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On a new goodness-of-fit test for multivariate normality with fixed parameters based on the David-Hellwig test idea

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Abstract

The article presents a proposal for a goodness-of-fit test for multivariate normality. The idea of the test is based on the empty cells test, which is well known in the literature. In the empty cells test, the area of the random variable's variability is divided into m disjoint cells. Assuming the truth of hypothesis H_0, which proclaims the multivariate normality of the distribution with given parameters, disjoint cells are arranged in such a way that random values with equal probabilities are in each cell. Based on the n-element sample, the number of empty cells, i.e. the cells without any elements from the sample, is determined. Crucial to the proposed procedure is the division of the multidimensional area of variation into disjoint cells. The advantage of this test is that it can be used for relatively small samples. In the article, a simulation comparison of the proposed test's properties and the Kolmogorov-Smirnov test's multivariate version is carried out.

Key words: multivariate normality, inferential statistics, empty cells test, Monte Carlo study.

1. Introduction

Evaluating normality is essential for various statistical methods that rely on the assumption of data being normally distributed. Normality assessments aim to ascertain whether a data assemblage deviates markedly from the Gaussian distribution (Hernandez, 2021). A variety of normality tests are employed in practice, with the most prevalently utilized being the Shapiro-Wilk test (Shapiro and Wilk, 1965), the Kolmogorov-Smirnov test (Kołmogorov, 1933; Smirnov, 1939), the Lilliefors test (Lilliefors, 1967), the Anderson-Darling test (Anderson and Darling, 1952), and the Jarque-Bera test (Jarque and Bera, 1980). The normality tests mentioned above represent merely a small component of such methodologies. Hernandez (2021) delineates a comparative analysis of 55 normality assessments. This plethora of normality assessments arises from the fact that deviations from normality can exhibit a wide variety.

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The evaluation of multivariate normality is considerably more complex. Multivariate normality constitutes a vital presumption in numerous multivariate statistical analyses, ensuring the integrity of methodologies such as MANOVA and PCA. Many statistical tests of multivariate normality have been documented in the literature. Traditional assessments for multivariate normality, such as those formulated by Mardia (1975), concentrate on evaluating skewness and kurtosis within multivariate datasets. These assessments have been foundational in discerning deviations from normality. Mardia (1975) elaborates on various methodologies for assessing multivariate normality, underscoring significant advancements such as techniques based on Mahalanobis angles and distances. Malkovich and Afifi (1973) discuss the generalization of univariate skewness and kurtosis statistics, in conjunction with the W statistic by Shapiro and Wilk, to evaluate multivariate normality using Roy's union-intersection principle. Kankainen, Taskinen, and Oja (2007) introduced assessments for multivariate normality that extend classical univariate measures to a multivariate context. These assessments are based on the Mahalanobis distance between multivariate location vector estimates and the distance between scatter matrix estimates. The authors developed an asymptotic theory to provide approximate null distributions and assess the asymptotic efficiencies of these assessments. These evaluations are particularly advantageous in practical applications where data may not strictly conform to normality assumptions, offering a reliable alternative to classical methodologies.

Liang and Yang (2022) proposed a novel assessment that amalgamates necessary-only characterization and statistical representative points. They also present an illustrative example, accentuating the assessment's supplementary function alongside existing assessments in the literature. Mudholkar, McDermo, and Srivastava (2024) presented a multivariate adaptation of the Lin and Mudholkar (1980) z-assessment for evaluating univariate normality, offering an accessible methodology. The research contributes to the understanding of multivariate normality evaluation and provides empirical refinements for the Zp assessment. Székely and Rizzo (2013) examined the theory and applications of energy statistics, showing their efficacy in inference and multivariate analysis. They discuss energy distance, a statistical distance that characterizes the equivalence of distributions of random vectors, drawing an analogy to Newton's gravitational potential energy. Mardia (2024) discussed the statistical properties of multivariate distributions, mainly focusing on multivariate skewness and kurtosis measures. It presented alternative forms of these measures and their applications in testing multivariate normality. The author derived exact moments and new approximations for the distributions of certain bilinear forms under multivariate normality. Additionally, the paper examines the impact of non-normality on the size of normality theory tests for covariance matrices.

1.1. Types of departures from the multivariate normality

Deviations from multivariate normality can take various forms. There are some typical departures from multivariate normality. Below, we present several typical examples of departures from multivariate normality distribution (Domański, 2009; Domański, 2011; Joenssen and Vogel, 2014; Ebner and Henze, 2020; Mardia, 2024):

- Uneven distribution of data in multidimensional space
  When data comes from a mixture of normal distributions, clusters with distinct locations and scales may occur. For example, a mixture of two normal distributions may generate bimodal data, i.e. with two distinct maxima. Quantile-quantile plots (*qq-plot*), probability plots, perspective plots, and contour plots can reveal such uneven distributions.
- Deviations from linearity on diagnostic plots
  On quantile-quantile and probability plots, deviations from linearity suggest departures from normality. For instance, a curved shape on a *qq-plot* may show skewness or kurtosis in the distribution.
- Non-zero values of multivariate measures of skewness and kurtosis
  Multivariate measures of skewness $b_{1\mathrm{p}}$ and kurtosis $b_{2\mathrm{p}}$ measure deviations from symmetry and the shape of the normal distribution. Non-zero values of these measures suggest departures from multivariate normality.
- Presence of outliers
  Outliers can significantly affect the results of multivariate normality tests. For example, on *qq-plot*, outliers often manifest as points far from the regression line. Removing outliers may lead to changes in test results and better fit to the normal distribution.
- Elliptical distributions
  Elliptical distributions form a broader class of distributions to which the normal distribution belongs. Multivariate normality tests may not be sensitive to deviations from normality within this class.
- Non-normal marginal distributions
  Therefore, it is essential to use multivariate normality tests, not just assessing the normality of particular variables.

