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Horvitz-Thompson estimator based on theauxiliary variable

J. Al-Jararha[1], Mazen Sulaiman[2]

ABSTRACT

In this paper, the Horvitz and Thompson (1952) estimator will be modified; so that, the modified estimators will use the availability of the auxiliary variable. Furthermore, the modified estimators are extended to be used in stratified sampling designs. Empirical studies are given for comparison purposes.

Key words: Horvitz-Thompson Estimator, Stratified Sampling Designs, Dual Calibration, GREG Type Estimator.

1. Introduction

Consider the finite population U of N units indexed by the set $\{1,2,\cdots,N\}$. For the ith unit, let y_i be the value of the interest variable Y, and x_i be the value of the auxiliary variable X. The values of X are known for all the units in the population and correlated with the study variable Y. Without loss of generality, we can assume that $x_i > 0$ for $i = 1, 2, \ldots, N$. Based on a probability sampling design $p(.)$, draw a random sample s from U. The first order inclusion probability π_i is defined by $\pi_i = \sum_{s\ni i} p(s)$, and the second inclusion probability π_{ij} is defined by $\pi_{ij} = \sum_{s\ni i,\, j} p(s)$, for $i \neq j$, and $\pi_{ij} = \pi_i$ when $i = j$. The probability sampling design $p(.)$ is assumed to be a measurable design. The population total for the auxiliary variable X is $t_x = \sum_{i\in U} x_i$.

Horvitz and Thompson (1952) proposed the following estimator

$$\begin{aligned} \hat{t}_{y\pi} &= \sum_{i\in U} \frac{y_i}{\pi_i} I_{\{i\in s\}} \\ &= \sum_{i\in s} d_i y_i \end{aligned} \qquad (1)$$

to estimate the finite population total $t_y = \sum_{i\in U} y_i$, where $d_i = 1/\pi_i$ are the sampling design weights and $I_{\{i\in s\}}$ is one if $i \in s$ and zero otherwise. The $\hat{t}_{y\pi}$ is exactly an unbiased estimator for t_y.

Remark 1.1 *The availability and the calibration on the auxiliary variables can be used to increase the precision of estimators. However, the Horvitz and Thompson (1952) estimator does not use the availability of the auxiliary variables. Therefore, the Horvitz and Thompson (1952) estimator will be modified, so that the modified estimators will use the availability of the auxiliary variable.*

[1]Department of Statistics, Yarmouk University, Irbid, Jordan. E-mail: jehad@yu.edu.jo. ORCID: https://orcid.org/0000-0001-8233-9849.
[2]Department of Statistics, Yarmouk University, Irbid, Jordan . E-mail: jomaa_mazen@yahoo.com

Deville and Särndal proposed the following estimator

$$\hat{t}_{y.ds} = \sum_{i \in U} w_i y_i I_{\{i \in s\}} = \sum_{i \in s} w_i y_i, \tag{2}$$

for estimating t_y, where w_i, $i \in s$, are the new sampling design weights that calibrated the sampling design weights d_i defined by Eq.(1) based on the calibration on the known population total for the auxiliary variable X and the chi-square distance. The calibrated weights w_i are obtained by minimizing the chi-square distance, subject to the side condition. As a result of this, the calibrated weights w_i are given by

$$w_i = d_i + \frac{t_x - \hat{t}_{x\pi}}{\sum_{i \in s} d_i q_i x_i^2} d_i q_i x_i, \tag{3}$$

Therefore, Eq. (2) is reduced to

$$\hat{t}_{y.ds} = \hat{t}_{y\pi} + \hat{\beta}_{ds} \left(t_x - \hat{t}_{x\pi} \right) \tag{4}$$

which is a GREG type estimator, where q_i's are known positive weights unrelated to d_i, $\hat{\beta}_{ds} = \frac{\sum_{i \in s} d_i q_i x_i y_i}{\sum_{i \in s} d_i q_i x_i^2}$, and $\hat{t}_{x\pi}$ is the Horvitz and Thompson (1952) estimator of t_x.

Stearns and Singh (2008) summarized the developments by several researchers on the GREG estimators and used the calibration idea to propose three new estimators of the variance of the GREG estimators.

Singh (2013) estimated t_y based on the dual calibration approach and his approach is summarized by the following.

