \\ 0 & -80 & -140 \\ -25 & 0 & 0 \end{bmatrix}. \tag{15}$$

The elements of the matrix M_r, the numerical values of the regret function, express the scale of regret at not choosing a better option. For instance, $r_{11} = -7$ is equivalent to the size of regret in the first world state relative to the decision d_1 producing the utility $u(x_{11}) = 8$, with the maximum possible utility in this world state being $\max_{1\le i\le 4} u(x_{i1}) = 15$. In the world state s_1 for the decision d_3, the value of regret is zero (i.e., this feeling is absent), since the value of utility is $u(x_{31}) = 15$; that is, it is the maximum possible utility to be achieved in this world state.

Besides regret, satisfaction with choosing a potentially better option than the worst in a given external world state is another strong emotion. It is contrary to the feeling of regret. Like in the case of regret, the level of satisfaction can be measured.

Definition 2.3 *Let*

$$s : U \to \mathbf{R}_+ \cup \{0\}, \tag{16}$$

be a continuous function that attributes to the utility $u(x_{ij})$ of making i-th decision in j-th world state the difference between this utility and the minimum possible value of the utility in the j-th world state:

$$s(u(x_{ij})) = u(x_{ij}) - \min_{1\le i\le k} u(x_{ij}) := s_{ij}. \tag{17}$$

The function s, defined with (17), will be called ***the satisfaction function****.*

Note that the values of the satisfaction function are non-negative; that is, they are either positive, where the utility of a decision is greater than the minimum utility of another decision (this positive number is then a measure of satisfaction) in a given world state, or zero if the utility of benefit from a given decision is the lowest in a given world state (satisfaction is absent in this case). In the case of a finite, discrete set of decisions and a finite, discrete set of external world states, the values of the satisfaction function for the particular decisions can be grouped into a matrix that will be referred to as ***the satisfaction matrix***:

$$M_s = [s_{ij}]_{i=1,\dots,k,\ j=1,\dots,n}. \tag{18}$$

Example 2 *As far as example 1 is concerned, satisfaction levels will be determined when considering a decision from the set of acceptable decisions $M = \{d_1, d_2, d_3, d_4\}$ given the set of external world states $S = \{s_1, s_2, s_3\}$, where the utilities of making the particular decisions in the successive world states are given in the utility matrix (Table 5).*

Since the utility of benefits from the decision d_4 is the minimum utility in the first world state s_1:

$$\min_{1 \le i \le 4} u(x_{i1}) = -10,$$

relying on (17), the first column of the satisfaction matrix can be derived:

$$\begin{aligned} s_{11} &= u(x_{11}) - \min_{1 \le i \le 4} u(x_{i1}) = 8 - (-10) = 18, \\ s_{21} &= u(x_{21}) - \min_{1 \le i \le 4} u(x_{i1}) = 10 - (-10) = 20, \\ s_{31} &= u(x_{31}) - \min_{1 \le i \le 4} u(x_{i1}) = 15 - (-10) = 25, \\ s_{41} &= u(x_{41}) - \min_{1 \le i \le 4} u(x_{i1}) = -10 - (-10) = 0. \end{aligned} \tag{19}$$

For the second world state s_2, where utility is minimum for decision d_3 and equals zero:

$$\min_{1 \le i \le 4} u(x_{i2}) = 0,$$

the second column of the satisfaction matrix is the same as the second column of the utility matrix (Table 5):

$$\begin{aligned} s_{12} &= u(x_{12}) - \min_{1 \le i \le 4} u(x_{i2}) = 14, \\ s_{22} &= u(x_{22}) - \min_{1 \le i \le 4} u(x_{i2}) = 10, \\ s_{32} &= u(x_{32}) - \min_{1 \le i \le 4} u(x_{i2}) = 0, \\ s_{42} &= u(x_{42}) - \min_{1 \le i \le 4} u(x_{i2}) = 80. \end{aligned} \tag{20}$$