Outliers can significantly affect the results of multivariate normality tests, and their removal can lead to a better fit for the normal distribution. It should be noted that even if the multivariate distribution is not normal, individual marginal variables may show a normal distribution.

There are many ways in which data can deviate from multivariate normality, including the uneven distribution of data in multivariate space, deviations from linearity in diagnostic charts, outliers, and elliptical distributions.

1.2. Types of tests for multivariate normality

Numerous methodologies for testing multivariate normality represent advancements in univariate normality testing. The majority of existing multivariate probability tests can be classified into four distinct categories (Domański and Pruska, 2000; Cramer and Howitt, 2004; Domański, 2009):

1. Procedures predicated upon graphical representations and correlation metrics,
2. Goodness-of-fit assessment methodologies,
3. Tests formulated based on skewness and kurtosis statistics,
4. Consistent methodologies derived from the empirical characteristic function.

The first of these categories involves visual data analysis using graphs, such as quantile-quantile charts (Royston, 1982). Perspective and contour charts can also be used for two-dimensional data (Korkmaz, Goksuluk, and Zararsiz, 2014). These charts allow one to assess whether the data follow the expected normal distribution (Koziol, 1993; Liang and Ng, 2009). For example, deviations from linearity in a qq-plot suggest deviations from normality.

Generalized goodness-of-fit tests make it possible to compare the empirical distribution of data with a theoretical normal distribution. Examples of such tests include the extension of the Anderson-Darling test proposed by Hawkins (Domański, 2011), the Kolmogorov-Smirnov test (Kesemen et al., 2021), the Cramér-von Mises test (Hernandez, 2021), and the chi-square test.

Tests based on measures of skewness and flattening use multivariate generalizations of skewness and flattening statistics to assess normality. Tests based on the measures mentioned above include Mardia (Mardia, 1975; Domański, 2009), the Doornik and Hansen omnibus test (Doornik and Hansen, 2008), and tests based on a multivariate version of the Shapiro-Wilk test (Domański, Gadecki, and Wagner, 1989).

Procedures based on empirical characteristic functions use an empirical characteristic function to compare the sample distribution with a normal distribution. An example of such a test is the Henze-Zirkler test (Szekely and Rizzo, 2013).

Many multivariate normality tests are extensions of one-dimensional normality tests. The appropriate test choice depends on the specifics of the research problem, sample size, and expected deviations from normality (Joenssen and Vogel, 2014).

1.3. Testing multivariate normality

Multivariate normality tests allow us to assess whether a multivariate dataset follows a multivariate normal distribution. The multivariate normal distribution is an essential assumption in many statistical modeling and inference methods. Many multivariate normality tests are available, each with its advantages and disadvantages.

Researchers have extensively explored various statistical approaches to evaluate the multivariate normality assumption due to its critical role in ensuring the validity of statistical inferences. The literature on multivariate normality tests encompasses a range of methodologies developed to assess the assumption of normality in multivariate distributions. Traditional tests such as Mardia's test (Mardia, 1974) focus on multivariate skewness and kurtosis, providing a measure that helps identify deviations from normality based on these higher moments. Henze-Zirkler's test (Henze and Zirkler, 1990) employs statistics based on the Mahalanobis distance, offering robustness against sample size variations and good power against alternatives. The Royston test (Royston, 1983) for multivariate normality extends the Shapiro-Wilk test for univariate normality. It involves transforming the Shapiro-Wilk statistic into one that Royston claims to be approximately chi-squared distributed with equivalent degrees of freedom, calculated based on the cumulative distribution function for the standard normal distribution. The test statistic combines the Shapiro-Wilk statistics for the separate variables.

The energy multivariate normality test (Székely and Rizzo, 2013, 2017; Móri, Székely, and Rizzo, 2021) is a distance-based test that assesses multivariate normality by comparing the energy of the observed data with the energy of data sampled from a multivariate normal distribution. The energy of a dataset is a measure of dispersion or spread, calculated based on the distances between observations in the dataset. This test computes a test statistic based on the difference in energy between the observed data and simulated multivariate normal data. Departures from multivariate normality are detected if the observed energy significantly differs from the expected energy under the null hypothesis.

Recent approaches include copulas and bootstrap methods, allowing for more flexible testing in complex data structures. These modern techniques address the limitations of classical tests, such as their sensitivity to outliers and dependency on large sample sizes. The literature also discusses the practical applications of these tests in fields ranging from finance and economics to the biological sciences, where the correct identification of data normality significantly impacts the conclusions of empirical research. Overall, the evolution of multivariate normality tests reflects a broader trend toward more computationally intensive and more accurate statistical methodologies in the face of increasingly complex data.