Let

$$\hat{t}_{sin} = \sum_{i \in s} \omega_i x_i \tag{5}$$

subject to

$$\sum_{i \in s} d_i = \sum_{i \in s} \omega_i \tag{6}$$

and a new constraint α defined by

$$\alpha = \frac{1}{2} \sum_{i \in s} \frac{(\omega_i - d_i)^2}{d_i q_i} \tag{7}$$

As a result of this, the proposed estimator is

$$\hat{t}_{y.sin} = \hat{t}_{y\pi} + \hat{\beta}_{sin} \left(t_x - \hat{t}_{x\pi} \right) \tag{8}$$

to estimate the finite population total t_y; $\hat{t}_{y.sin}$ is a GREG type estimator, where

$$\hat{\beta}_{sin} = \frac{S_{xy}}{S_{xx}}, \tag{9}$$

where

$$S_{xy} = \sum_{i\in s} d_i q_i \left(y_i - \frac{\sum_{i\in s} d_i q_i y_i}{\sum_{i\in s} d_i q_i} \right) \left(x_i - \frac{\sum_{i\in s} d_i q_i x_i}{\sum_{i\in s} d_i q_i} \right) \tag{10}$$

and

$$S_{xx} \quad = \quad \sum_{i\in s} d_i q_i \left(x_i - \sum_{i\in s} d_i q_i x_i / \sum_{i\in s} d_i q_i \right)^2 \tag{11}$$

Two concerns about Eq.(8) are raised by Singh (2013), Remark 1 and Remark 2. Al-Yaseen (2014) showed that the estimator given by Eq.(8) can be obtained theoretically, which clarifies the first concern mentioned in Remark 1. Al-Jararha (2015) made an attempt to suggest a way to use the dual calibration of the design weights in the case of multi-auxiliary variables; in other words, an attempt to give an answer to the second concern in Remark 2.

Sugden and Smith (2002) defined the term strictly linear estimator and proposed two exactly unbiased estimators for the general linear estimates. The possibility of construction an exactly unbiased estimator from a general linear estimator, the constructed unbiased estimator is called a strictly linear estimate. Consider the general linear estimates of t_y, defined by Godambe (1955), to be of the form

$$\hat{t}_y = \sum_{i\in s} b_{si} y_i. \tag{12}$$

The exactly unbiased estimators, based on the Sugden and Smith (2002) approach, from $\hat{t}_y$ are defined by

$$\hat{t}_{y(1)} = \hat{t}_y - \sum_{i\in s} (B_i - 1) y_i / \pi_i \tag{13}$$

and

$$\hat{t}_{y(2)} = \sum_{i\in s} b_{si} y_i / B_i \tag{14}$$

for estimating the finite population total t_y, where

$$B_i = \sum_{s\ni i} p(s) b_{si}. \tag{15}$$

Recently, different authors have adopted the calibration technique to modify the original weights in stratified sampling designs. In the case of stratified sampling designs, Nidhi, Sisodia, Singh and Singh (2017) proposed a class of calibration estimators for estimating the population mean. Based on the availability of two auxiliary variables in the study and in the case of stratified sampling designs, Ozgul (2018) proposed a calibration estimator for estimating the population mean.

The Horvitz and Thompson (1952) estimator is well known in survey sampling for estimating the finite population total t_y. However, this estimator does not use the availability of the auxiliary variable. In order to improve the precision of this estimator, an attempt to generalize this estimator will be given, so that the modified Horvitz and Thompson (1952) estimators will use the availability of the auxiliary variable. Furthermore, our approach can be applied in the case of stratified sampling designs.

2. Proposed Approach

Based on the dual calibration approach, the estimator

$$\hat{t}_{y.new} = \sum_{i \in S} \omega_i y_i \tag{16}$$

is proposed to estimate the finite population total t_y, by modifying the constraint α of the Singh (2013) approach. In other words, redefine α as

$$\alpha = \frac{1}{2} \sum_{i \in s} \frac{(\omega_i - d_i)^2}{d_i q_i} + \frac{1}{2} \phi^2 \sum_{i \in s} \frac{\omega_i^2}{d_i q_i}, \tag{17}$$

where ϕ is a positive quantity.

The problem now is to minimize

$$\hat{t}_x = \sum_{i \in s} \omega_i x_i \tag{18}$$

with respect to ω_i subject to

$$\sum_{i \in s} \omega_i = \sum_{i \in s} d_i \tag{19}$$

and a new constraint α defined by Eq.(17).

The Lagrange function is defined by

$$l = \sum_{i \in s} \omega_i x_i - \lambda_1 \left(\sum_{i \in s} \omega_i - \sum_{i \in s} d_i \right) - \lambda_2 \left(\frac{1}{2} \sum_{i \in s} \frac{(\omega_i - d_i)^2}{d_i q_i} + \frac{1}{2} \phi^2 \sum_{i \in s} \frac{\omega_i^2}{d_i q_i} - \alpha \right) \tag{20}$$

where λ_1 and λ_2 are the Lagrange multipliers.