For the third world state s_3, in which utility is minimum for decision d_3:

$$\min_{1 \le i \le 4} u(x_{i3}) = -40,$$

the third column of the satisfaction matrix results:

$$\begin{aligned} s_{13} &= u(x_{13}) - \min_{1\le i\le 4} u(x_{i3}) = 20-(-40) = 60, \\ s_{23} &= u(x_{23}) - \min_{1\le i\le 4} u(x_{i3}) = 10-(-40) = 50, \\ s_{33} &= u(x_{33}) - \min_{1\le i\le 4} u(x_{i3}) = -40-(-40) = 0, \\ s_{43} &= u(x_{43}) - \min_{1\le i\le 4} u(x_{i3}) = 100-(-40) = 140. \end{aligned} \tag{21}$$

On substituting the calculated values (19), (20), and (21) of the satisfaction function into (18), the following satisfaction matrix is generated:

$$M_s = \begin{bmatrix} 18 & 14 & 60 \\ 20 & 10 & 50 \\ 25 & 0 & 0 \\ 0 & 80 & 140 \end{bmatrix}. \tag{22}$$

The elements of the matrix M_s, the numerical values of the satisfaction function, express the extent of satisfaction with choosing a better-than-the-worst option. For example, $s_{11} = 18$ is the scale of satisfaction in the first world state s_1 when considering the decision d_1 that produces the utility $u(x_{11}) = 8$, where the lowest possible utility in this world state is $\min_{1\le i\le 4} u(x_{i1}) = -10$. Satisfaction is zero in the world state s_1 for decision d_4, since utility is $u(x_{41}) = -10$, i.e., the lowest possible utility in this world state.

Some new notions are defined in the decision-making model introduced here: relative utility and expected relative utility.

Definition 2.4 ***Relative utility of i-th decision*** d_i ***in the*** j***-th world state*** s_j *is the total sum of utility* $u(x_{ij})$ *and the values of regret function* $r(u(x_{ij}))$ *and the sum total* $s(u(x_{ij}))$ *with making the decision:*

$$u_w(x_{ij}) = u(x_{ij}) + \alpha \cdot r(u(x_{ij})) + (1-\alpha)\cdot s(u(x_{ij})), \tag{23}$$

where $0 \le \alpha \le 1$ *is loss-aversion factor,* $i = 1,\dots,k,\ j = 1,\dots,n$*. The expression* $u(x_{ij})$ *will be referred to as* ***the rational part*** *and* $\alpha \cdot r(u(x_{ij})) + (1-\alpha)\cdot s(u(x_{ij}))$ *-* ***as the behavioral part*** *of the relative utility of i-th decision in the j-th world state (or, to be more brief - if it does not give rise to misunderstandings - the rational and behavioral parts of the relative utility).*

Denoted by

$$\mathrm{Ra}(u_w(x_{ij})) := u(x_{ij}) \ \text{ and } \ \mathrm{Be}(u_w(x_{ij})) := \alpha \cdot r(u(x_{ij})) + (1-\alpha)\cdot s(u(x_{ij})), \tag{24}$$

relative utility can be expressed briefly as the sum of the rational and behavioral parts:

$$u_w(x_{ij}) = \mathrm{Ra}(u_w(x_{ij})) + \mathrm{Be}(u_w(x_{ij})). \tag{25}$$

The behavioral part of relative utility

$$\mathrm{Be}(u_w(x_{ij})) = \alpha \cdot r(u(x_{ij})) + (1-\alpha) \cdot s(u(x_{ij})), \tag{26}$$

for $0 \leq \alpha \leq 1$, is a convex combination of the values of regret and satisfaction function when making decision d_i in the world state s_j.