Commonly employed methods include multivariate extensions of univariate normality, distance-based, and transformation-based tests. Multivariate normality tests typically examine the distributional properties of multivariate data while considering the interrelationships among variables. Among these, tests such as Mardia's, Henze-Zirkler, and Royston's tests are widely used, leveraging sample moments and distributional properties to assess departures from multivariate normality. On the other hand, distance-based tests rely on measures of distances or dissimilarities between observations to evaluate deviations from multivariate normality. Examples include the Mahalanobis distance criterion and the correlation ratio criterion, which assess the adequacy of the observed data distribution compared to the multivariate normal distribution.

Furthermore, transformation-based tests involve transforming multivariate data to conform more closely to a multivariate normal distribution and assessing the goodness of fit of the transformed data. Despite the availability of these diverse methodologies, challenges persist in their application, including the sensitivity to sample size, dimensionality, and underlying distributional characteristics. Additionally, the choice of a proper test depends on the specific characteristics of the dataset and the research context. As such, further research is warranted to develop robust and versatile methodologies for assessing multivariate normality across diverse research domains.

In statistical research, the inference of unknown population parameters is often carried out based on random samples. In the case of characteristics measured on strong scales (interval and ratio), samples of relatively small sizes are sufficient to carry out the inference effectively.

The multivariate goodness-of-fit multivariate normality test is proposed in the paper. The hypothesis regarding the distribution of the multivariate random variable will be examined. The idea of this proposal is based on the empty cells test.

2. The proposal of a goodness-of-fit multivariate normality test based on the idea of David-Hellwig empty cells

The empty cells test (David, 1950; Hellwig, 1965; Domański, 2012) is one of the goodness-of-fit tests. The hypothesis regarding the form of the distribution of random variables can be assessed with this test. A random variable's entire area of variability is divided into m cells, and the number of elements from the random sample in each cell is counted. Then, the number of empty cells is counted. This number of empty cells is compared to the critical value. Domański and Pruska (2000) presented the interpolated critical values for the empty cells test statistic.

Let us assume that an n-elements random sample was taken from the population. We will test the hypothesis about the form of the distribution

$$H_0\colon F(x) = F_0(x)$$

against the alternative

$$H_1\colon F(x) \neq F_0(x)$$

where F_0 is the specified distribution.

First, we divide the random variable's variability area $\mathbb{X}$ into m disconnected cells Mi. It could be written as (Kończak, 2005, 2008):

$$\mathbb{X} = \bigcup_{i=1}^{m} M_i \tag{1}$$

where $\mathbb{X}$ is the area of variability $\mathbb{X}$ of the random variable X, and

$$M_i \cap M_j = \emptyset, for\ , i, j\ \in \{1, 2, \dots, m\}, i \neq j \text{ for } i = 1, 2, \dots, m, \text{ and}$$

$$P(x \in M_i) = \frac{1}{m}, \tag{2}$$

for $i = 1, 2, \dots, m$, if the hypothesis H_0 is true.

The test statistic in the empty cells test has the following form:

$$K_n = card\{j: m_j = 0\}$$

where m_j means the number of elements from a random sample in j-th cell.

The critical area of the empty target test is right-sided.

Under the assumption of the truth of the hypothesis H_0, for a division into m targets, the probability of k empty cells, when an n-element sample is taken, is expressed by the following formula (Hellwig, 1965):

$$p_k(n, m) = \binom{m}{k} \sum_{r=0}^{m-k} (-1)^r \binom{m-k}{r} \left(1 - \frac{k+r}{m}\right)^n \tag{3}$$

and the cumulative probability function has the following form:

$$P_k(n, m) = \sum_{s=0}^{k} \binom{m}{s} \sum_{r=0}^{m-s} (-1)^r \binom{m-s}{r} \left(1 - \frac{s+r}{m}\right)^n \tag{4}$$

For the assumed significance level α the rejection region could be denoted as:

$$K = \{k: k \geq K_{n,\alpha}\}$$

where $K_{n,a}$ are taken from tables (e.g. David (1950), Hellwig (1965), Domański and Pruska (2000).

Domański (2011) points out the superiority of the power of the empty cells test over the Shapiro-Wilk test in testing unidimensional normality for sample sizes of n≤20.

The one-dimensional and multivariate normality tests presented above overwhelmingly do not consider the case of known parameters. Among the few exceptions is the Kolmogorov-Smirnov test, which checks for conformity to a distribution, particularly a normal distribution, with fixed parameters. There are few goodness-of-fit tests for multivariate normality, and those that exist are highly constrained. McAssey (2013) presented a multivariate goodness-of-fit test based on Mahalanobis distances. Some tests are extremely complex to implement, so much so that they can handle only three or four p-variates (Romeu and Ozturk, 1993). Hanusz, Tarasińska, and Zieliński (2012) modified the Shapiro-Wilk test for the case of normality with a known mean.

Let $\boldsymbol{X} = (X_1, X_2, \cdots, X_k)$ be the k-dimensional continuous vector random variable with the cumulative distribution function $F(\boldsymbol{x})$. Let $x_1, x_2, \cdots, x_n$ where $\boldsymbol{x_i} = (x_{i1}, x_{i2}, \cdots, x_{ik})$ for $i = 1, 2, \cdots, n$, be an n-element sample.