Differentiating the right hand side of Eq.(20) with respect to ω_i, equate to zero, and solving for ω_i, we have

$$\omega_i = \frac{1}{1+\phi^2}\left(d_i + \frac{d_i q_i}{\lambda_2}\left(x_i - \lambda_1\right)\right) \tag{21}$$

Summing both sides of Eq.(21) over all possible sampled values and using Eq.(19), we have

$$\lambda_1 = \frac{1}{\sum_{i\in s} d_i q_i}\left(\sum_{i\in s} d_i q_i x_i - \phi^2 \lambda_2 \sum_{i\in s} di\right) \tag{22}$$

Now, substituting Eq.(21) into Eq.(17), we have

$$2\alpha\left(1+\phi^2\right)\lambda_2^2 = \phi^2\lambda_2^2 \sum_{i\in s}\frac{d_i}{q_i} + \sum_{i\in s} d_i q_i x_i^2 - 2\lambda_1 \sum_{i\in s} d_i q_i x_i + \lambda_1^2 \sum_{i\in s} d_i q_i \tag{23}$$

Substituting Eq.(22) into Eq.(23), we have

$$\lambda_2 = \pm\frac{1}{c}\sqrt{\sum_{i\in s} d_i q_i \left(x_i - \sum_{i\in s} d_i q_i x_i / \sum_{i\in s} d_i q_i\right)^2} \tag{24}$$

where

$$c = \sqrt{2\alpha\left(1+\phi^2\right) - \phi^2 \sum_{i\in s} d_i/q_i - \phi^4\left(\sum_{i\in s} d_i\right)^2 / \sum_{i\in s} d_i q_i}. \tag{25}$$

Ignore the negative sign, where the sign is to be determined by the choice of the sign of c. Substituting Eq.(24) into Eq.(22) and using the result in Eq.(21), multiplying ω_i by y_i and summing over $i \in s$ we have

$$\hat{t}_{y.new} = \frac{1}{1+\phi^2}\left(\sum_{i\in s} d_i y_i + \phi^2\left(\sum_{i\in s} d_i / \sum_{i\in s} d_i q_i\right)\sum_{i\in s} d_i q_i y_i + \delta\, c\right) \tag{26}$$

where

$$\delta \quad = \quad S_{xy}/\sqrt{S_{xx}}, \tag{27}$$

where c, S_{xy}, and S_{xy} are given by Eq.(25), Eq.(10), and Eq.(11) respectively. With the same reasons adopted by Singh (2013), the best choice of c is

$$c = \frac{t_x - \hat{t}_{x\pi}}{\sqrt{S_{xx}}} \sim N(0,\, 1);$$

therefore,

$$\hat{t}_{y.new} = \lambda \hat{t}_{y.sin} + (1-\lambda)\widetilde{t}_{y\pi}, \tag{28}$$

where $\lambda = 1/\left(1+\phi^2\right)$, $\hat{t}_{y.sin}$ is defined by Eq.(8), and

$$\widetilde{t}_{y\pi} = \frac{\hat{t}_{1\pi}}{\hat{t}_{q\pi}} \hat{t}_{qy\pi}. \tag{29}$$

Furthermore, $\hat{t}_{1\pi} = \sum_{i\in s}(1/\pi_i)$, $\hat{t}_{q\pi} = \sum_{i\in s}(q_i/\pi_i)$, and $\hat{t}_{qy\pi} = \sum_{i\in s}(q_i y_i/\pi_i)$ be the Horvitz and Thompson (1952) estimators for N, t_q, and t_{qy}, respectively.

Remark 2.1 *Since $\lambda \in (0,1)$, Eq.(28) is a convex transformation between $\hat{t}_{y.sin}$ and $\widetilde{t}_{y\pi}$, defined by Eq.(8) and Eq.(29) respectively. At the same time, as $\phi^2 \to \infty \Rightarrow \lambda \to 0 \Rightarrow \hat{t}_{y.new} \to \widetilde{t}_{y\pi}$; moreover, as $\phi^2 \to 0 \Rightarrow \lambda \to 1 \Rightarrow \hat{t}_{y.new} \to \hat{t}_{y.sin}$.*

The performance of $\hat{t}_{y.new}$ will be discussed through simulations from real data set. We will compare $\hat{t}_{y.new}$, $\hat{t}_{y.sin}$, and $\widetilde{t}_{y\pi}$. Consider the FEV data set which was used by Singh (2013) and downloaded from http://www.amstat.org/publications/jse/datasets/fev.dat.txt. Let Y be the Forced expiratory volume, $t_y = 1724$; and the auxiliary variable X be the Children height in inches, $t_x = 39988$. Our aim is to estimate t_y by using $\hat{t}_{y.new}$, $\hat{t}_{y.sin}$, and $\widetilde{t}_{y\pi}$. To achieve our aim, simulate $\upsilon = 3000$ independent random samples from the FEV data set by using procedure `surveyselect` of SAS Institute, under SRSWR design. For $q_i = x_i$ and based on the random samples, estimate t_y by $\hat{t}_{y.new}$, $\hat{t}_{y.sin}$, and $\widetilde{t}_{y\pi}$. Furthermore, compute the empirical mean (Em.Mean), relative bias (RB), and empirical relative mean squares error (REMSE) of the estimators $\hat{t}_{y.new}$, $\hat{t}_{y.sin}$, and $\widetilde{t}_{y\pi}$; where