Remark 3 *The loss-aversion factor α, depending on its value in the range $[0;1]$, may enhance or weaken the values of regret and satisfaction functions. Thus, when considering a given decision-making variant in a specified world state:*

1. *if $0 \leq \alpha < 0.5$, the feeling of satisfaction prevails over regret in the decision-making process for the decision-maker,*
2. *if $0.5 < \alpha \leq 1$, the feeling of regret prevails over satisfaction,*
3. *if $\alpha = 0.5$, the feelings of regret and satisfaction are balanced when making decisions.*

Remark 4 *In some economic studies, the $\alpha : (1-\alpha)$ ratio is like 2:1 (Kahneman, 2012).*

Once the values of regret $r(u(x_{ij}))$ and satisfaction $s(u(x_{ij}))$ functions are replaced with the formulas from (10) and (17) in the equation (23) as appropriate, the relative utility of the i-th decision ($i = 1,\ldots,k$) in the j-th world state ($j = 1,\ldots,n$) becomes:

$$u_w(x_{ij}) = u(x_{ij}) + \alpha \cdot (u(x_{ij}) - \max_{1\leq i\leq k} u(x_{ij})) + (1-\alpha) \cdot (u(x_{ij}) - \min_{1\leq i\leq k} u(x_{ij})). \tag{27}$$

Relative utility is thus understood as utility relative to the maximum or minimum utility, with reference to the extent of regret or satisfaction in a given world state.

Transforming (27) produces yet another equivalent form of relative utility of i-th decision in the j-th world state:

$$u_w(x_{ij}) = 2u(x_{ij}) - \alpha \cdot \max_{1\leq i\leq k} u(x_{ij}) - (1-\alpha) \cdot \min_{1\leq i\leq k} u(x_{ij}). \tag{28}$$

Definition 2.5 *The system (M,S,P,U_w), where $U_w = [u_w(x_{ij})]_{i=1,\ldots,k,\ j=1,\ldots,n}$ is a relative utility matrix, will be called **the behavioral model of decision-making under risk** or shortly **BMR**.*

Based on definition 2.4 of relative utility of i-th decision in the j-th world state, another notion is constructed that is needed to build the optimization criterion of behavioral decisions:

Definition 2.6 ***Expected relative utility of i-th decision** d_i for $i = 1,\ldots,k$ is defined as:*

$$Eu_w(d_i) = \sum_{j=1}^{n} u_w(x_{ij}) \cdot p_j, \tag{29}$$

where p_j is the probability of the world state s_j, $j = 1,\ldots,n$.

Remark 5 *In the BMR model, the simple probabilities $p_1,\dots,p_n$ can be replaced by the so-called weighting function (from the prospect theory, (Jajuga 2008)) transforming the probabilities:*

$$\pi : [0;1] \to [0;1],$$

$$\pi(p_j) = \frac{p_j^b}{\sum_{j=1}^n p_j^b}, \ \ 0 \le b \le 1, \tag{30}$$

Kahneman and Tversky (1979) noted that low probabilities close to zero are normally inflated, whereas those high, approaching one, are depressed. The parameter b in (30) of the weighting function fulfills the role of subjectively distorting the probabilities:

1. *for $b = 0$, the weighting function (30) distorts the probabilities into a uniform distribution, i.e.*
$$\forall_{j=1,\dots,n} \ \pi(p_j) = \frac{1}{n},$$
2. *for $b = 1$, the distribution remains unchanged, that is*
$$\forall_{j=1,\dots,n} \ \pi(p_j) = p_j.$$

<u>The criterion of decision optimization under risk in the BMR model:</u>

Let the behavioral decision-making model under risk, described above, (M,S,P,U_w), be given. For any decision d_i, $i = 1,\dots,k$, the expected relative utility can be calculated (eq. (29)). The decision d_0 for which the expected relative utility is maximum is the optimum decision:

$$E_0 = \max_{i=1,\dots,k} Eu_w(d_i). \tag{31}$$

The foregoing criterion, besides rational factors represented in relative utility u_w by utility in its classic sense, also considers behavioral factors represented by the values of the regret and satisfaction functions. If, in equation (23), the relative utility of the i-th decision in the j-th world state described by (25) is adopted, it becomes clear that the optimum decision is d_0, for which

$$E_0 = \max_{i=1,\dots,k} Eu_w(d_i) = \max_{i=1,\dots,k} \sum_{j=1}^n u_w(x_{ij}) \cdot p_j = \max_{i=1,\dots,k} \sum_{j=1}^n (\mathrm{Ra}(u_w(x_{ij})) + \mathrm{Be}(u_w(x_{ij}))) \cdot p_j. \tag{32}$$