Let us consider the simple null hypothesis of multivariate normality denoted by the form:

$$H_0: F_p(\boldsymbol{x}) = F_{0p}(\boldsymbol{x})$$

where $F_{0p}(\boldsymbol{x}) \sim N_p(\boldsymbol{x};\ \boldsymbol{\mu_0}, \boldsymbol{\Sigma_0})$, $\boldsymbol{\mu_0}$ and $\boldsymbol{\Sigma_0}$ are given vector of means and covariance matrix,

against the alternative hypothesis

$$H_1: \sim H_0$$

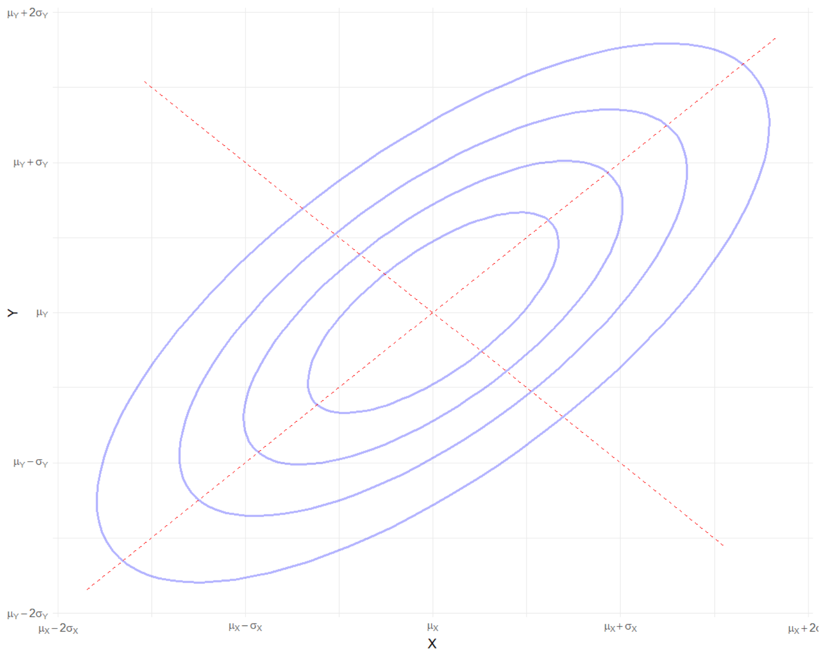

Figure 1. The idea of constructing cells for goodness-of-fit test for bivariate normality (m = 20)

For the application of the empty cells test for multivariate normality, it is necessary to identify disjoint cells that satisfy conditions (1) and (2). The idea of cells construction is based on using confidence ellipsoids and then dividing the created areas into 2^p targets, where p is the dimension of the space. A visualization of how the area of variation is divided into cells is shown only for a two-dimensional distribution (p = 2). Figure 1 shows the idea of target construction for the case of a two-dimensional normal distribution with a vector of expected values $\boldsymbol{\mu_0} = (\mu_X, \mu_Y)$ and a covariance matrix $\boldsymbol{\Sigma_0} = \begin{bmatrix} \sigma_X^2 & \sigma_X\sigma_Y \\ \sigma_X\sigma_Y & \sigma_Y^2 \end{bmatrix}$. The division of the area of variation into m = 20 cells is shown. For the indicated cell design, the probabilities of observations occurring in all cells under H_0 are the same.

The p-dimensional ellipsoids are determined by:

$$(\boldsymbol{x} - \boldsymbol{\mu_0})^T \boldsymbol{\Sigma}_0^{-1} (\boldsymbol{x} - \boldsymbol{\mu_0}) \leq \chi^2_{p,1-\alpha} \tag{5}$$

where

$\boldsymbol{x} \in \mathbb{R}^p$,

$\boldsymbol{\mu_0}$ - given vector of expected values,

$\boldsymbol{\Sigma}_0$ - given covariance matrix (dimension $p \times p$),

$\chi^2_{p,1-\alpha}$ - quantile of the chi-squared distribution with p degrees of freedom at confidence level $1 - \alpha$.

The division of the elliptic strip into 2^p cells is determined by the eigenvectors of the $\boldsymbol{\Sigma}_0$ matrix.

Figure 2 shows the partitioning into $m = 20$ cells for a two-dimensional normal distribution with an expectation vector $\boldsymbol{\mu_0} = (0,0)$ and a covariance matrix $\boldsymbol{\Sigma_0} = \begin{bmatrix} 1 & 0.7 \\ 0.7 & 1 \end{bmatrix}$. The figure shows $n = 20$ element random samples from a two-dimensional normal distribution. In the general case, the number of cells m may differ from the sample size n. If $m \neq n$, the critical values can be determined based on (3) and (4). In the case of $m = n$, critical values can be obtained from Domanski and Pruska (2000). The theoretical contour plot represent the bivariate normal distribution with parameters $\boldsymbol{\mu_0}$ and $\boldsymbol{\Sigma_0}$. The empirical contour plots in Figure 2 show estimates from an $n = 20$ element sample of the density of the distribution. Two variants of expected values of X and Y variables were considered:

a) $\boldsymbol{\mu} = (1, 0)$

b) $\boldsymbol{\mu} = (2, 2)$

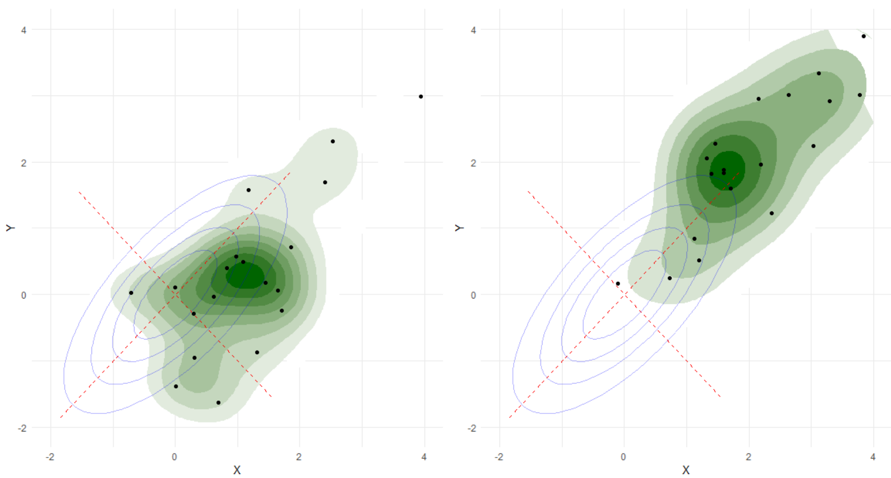

Figure 2. Sample of $n = 20$ elements with empirical and theoretical, under H_0, contour plot

For the sample size n = 20, the number of cells m = 20, and the significance level α=0.05, the interpolated critical value in the empty cells test equals 9.96 (Domański and Pruska, 2000). This means that hypothesis H_0 is rejected if the number of empty cells is greater than or equal to 10. In both cases shown in Figure 2, the number of empty cells is k = 11 (left) and k = 15 (right). In both cases presented in Figure 2, the hypothesis H_0 should be rejected for the assumed significance level $\alpha = 0.05$.

3. The properties of the test - Monte Carlo study

The simulation analyses carried out evaluated the probability of rejecting the H_0 hypothesis. The size and power of this test were analyzed. Simulation analyses were conducted for a two-dimensional normal distribution. Two variants were considered:

a) $N_2(x;\ \mu_0, \Sigma_0)$, where $\mu_0 = (0{,}0), \Sigma_0 = \begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 1.0 \end{bmatrix}$

b) $N_2(x;\ \mu_0, \Sigma_0)$, where $\mu_0 = (0{,}0), \Sigma_0 = \begin{bmatrix} 1.0 & 0.7 \\ 0.7 & 1.0 \end{bmatrix}$

Random values were generated for variant (a) or (b) assuming changes in expected values

$$\delta_X = 0.0, 0.1, \dots, 1.5;\ \delta_Y = 0.0, 0.1, \dots, 1.5.$$

If $\delta_X = \delta_Y = 0.0$, then the hypothesis H_0 was true, otherwise H_0 was false. The estimated probabilities of rejecting H_0 were obtained by a Monte Carlo study. The sample sizes considered were n = 12, 20, and 28. In all considered cases, the number of cells was set to $m = n$, and the number of simulations was established to N = 1,000. The largest possible standard error when considering rejection proportions of multivariate normality with 1,000 simulations is $\sigma_{\hat{p}} = \sqrt{\frac{\hat{p}(1-\hat{p})}{N}} = \sqrt{\frac{0.5(1-0.5)}{1000}} \approx 0.0158$.

In generating random samples from a two-dimensional normal distribution, the *rmvnorm* function from the **mvtnorm** package in *R* was used (mvtnorm 2025). The function allows to generate random values from a multivariate normal distribution with a given vector of expected values $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}_0$.

The probabilities of rejecting H_0 were estimated for the multivariate Kolmogorov-Smirnov test (KS) and the described proposal multivariate normality test (mvnDH). Figure 3 shows the power function of the mvnDH test of concordance and the KS test for a two-dimensional normal distribution for independent variables (variant a). The presentation includes sample sizes of n = 12, 20, and 28. It is noticeable that the power of both tests increases as the sample size increases. In all cases, the KS test features a slightly higher power. Figure 4 shows the power function of the mvnDH concordance test and the KS test for a two-dimensional normal distribution for the dependent variables (variant b). The presentation includes sample sizes of n = 12, 20, and 28. It is noticeable that the power of both tests increases as the sample size increases. For simultaneous increases in the expected values of both variables (X and Y), the probability of rejecting the H_0 hypothesis for the KS test is higher than for the mvnDH test. However, when increasing the expected value of only one variable (X or Y), the probability of rejecting the H_0 hypothesis for the mvnDH test is greater than for the KS test. Depending on the type of deviation from H_0, the effectiveness of the analyzed tests is not the same.