$$\text{EM.Mean}\left(\hat{t}_y^*\right) = \frac{1}{\upsilon}\sum_{i=1}^{\upsilon}\left(\hat{t}_y^*\right)_i \tag{30}$$

$$\text{RB}\left(\hat{t}_y^*\right) = \frac{\text{EM.Mean}\left(\hat{t}_y^*\right) - t_y}{t_y} \times 100\% \tag{31}$$

$$\text{REMSE}\left(\hat{t}_y^*\right) = \frac{\sum_{i=1}^{\upsilon}\left(\hat{t}_y^* - t_y\right)^2}{\sum_{i=1}^{\upsilon}\left(\hat{t}_{y.new} - t_y\right)^2}, \tag{32}$$

where EM.Mean $\left(\hat{t}_y^*\right)$, RB $\left(\hat{t}_y^*\right)$, and REMSE $\left(\hat{t}_y^*\right)$ are the empirical mean, relative bias, and relative mean squares error of the estimator $\hat{t}_y^*$. For $n = 25, 35, 45, 55, 65,$ and 75. The results are summarized in Table (1).

From Table (1), in the sense of REMSE, the estimator $\hat{t}_{y.sin}$ performs better than $\hat{t}_{y.new}$ and $\widetilde{t}_{y\pi}$ for all values of n and for the different values of $\lambda = 0, 0.25, 0.5, 0.75,$ and 1. However, REMSE $\left(\widetilde{t}_{y\pi}\right)$ varies from 1 to 6.74; at the same time, REMSE $\left(\widetilde{t}_{y\pi}\right) = 6.74$ is attainable for large $n = 75$. From this point, concentrations will be focused on the performance of $\widetilde{t}_{y\pi}$ in order to improve the performance of $\hat{t}_{y.new}$. The remaining of this article will be focused on the improvement of $\widetilde{t}_{y\pi}$.

Remark 2.2 *The Horvitz and Thompson (1952) estimator defined by Eq.(1) is a special case from Eq.(29), namely when $q_i = 1$ (or a positive constant). Hence, $\widetilde{t}_{y\pi}$ is modified $\hat{t}_{y\pi}$ for estimating the finite population total t_y. Further, $\widetilde{t}_{y\pi}$ uses the availability of the auxiliary variable through q_i's.*

To the first order and by using Taylor expansion, expanding the right hand side of Eq.(29), we have

$$\widetilde{t}_{y\pi} \simeq \frac{t_1}{t_q} t_{qy} + \frac{t_{qy}}{t_q}\left(\hat{t}_{1\pi} - t_1\right) + \frac{t_1}{t_q}\left(\hat{t}_{qy\pi} - t_{qy}\right) - \frac{t_1 t_{qy}}{t_q^2}\left(\hat{t}_{q\pi} - t_q\right) \tag{33}$$

Therefore, the bias of $\widetilde{t}_{y\pi}$ is given by

$$Bias\left(\widetilde{t}_{y\pi}\right) = t_y - \frac{t_1}{t_q} t_{qy}. \tag{34}$$

Remark 2.3 *It is clear from Eq.(34) that $\widetilde{t}_{y\pi}$ is a biased estimator for estimating the finite population total t_y. However, $\widetilde{t}_{y\pi}$ is a strictly linear estimator; therefore, we can deduce two exactly unbiased estimators from $\widetilde{t}_{y\pi}$ based on Sugden and Smith (2002).*

From Eq.(29), rewrite $\widetilde{t}_{y\pi}$ as

$$\widetilde{t}_{y\pi} = \sum_{i \in s} b_{si} y_i, \tag{35}$$

where

$$b_{si} = \frac{q_i/\pi_i}{\sum_{i \in s}\left(q_i/\pi_i\right)/\sum_{i \in s}\left(1/\pi_i\right)}. \tag{36}$$

From Eq.(15), recall the definition of B_i,

$$\begin{aligned} B_i &= \sum_{s \ni i} p(s)\, b_{si} \\ &= \frac{q_i}{\pi_i} \sum_{s \ni i}\left[p(s)\frac{\sum_{i \in s}\left(1/\pi_i\right)}{\sum_{i \in s}\left(q_i/\pi_i\right)}\right] \end{aligned} \tag{37}$$

Based on Sugden and Smith (2002) approach, the two exactly unbiased estimators deduced from $\widetilde{t}_{y\pi}$ for t_y are

$$\widetilde{t}_{y\pi(1)} = \widetilde{t}_{y\pi} - \sum_{i \in s}\left(B_i - 1\right) y_i/\pi_i \tag{38}$$

and

$$\widetilde{t}_{y\pi(2)} = \sum_{i \in s} \frac{b_{si}}{B_i} y_i \tag{39}$$

where b_{si} and B_i are defined by Eq.(36) and Eq.(37) respectively.