Remark 6 *If there exist two different decisions $d_l \neq d_m$, $l,m = 1,\dots,k$, for which*

$$E_0 = \max_{i=1,\dots,k} Eu_w(d_i) = Eu_w(d_l) = Eu_w(d_m), \tag{33}$$

then, in order to decide which decision is optimal for the decision-maker, the standard deviations for both decisions should be calculated and, using their analysis, the optimal decision should be selected. The standard deviation for i-th decision is understood here as the *standard deviation from the relative expected value:*

$$su_w(d_i) = \sqrt{\sum_{j=1}^{n} (u_w(x_{ij}) - Eu_w(d_i))^2 p_j}. \tag{34}$$

3. BMR model in the matrix form

The behavioral decision-making model under risk described above, addressing the emotions of regret and satisfaction, can be expressed in an equivalent matrix form.
Let (M, S, P, U_w) be the behavioral decision-making model under risk (BMR), where $M = \{d_1, \ldots, d_k\}$ is a set of acceptable decisions, $S = \{s_1, \ldots, s_n\}$ a set of external world states with a corresponding system of probabilities $p_1, \ldots, p_n$, where $p_j = P(s_j)$, $j = 1, \ldots, n$, and $\sum_{j=1}^{n} p_j = 1$ are given. Let $u : K \to \mathbf{R}$ be the utility function for making a decision d_i, $i = 1, \ldots, k$ in the world state s_j, $j = 1, \ldots, n$, with properties 1. - 6. referring to Kahneman's and Tversky's prospect theory. Assuming that, for any benefit $x_{ij} \in K$ of making the i-th decision, $i = 1, \ldots, k$, in the j-th world state, $j = 1, \ldots, n$, the benefit's utility is formulated as

$$u(x_{ij}) = u_{ij}, \tag{35}$$

a utility matrix can be created:

$$M = [u_{ij}]_{i=1,\ldots,k,\ j=1,\ldots,n} = \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ u_{21} & u_{22} & \cdots & u_{2n} \\ \vdots & \vdots & & \vdots \\ u_{k1} & u_{k2} & \cdots & u_{kn} \end{bmatrix} \in M_{k \times n}(\mathbf{R}). \tag{36}$$

On defining the regret function $r : U \to \mathbf{R}_- \cup \{0\}$ by means of (10), where U is the set of utility functions, the values of regret r_{ij} at making the i-th decision, $i = 1, \ldots, k$, in the j-th world state, $j = 1, \ldots, n$, are inputs to the regret matrix described with (11), expanded as follows:

$$M_r = [r(u_{ij})]_{i=1,\ldots,k,\ j=1,\ldots,n} = \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ r_{21} & r_{22} & \cdots & r_{2n} \\ \vdots & \vdots & & \vdots \\ r_{k1} & r_{k2} & \cdots & r_{kn} \end{bmatrix} \in M_{k \times n}(\mathbf{R}). \tag{37}$$

By analogy, when defining the satisfaction function $s : U \to \mathbf{R}_+ \cup \{0\}$ using the formula (17), the values of satisfaction s_{ij} resulting from the i-th decision $i = 1, \ldots, k$ in the j-th world state $j = 1, \ldots, n$ are inputs to the satisfaction matrix described by (18), which is expanded as follows:

$$M_s = [s(u_{ij})]_{i=1,\ldots,k,\ j=1,\ldots,n} = \begin{bmatrix} s_{11} & s_{12} & \ldots & s_{1n} \\ s_{21} & s_{22} & \ldots & s_{2n} \\ \vdots & \vdots & & \vdots \\ s_{k1} & s_{k2} & \ldots & s_{kn} \end{bmatrix} \in M_{k\times n}(\mathbf{R}). \tag{38}$$