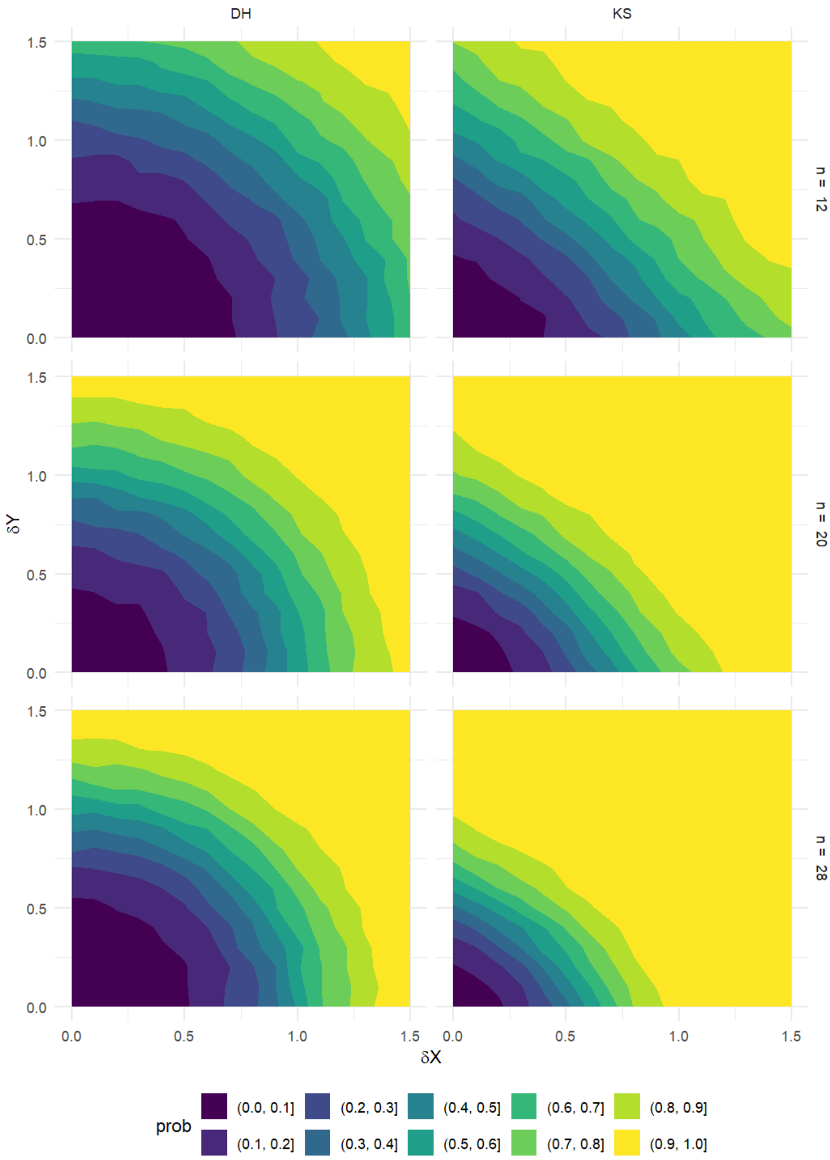

Figure 3. Estimated probabilities of rejecting H_0 for the case a) - independent variables