Remark 2.4 *Eq.(35) shows that $\widetilde{t}_{y\pi}$ is a general linear estimator of t_y. Furthermore, $\widetilde{t}_{y\pi(1)}$ and $\widetilde{t}_{y\pi(2)}$ are two exactly unbiased estimators for t_y deduced from $\widetilde{t}_{y\pi}$; therefore, $\widetilde{t}_{y\pi}$ is a strictly linear estimator based on the Sugden and Smith (2002) definition. Hence, the estimators $\widetilde{t}_{y\pi(1)}$ and $\widetilde{t}_{y\pi(2)}$ are generalization of the Horvitz and Thompson (1952) estimator and use the availability of the auxiliary variable.*

Since $\widetilde{t}_{y\pi(1)}$ and $\widetilde{t}_{y\pi(2)}$ are exactly unbiased estimators for t_y, the infinite number of exactly unbiased estimators is defined by

$$\widetilde{t}_{y\pi} = \omega\widetilde{t}_{y\pi(1)} + (1-\omega)\widetilde{t}_{y\pi(2)}, \quad \text{for} \quad 0 \le \omega \le 1. \tag{40}$$

Remark 2.5 *The estimator $\widetilde{t}_{y\pi}$ is a convex transformation and an unbiased estimator for estimating the population total t_y.*

2.1. Modified Horvitz-Thompson and Stratified Sampling Designs

The finite population U of size N is divided into L non-overlapping strata $U_1, U_2, \ldots, U_L$; $U = \bigcup_{h=1}^{L} U_h$. The population total for the h^{th} stratum is $t_{yh} = \sum_{i\in U_h} y_i$. Furthermore, the h^{th} stratum is of size N_h and $N = \sum_{h=1}^{L} N_h$. The population total t_y is redefined as

$$t_y = \sum_{h=1}^{L} t_{yh}. \tag{41}$$

For the h^{th} stratum and based on a measurable sampling design $p_h(.)$, draw a random sample s_h of size n_h from U_h. Assume $\bar{x}_h = \sum_{i\in U_h} x_i/N_h$ is known for $h = 1, 2, \ldots, L$. Apply $\widetilde{t}_{y\pi(1)}$ and $\widetilde{t}_{y\pi(2)}$ to the h^{th} stratum. In other words, estimate t_{yh} by

$$\widetilde{t}_{y\pi(1).h} = \widetilde{t}_{y\pi.st} - \sum_{i\in s_h} (B_i - 1) y_i/\pi_i, \tag{42}$$

or by

$$\widetilde{t}_{y\pi(2).h} = \sum_{i\in s_h} \frac{b_{s_{hi}}}{B_i} y_i. \tag{43}$$

In this case, $\widetilde{t}_{y\pi(1).h}$ and $\widetilde{t}_{y\pi(2).h}$ are exactly two unbiased estimators for t_{yh}, where

$$\widetilde{t}_{y\pi.st} = \sum_{h=1}^{L} \frac{\hat{t}_{1\pi.h}}{\hat{t}_{q\pi.h}} \hat{t}_{qy\pi.h}; \tag{44}$$

$\hat{t}_{1\pi.h} = \sum_{i\in s_h} (1/\pi_i)$, $\hat{t}_{q\pi.h} = \sum_{i\in s_h} (q_i/\pi_i)$, and $\hat{t}_{qy\pi.h} = \sum_{i\in s_h} (q_i y_i/\pi_i)$ be the Horvitz and Thompson (1952) estimators for N_h, $t_{q.h}$, and $t_{qy.h}$, respectively.

From Eq.(41), estimate t_y by

$$\widetilde{t}_{y\pi(1).st} = \sum_{h=1}^{L} \widetilde{t}_{y\pi(1).h}, \tag{45}$$

or by

$$\widetilde{t}_{yh\pi(2).st} \quad = \quad \sum_{h=1}^{L} \widetilde{t}_{y\pi(2).h}, \tag{46}$$

where $\widetilde{t}_{y\pi(1).h}$ and $\widetilde{t}_{y\pi(2).h}$ are defined in Eq.(42) and Eq.(43), respectively.

Remark 2.6 *The two estimators $\widetilde{t}_{y\pi(1).h}$ and $\widetilde{t}_{y\pi(2).h}$ are two exactly unbiased estimators for t_{yh}. Based on this idea, the two estimators $\widetilde{t}_{y\pi(1).st}$ and $\widetilde{t}_{y\pi(2).st}$ are exactly unbiased estimators for t_y; therefore, the accumulation of bias across strata is avoided.*

2.2. Special Cases

The exactly unbiased estimators $\widetilde{t}_{y\pi(1)}$ and $\widetilde{t}_{y\pi(2)}$ are given by Eq.(38) and Eq.(39) respectively, deduced from the modified HT estimator $\widetilde{t}_{y\pi}$, depending on the weight q_i. Therefore, $\widetilde{t}_{y\pi(1)}$ and $\widetilde{t}_{y\pi(2)}$ can use the availability of the auxiliary variables through q_i. In this section, different special cases are considered.