Considering definition 2.4 of the relative utility of the i-th decision d_i, in the j-th world state s_j and (23), ***the matrix of relative utility*** is given by:

$$U_w = [u_w(x_{ij})]_{i=1,\ldots,k,\ j=1,\ldots,n} = M + \alpha \cdot M_r + (1-\alpha)\cdot M_s \in M_{k\times n}(\mathbf{R}), \tag{39}$$

where $0 \leq \alpha \leq 1$. In (39), the matrix M accounts for the rational part of the decision, whereas the convex combination of the matrices M_r and M_s, i.e., $\alpha \cdot M_r + (1-\alpha)\cdot M_s$, accounts for the behavioral portion. On developing (39), the matrix of relative utility becomes:

$$\begin{aligned} U_w &= \begin{bmatrix} u_w(x_{11}) & u_w(x_{12}) & \ldots & u_w(x_{1n}) \\ u_w(x_{21}) & u_w(x_{22}) & \ldots & u_w(x_{2n}) \\ \vdots & \vdots & & \vdots \\ u_w(x_{k1}) & u_w(x_{k2}) & \ldots & u_w(x_{kn}) \end{bmatrix} = \\ &= \begin{bmatrix} u_{11} & u_{12} & \ldots & u_{1n} \\ u_{21} & u_{22} & \ldots & u_{2n} \\ \vdots & \vdots & & \vdots \\ u_{k1} & u_{k2} & \ldots & u_{kn} \end{bmatrix} + \alpha \cdot \begin{bmatrix} r_{11} & r_{12} & \ldots & r_{1n} \\ r_{21} & r_{22} & \ldots & r_{2n} \\ \vdots & \vdots & & \vdots \\ r_{k1} & r_{k2} & \ldots & r_{kn} \end{bmatrix} + \\ &\quad +(1-\alpha)\cdot \begin{bmatrix} s_{11} & s_{12} & \ldots & s_{1n} \\ s_{21} & s_{22} & \ldots & s_{2n} \\ \vdots & \vdots & & \vdots \\ s_{k1} & s_{k2} & \ldots & s_{kn} \end{bmatrix}. \end{aligned} \tag{40}$$

Based further on definition 2.6 of the expected relative utility of the i-th decision d_i in the j-th world state s_j, and using (29), ***the vector of the expected relative utilities of decisions*** $d_1,\ldots,d_k$ is defined:

$$E_w = [Eu_w(d_i)]_{i=1,\ldots,k} = U_w \cdot p = (M + \alpha \cdot M_r + (1-\alpha)\cdot M_s)\cdot p \in \mathbf{R}^k, \tag{41}$$

or in its expanded form:

$$E_w = \begin{bmatrix} Eu_w(d_1) \\ Eu_w(d_2) \\ \vdots \\ Eu_w(d_k) \end{bmatrix}, \tag{42}$$

where $p \in \mathbf{R}^n$ is the vector of the probabilities $p_1,\dots,p_n$ of the world states $s_1,\dots,s_n$:

$$p = [p_j]_{j=1,\dots,n} = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix}. \tag{43}$$

The optimum decision d_0 is signaled by the maximum coordinate of the vector of the expected relative utilities, i.e.:

$$E_0 = \max_{i=1,\dots,k} Eu_w(d_i). \tag{44}$$

Example 3 *With reference to examples 1 and 2, the optimum decision will be determined for the decision-maker considering the choice of a decision from the set $M = \{d_1,d_2,d_3,d_4\}$ in the external world states from $S = \{s_1,s_2,s_3\}$, whose utilities from making the particular decision in the successive world states are given in the matrix of utilities (Table 5).*