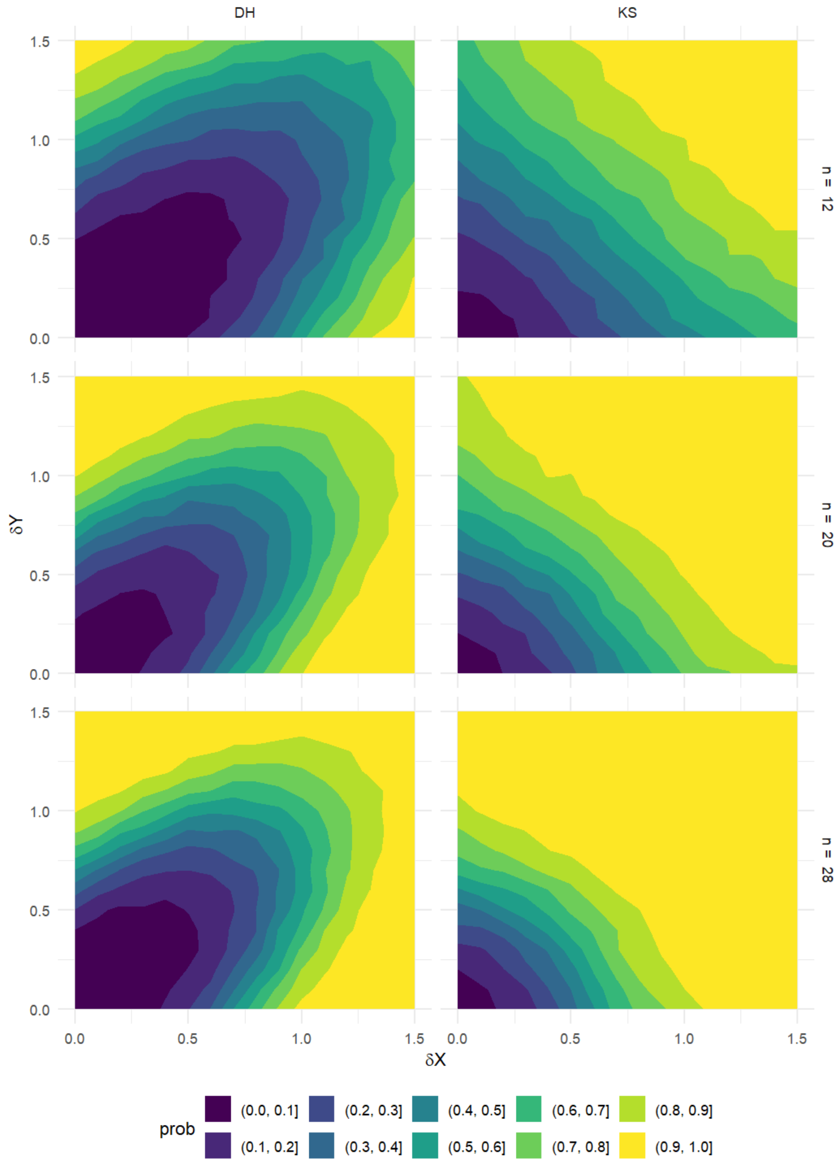

Figure 4. Estimated probabilities of rejecting H_0 for the case b) - dependent variables

4. Conclusions

In reality, many economic, social, or medical issues are complex and involve multiple dimensions. This article addresses the issue of statistical inference for multivariate normality tests. A proposal for using the empty cells test to assess multivariate normality is presented. The idea of this test is based on the concept of empty cells. In conducting an empty cells test, it is crucial to partition the area of variation into disjoint cells, which is particularly significant in multivariate analysis. In the proposed testing procedure, cells are defined by confidence ellipsoids and are further segmented along the eigenvectors of the variance-covariance matrix. This division of the area of variation ensures equal probabilities of observations in the cells under H_0. The proposed test validates the conformity of the empirical distribution to a preset, precisely specified, multivariate normal distribution. Accordingly, it reacts not only to changes in the parameters of the distribution, but also to deviations from normality, in particular to the occurrence of a distribution type other than normal.

In simultaneous increases in the expected values of both variables (X and Y), the probabilities of rejecting the H_0 hypothesis for the KS test are greater than those for the mvnDH test. However, increasing the expected value of either variable (X or Y) enhances the likelihood of rejecting the H_0 hypothesis in the mvnDH analysis compared to the KS test. The performance of the analyzed tests is inconsistent, contingent upon the form of deviation from H_0. In some cases of deviation from multivariate normality, the mvnDH test has greater power than the KS test. Due to the discrete nature of the test statistic (number of empty cells), it is recommended that the researcher specify a larger number of cells.

The advantage of the proposed mvnDH test is its intuitive idea and the possibility of relatively easy programming of the relevant functions.

References

Anderson, T. W., Darling, D. A., (1952). Asymptotic Theory of Certain Nonparametric Tests, *Annals of Mathematical Statistics*, t. 23, 3, pp. 193–212.

Cramer, D., Howitt, D., (2004). *The Sage dictionary of statistics: A practical resource for students in the social sciences*. Sage Publications.

David, F. N., (1950). Order Statistics. J. Wiley & Sons Inc. New York.

Domański, C., (2009). Attempt to Assess Multivariate Normality Tests. *Acta Universitatis Lodziensis. Folia Oeconomica*, 225, pp. 76–90.

Domański, C., (2011). Własności testu Davida-Hellwiga. *Prace Naukowe Uniwersytetu Ekonomicznego we Wrocławiu*, 165, pp. 40–49.

Domański, C., (2012). Statistical Tests based on Empty Cells. *Acta Universitatis Lodziensis. Folia Oeconomica*, 269, pp. 39–47.

Domański Cz., Gadecki H. and Wagner W. (1989). Wartości krytyczne uogólnionego testu normalności Shapiro-Wilka. *Przegląd Statystyczny*, 36, pp. 107–112.

Domański, Cz., Pruska, K., (2000). Nieklasyczne metody statystyczne. *Polskie Wydawnictwo Ekonomiczne*, Warszawa.

Doornik, J. A., Hansen, H., (2008). An Omnibus test for univariate and multivariate normality. *Oxford Bulletin of Economics and Statistics,* 70, pp. 927–939.

Ebner, B., Henze, N., (2020). Tests for Multivariate Normality – a Critical Review with Emphasis on Weighted L^2-Statistics. *TEST (2020)*, 29, pp. 845–892, https://doi.org/10.1007/s11749-020-00740-0

Hanusz, Z., Tarasińska, J. and Zieliński, W., (2012). Adaptation of Shapiro-Wilk Test to the Case of Known Mean. *Colloquium Biometricum*, 42, pp. 43–50.

Hellwig, Z., (1965). *Test zgodności dla małej próby*. Przegląd Statystyczny, 12, pp. 99–112.

Hernandez, H., (2021). *Testing for Normality: What is the Best Method?*

Henze, N., Zirkler, B., (1990). A Class of Invariant Consistent Tests for Multivariate Normality. *Commun. Statist.-Theor. Meth*., 19(10), pp. 35953618.

Jarque, C. M., Bera, A. K., (1980). Efficient tests for normality, homoscedasticity and serial independence of regression residuals. *Economics Letters*, 6(3), 1980, pp. 255–259.

Joenssen, D. W., Vogel, J., (2014). A power study of goodness-of-fit tests for multivariate normality implemented in R. *Journal of Statistical Computation and Simulation*, 84(5), pp. 1055–1078. https://doi.org/10.1080/00949655.2012.739620.