As we mentioned earlier, $\widetilde{t}_{y\pi}$ reduces to $\hat{t}_{y\pi}$, the ordinary Horvitz and Thompson (1952) estimator, when q_i's are one or positive constant. Furthermore, from Eq.(36), $b_{si} = 1/\pi_i$ and from Eq.(37), $B_i = 1$. Therefore,

$$\widetilde{t}_{y\pi(1)} = \widetilde{t}_{y\pi(2)} = \hat{t}_{y\pi}, \tag{47}$$

i.e. $\widetilde{t}_{y\pi(1)}$ and $\widetilde{t}_{y\pi(2)}$ are identical; in other words, $\widetilde{t}_{y\pi}$ is exactly an unbiased estimator for t_y. In this case, the Sugden and Smith (2002) approach gives exactly one unbiased estimator for estimating t_y.

Draw a random sample s of size n from the population U of size N by using the simple random sample without replacement (SRSWR) design. Under SRSWR design, $p(s) = 1/\begin{pmatrix} N \\ n \end{pmatrix}$ and $\pi_i = n/N$. Consider the following two cases:

a. $q_i = \pi_i$.

In this case, $b_{si} = \frac{N}{n}$ and $B_i = 1$. Therefore,

$$\hat{t}_{y\pi} = \widetilde{t}_{y\pi(1)} = \widetilde{t}_{y\pi(2)} = N\bar{y}_s, \tag{48}$$

which is well-known estimator for estimating t_y, where $\bar{y}_s = \sum_{i=1}^{n} y_i/n$. In this case, the two exactly unbiased estimators based on Sugden and Smith (2002) are reduced to one unbiased estimator, i.e. the Sugden and Smith (2002) approach produces exactly only one unbiased estimator.

b. $q_i = x_i, x_i > 0$.

In this case, $b_{si} = Nx_i / \sum_{j\in s} x_j$ and $B_i = Nx_i p(s) \sum_{s\ni i} \left(\sum_{j\in s} x_j\right)^{-1}$. Therefore,

$$\widetilde{t}_{y\pi(1)} = N \left[\frac{\sum_{i\in s} x_i y_i}{\sum_{i\in s} x_i} + \bar{y}_s - \frac{1}{\binom{N-1}{n-1}} \sum_{i\in s} \left\{ \sum_{s\ni i} \left(\sum_{j\in s} x_j \right)^{-1} \right\} x_i y_i \right], \quad (49)$$

and

$$\widetilde{t}_{y\pi(2)} = \binom{N}{n} \sum_{i\in s} \left[\frac{y_i}{\sum_{s\ni i} \left(\sum_{j\in s} x_j\right)^{-1}} \right] / \sum_{j\in s} x_j \quad (50)$$

$$= \hat{T}_{R(2)}, \quad (51)$$

where $\hat{T}_{R(2)}$ is an estimate of t_y defined by Sugden and Smith (2002), Eq.(4.5).

3. Empirical Studies

Sugden and Smith (2002) considered the ratio estimator

$$\hat{T}_R = t_x \frac{\hat{t}_{y\pi}}{\hat{t}_{x\pi}} \quad (52)$$

as a general linear estimator for the population total t_y. $\hat{T}_R$ is asymptotically an unbiased estimator of t_y. Since $\hat{T}_R$ produces two exactly unbiased estimators of t_y, $\hat{T}_R$ is a strictly linear estimator for t_y. Under SRSWR, $B_{Ri} = t_x \sum_{s\ni i} \left(\sum_{j\in s} x_j\right)^{-1} / \binom{N}{n}$. In this case, the exactly unbiased estimators are

$$\hat{T}_{R(1)} = \hat{T}_R - \frac{N}{n} \sum_{i\in s} (B_{Ri} - 1) y_i, \quad (53)$$

and $\widetilde{T}_{R(2)}$, defined by Eq.(51).

Assume all the values of the auxiliary variable are available in the study; under SRSWR design, the estimators $\hat{t}_{y\pi}$, $\widetilde{t}_{y\pi(1)}$, $\widetilde{t}_{y\pi(2)}$, $\hat{T}_{R(2)}$, $\hat{T}_R$, and $\hat{T}_{R(1)}$ defined by Eq.(48), (49), (50), (51), (52), (53) respectively, will be used in the empirical studies.

Consider the data set given by Example(4.9), Page 139, Lohr (2010). In this example, X is the photo counts of dead trees and Y is the field counts of dead trees; $N = 25$, $t_x = 265$, and $t_y = 289$. From this data set, under SRSWR, draw all random samples of sizes $n = 2, 3, 4$. The computations are implemented by using a SAS program written under the iml procedure. The number of all random samples is $m = 300, 2300, 12650$ for $n = 2, 3, 4$ respectively. The relative efficiency of the ratio family is defined by $MSE\left(\hat{T}_{R(i)}\right)/MSE\left(\hat{T}_R\right)$ and relative efficiency of the Horvitz and Thompson (1952) family is defined by $MSE\left(\widetilde{t}_{y\pi(i)}\right)/MSE\left(\hat{t}_{y\pi}\right)$ for $i = 1, 2$. The results are given in Table(2).