The matrix of relative utilities U_w derived from (39), where the matrices of regret M_r and satisfaction M_s are (15) and (22), respectively, is:

$$U_w = M + \alpha \cdot M_r + (1-\alpha)\cdot M_s =$$

$$= \begin{bmatrix} 8 & 14 & 20 \\ 10 & 10 & 10 \\ 15 & 0 & -40 \\ -10 & 80 & 100 \end{bmatrix} + \alpha \cdot \begin{bmatrix} -7 & -66 & -80 \\ -5 & -70 & -90 \\ 0 & -80 & -140 \\ -25 & 0 & 0 \end{bmatrix} + (1-\alpha)\cdot \begin{bmatrix} 18 & 14 & 60 \\ 20 & 10 & 50 \\ 25 & 0 & 0 \\ 0 & 80 & 140 \end{bmatrix},$$

where $0 \le \alpha \le 1$. Depending on the value of α, various matrices of relative utilities may be generated. Taking, for instance, $\alpha = 0.66$, which means that according to the classification in Remark 3, the feeling of regret prevails over satisfaction (2:1), the following matrix of relative utilities results:

$$U_w = \begin{bmatrix} 9.5 & -24.8 & -12.4 \\ 13.5 & -32.8 & -32.4 \\ 23.5 & -52.8 & -132.4 \\ -26.5 & 107.2 & 147.6 \end{bmatrix}.$$

Further, relying on (41), the vector of expected subjective relative utilities is computed:

$$E_w = U_w \cdot p = \begin{bmatrix} 9.5 & -24.8 & -12.4 \\ 13.5 & -32.8 & -32.4 \\ 23.5 & -52.8 & -132.4 \\ -26.5 & 107.2 & 147.6 \end{bmatrix} \cdot \begin{bmatrix} 0.8 \\ 0.15 \\ 0.05 \end{bmatrix} = \begin{bmatrix} 3.26 \\ 4.26 \\ 4.26 \\ 2.26 \end{bmatrix}.$$

There are two coordinates of the maximum value 4.26, corresponding to decisions d_2 and d_3:

$$Eu_w(d_2) = Eu_w(d_3) = 4.26.$$

Thus, using the formula (34) from Remark 6, we calculate the standard deviations from the relative expected values for decisions d_2 and d_3:

$$su_w(d_2) = \sqrt{\sum_{j=1}^{3}(u_w(x_{2j}) - 4.26)^2 p_j} = \sqrt{341.5164} = 18.48,$$
$$su_w(d_3) = \sqrt{\sum_{j=1}^{3}(u_w(x_{3j}) - 4.26)^2 p_j} = \sqrt{1718.316} = 41.45.$$

Comparing the standard deviations from the relative expected values of 4.26, it can be concluded that, from the perspective of a risk-averse decision-maker, decision d_2 is optimum because the standard deviation is lower. However, for a risk-taking decision-maker, decision d_3 appears to be optimal; due to the higher standard deviation, a higher payout is possible.

4. Conclusions

The BMR behavioral model of decision-making under risk demonstrates the method of selecting an optimum decision for the decision-maker wishing to choose the single best decision from a set of multiple acceptable decisions in a simple and clear manner. Introducing the concept of relative utility allows for considering not only rational factors, such as the utility of benefits, but also behavioral factors that address two opposing emotions important to decision-making: regret at a choice that is worse than the best possible option and satisfaction with a choice that is better than the worst in a given world state. This method goes beyond the framework of the rational choice theory. It does not reject rationality, although the relative utility introduced here is the total sum of the rational and behavioral parts (25). It emphasizes the fact that human choices are a complex interplay of rational and emotional elements. The study draws inspiration from the prospect theory (the utility function, which treats profits and losses differently, and the weighting function that transforms probabilities). Considering two strong emotions: regret, expressed by the values of the regret function, and satisfaction, expressed by the values of the satisfaction function, is an important addition to the prospect theory. These results are a starting-point for further development of the rational-behavioral theory of decision-making.

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