Kankainen, A, Taskinen, S. and Oja, H., (2007). Tests of multinormality based on location vectors and scatter matrices. *Statistical Methods & Applications,* 16, pp. 357–379.

Kesemen, O., Tiryaki̇, B. K., Tezel, Ö. and Özkul, E., (2021). A new goodness of fit test for multivariate normality. *Hacettepe Journal of Mathematics and Statistics*, 50(3), pp. 872–894.

Kołmogorov A., (1933). Sulla determinazione empirica di una legge di distribuzione. *Giornale dell'Istituto Italiano degli Attuari*, 4, pp. 83–91.

Kończak, G., (2005). On the Modification of David-Hellwig Test. [in:] *Innovation in classification, data science and information systems. Proceedings of the 27th Annual Conference of the Gesellschaft fur Klassifikation, Brandenburg University of Technology, Cottbus, March 12-14, 2003,* pp. 138–145.

Kończak, G., (2008). On the Multivariate Goodness-of-Fit test. Compstat. Proceedings in Computational Statistics. 18th Symposium held in Porto, Portugal. *Physica-Verlag*, pp. 751–758.

Korkmaz, S., Goksuluk, D. and Zararsiz, G., (2014). MVN: An R Package for Assessing Multivariate Normality. *The R Journal*, 6(2), 151. https://doi.org/10.32614/RJ-2014-031

Koziol, J. A., (1993). Probability Plots for Assessing Multivariate Normality. *The Statistician*, 42(2), 161. https://doi.org/10.2307/2348980.

Liang, J., Ng, K. W., (2009). A Multivariate Normal Plot to Detect Nonnormality. *Journal of Computational and Graphical Statistics*, 18(1), pp. 52–72. https://doi.org/10.1198/jcgs.2009.0004.

Liang, J., He, P. and Yang, J., (2022). Testing Multivariate Normality Based on t-Representative Points. *Axioms*, 11(11), p. 587. https://doi.org/10.3390/axioms11110587.

Lilliefors, H. W., (1967). On the Kolmogorov-Smirnov test for normality with mean and variance unknown. *Journal of the American Statistical Association*, 62(318), pp. 399–402.

Lin, C.-C., Mudholkar, G. S., (1980). A simple test for normality against asymmetric alternatives. *Biometrika*, 67, pp. 455–61.

Malkovich, J. F., Afifi, A. A., (1973). On Tests for Multivariate Normality. *Journal of the American Statistical Association*, *68*(31), pp. 176–179.

Mardia, K. V., (1974). Applications of some measures of multivariate skewness and kurtosis for testing normality and robustness studies. *Sankhy A*, 36, pp. 115–128.

Mardia, K. V., (1975). Assessment of Multinormality and the Robustness of Hotelling's T^2 Test. *Journal of the Royal Statistical Society. Series C (Applied Statistics)*, 24(2), pp. 163–171.

Mardia, K. V., (2024). *Applications of Some Measures of Multivariate Skewness and Kurtosis in Testing Normality and Robustness Studies.*

McAssey, M. P., (2013). An empirical goodness-of-fit test for multivariate distributions. *Journal of Applied Statistics*, 40(5), pp. 1120–1131. https://doi.org/10.1080/02664763.2013.780160.

Móri, T. F., Székely, G. J. and Rizzo, M. L., (2021). On energy tests of normality, *Journal of Statistical Planning and Inference*, Vol. 213, pp. 1–15, https://doi.org/10.1016/j.jspi.2020.11.001.

Mudholkar, G. S., McDermo, M. and Srivastava, D. K., (2024). *A Test of p-Variate Normality.*

Mvtnorm, (2025). https://cran.r-project.org/web/packages/mvtnorm/mvtnorm.pdf [25.05.2025].

Rizzo M. L., Székely G. J., (2016). Energy Distance. *WIRES Computational Statistics*, *Wiley*, Vol. 8, Issue 1, pp. 27–38.

Romeu, J. L., Özturk, A., (1993). A Comparative Study of Goodness-of-Fit Tests for Multivariate Normality. *Journal of Multivariate Analysis*, 46(2), pp. 309–334. https://doi.org/10.1006/jmva.1993.1063.

Royston, J. P., (1982). An Extension of Shapiro and Wilks W Test for Normality to Large Samples. *Applied Statistics*, 31(2), p. 115124.

Royston, J. P., (1983). Some Techniques for Assessing Multivariate Normality Based on the Shapiro-Wilk W. *Applied Statistics*, 32(2).

Shapiro, S. S., Wilk, M. B., (1965). An Analysis of Variance Test for Normality. *Biometrika*, Vol. 52, No. 3/4, pp. 591–611.

Smirnow, N., (1939). Estimate of the goodness of fit for a continuous distribution. *Bulletin de l'Académie des Sciences de l'URSS. Série Mathématique*, 7, pp. 2–14.

Székely, G. J., Rizzo, M. L., (2013). Energy statistics: A class of statistics based on distances. *Journal of Statistical Planning and Inference.*

Székely, G. J., Rizzo, M. L., (2017). The Energy of Data. *The Annual Review of Statistics and Its Application,* 4, pp. 447–79. https://doi.org/10.1146/annurev-statistics-060116-054026.