In the case of a stratification, consider the data set cars93 from Scheaffer, Mendenhall and Ott (2006). The data set cars93 consists of different variables; for our study, let $X := MPGCITY, Y := MPGHIGH$, and the stratifications based on the variable "typecode". The cars93 data set is summarized by the following table.

h^{th} stratum	1	2	3	4	5	6	total
N_h	20	16	22	11	14	9	$N = 92$
t_{xh}	598	363	430	202	305	153	$t_x = 2051$
t_{yh}	712	478	588	294	403	197	$t_y = 2672$

For the h^{th} stratum, $h = 1,\ldots,6$, the results are given in Tables (3),...,(8) respectively. Based on the stratified sampling design, the population total t_y is estimated by using the estimators $\hat{t}_{y\pi}$, $\widetilde{t}_{y\pi(1)}$, $\widetilde{t}_{y\pi(2)}$, $\hat{T}_{R(2)}$, $\hat{T}_R$, and $\hat{T}_{R(1)}$; for $n = 12, 18, 24$. The results are given in Table (9). At the same time, Table (9) is computed from Tables (3),...,(8).

4. Concluding Remarks

In this paper, the Horvitz and Thompson (1952) estimator is modified so that the modified estimators can use the availability of the auxiliary variable in the study. Based on the Sugden and Smith (2002) approach, two exactly unbiased estimators for estimating the population total t_y are deduced from the modified estimator. Furthermore, the exactly two unbiased estimators can be used in stratified sampling designs.

From Table(2), the deduced estimators $\widetilde{t}_{y\pi(1)}$ and $\widetilde{t}_{y\pi(2)}$ are exactly unbiased estimators for estimating t_y and perform better than the original Horvitz and Thompson (1952) estimator $\hat{t}_{y\pi}$, in the sense of relative efficiency. Moreover, Table(2) supports the same conclusion mentioned by Sugden and Smith (2002), i.e. the estimators $\hat{T}_{R(1)}$ and $\hat{T}_{R(2)}$ are exactly unbiased estimators and perform better than the original ratio estimator $\hat{T}_R$, in the sense of relative efficiency.

Based on the Sugden and Smith (2002) approach, the two exactly unbiased estimators based on their families for estimating t_y perform better than the original estimators even if the original estimators are asymptotically unbiased or unbiased estimators. Furthermore, the estimators deduced from Horvitz and Thompson (1952) perform better than the deduced estimators from the ratio estimator. Small sample sizes are usually selected in the case of stratified sampling design; moreover, the deduced estimators can be applied to every stratum and aggregated together to estimate the population total.

For $h = 1,\ldots,6$ the results are given by Tables (3),...,(8), respectively. Table (9) is computed from Tables (3),...,(8), and shows that $\widetilde{t}_{y\pi(1)}$, $\widetilde{t}_{y\pi(2)}$, $\hat{T}_{R(1)}$, and $\hat{T}_{R(2)}$ are exactly unbiased estimators for t_y. Furthermore, the bias of the ratio estimator $\hat{T}_R$, is negligible (asymptotically unbiased) and performs better than $\hat{t}_{y\pi}$ (exactly unbiased) in the sense of relative efficiency. $\hat{T}_{R(1)}$ and $\hat{T}_{R(2)}$ estimators perform better than the ratio estimator $\hat{T}_R$ for all $n = 12, 18, 24$. At the same time, the relative efficiency of $\hat{T}_{R(1)}$ and $\hat{T}_{R(2)}$ are approximately the same for $n = 12, 18, 24$. In the case of the Horvtiz-Thompson family, the deduced estimators $\widetilde{t}_{y\pi(1)}$ and $\widetilde{t}_{y\pi(2)}$ perform significantly better than the original estimator $\hat{t}_{y\pi}$, in the sense of relative efficiency. Furthermore, the estimators $\widetilde{t}_{y\pi(1)}$ and $\widetilde{t}_{y\pi(2)}$ deliver approximately the same performance, for all $n = 12, 18, 24$.

From Eq.(51), we have $\widetilde{t}_{y\pi(2)} = \hat{T}_{R(2)}$; therefore, the ratio family and the Horvitz-Thompson family can be compared. Tables (2), (3),..., (9) show that

$$\frac{MSE\left(\widetilde{t}_{y\pi(1)}\right)}{MSE\left(\widetilde{t}_{y\pi(2)}\right)} \simeq \frac{MSE\left(\hat{T}_{R(1)}\right)}{MSE\left(\hat{T}_{R(2)}\right)},$$

for all values of n. Therefore, the deduced estimators $\widetilde{t}_{y\pi(1)}$ and $\widetilde{t}_{y\pi(2)}$ from the Horvitz-Thompson family and $\hat{T}_{R(1)}$ and $\hat{T}_{R(2)}$ from the ratio family perform better than the original families even though the original families are unbiased or asymptotically unbiased estimators.

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		$\hat{t}_{y.sin}$	$\widetilde{t}_{y\pi}$	$\hat{t}_{y.new}$			$\hat{t}_{y.sin}$	$\widetilde{t}_{y\pi}$	$\hat{t}_{y.new}$
$n=25$	Em.Mean	1720.34	1770.59	1770.59	$n=55$	Em.Mean	1722.70	1768.77	1768.77
$\lambda \to 0$	RB	-0.24	2.68	2.68	$\lambda \to 0$	RB	-0.10	2.57	2.57
	REMSE	0.21	1.00	1.00		REMSE	0.17	1.00	1.00
$n=25$	Em.Mean	1720.34	1770.59	1758.03	$n=55$	Em.Mean	1722.70	1768.77	1757.25
$\lambda=0.25$	RB	-0.24	2.68	1.95	$\lambda=0.25$	RB	-0.10	2.57	1.90
	REMSE	0.32	1.54	1.00		REMSE	0.27	1.58	1.00
$n=25$	Em.Mean	1720.34	1770.59	1745.47	$n=55$	Em.Mean	1722.70	1768.77	1745.73
$\lambda=0.50$	RB	-0.24	2.68	1.22	$\lambda=0.50$	RB	-0.10	2.57	1.23
	REMSE	0.52	2.51	1.00		REMSE	0.45	2.68	1.00
$n=25$	Em.Mean	1720.34	1770.59	1732.90	$n=55$	Em.Mean	1722.70	1768.77	1734.21
$\lambda=0.75$	RB	-0.24	2.68	0.49	$\lambda=0.75$	RB	-0.10	2.57	0.57
	REMSE	0.83	3.98	1.00		REMSE	0.78	4.58	1.00
$n=25$	Em.Mean	1720.34	1770.59	1720.34	$n=55$	Em.Mean	1722.70	1768.77	1722.70
$\lambda \to 1$	RB	-0.24	2.68	-0.24	$\lambda \to 1$	RB	-0.10	2.57	-0.10
	REMSE	1.00	4.81	1.00		REMSE	1.00	5.89	1.00
$n=35$	Em.Mean	1722.08	1770.28	1770.28	$n=65$	Em.Mean	1722.93	1768.89	1768.89
$\lambda \to 0$	RB	-0.14	2.66	2.66	$\lambda \to 0$	RB	-0.09	2.58	2.58
	REMSE	0.19	1.00	1.00		REMSE	0.16	1.00	1.00
$n=35$	Em.Mean	1722.08	1770.28	1758.23	$n=65$	Em.Mean	1722.93	1768.89	1757.40
$\lambda=0.25$	RB	-0.14	2.66	1.96	$\lambda=0.25$	RB	-0.09	2.58	1.91
	REMSE	0.30	1.55	1.00		REMSE	0.26	1.57	1.00
$n=35$	Em.Mean	1722.08	1770.28	1746.18	$n=65$	Em.Mean	1722.93	1768.89	1745.91
$\lambda=0.50$	RB	-0.14	2.66	1.26	$\lambda=0.50$	RB	-0.09	2.58	1.24
	REMSE	0.49	2.56	1.00		REMSE	0.44	2.67	1.00
$n=35$	Em.Mean	1722.08	1770.28	1734.13	$n=65$	Em.Mean	1722.93	1768.89	1734.42
$\lambda=0.75$	RB	-0.14	2.66	0.56	$\lambda=0.75$	RB	-0.09	2.58	0.58
	REMSE	0.80	4.17	1.00		REMSE	0.76	4.59	1.00
$n=35$	Em.Mean	1722.08	1770.28	1722.08	$n=65$	Em.Mean	1722.93	1768.89	1722.93
$\lambda \to 1$	RB	-0.14	2.66	-0.14	$\lambda \to 1$	RB	-0.09	2.58	-0.09
	REMSE	1.00	5.20	1.00		REMSE	1.00	6.07	1.00
$n=45$	Em.Mean	1721.84	1769.24	1769.24	$n=75$	Em.Mean	1723.25	1770.39	1770.39
$\lambda \to 0$	RB	-0.15	2.60	2.60	$\lambda \to 0$	RB	-0.07	2.66	2.66
	REMSE	0.19	1.00	1.00		REMSE	0.15	1.00	1.00
$n=45$	Em.Mean	1721.84	1769.24	1757.39	$n=75$	Em.Mean	1723.25	1770.39	1758.61
$\lambda=0.25$	RB	-0.15	2.60	1.91	$\lambda=0.25$	RB	-0.07	2.66	1.98
	REMSE	0.29	1.55	1.00		REMSE	0.25	1.59	1.00
$n=45$	Em.Mean	1721.84	1769.24	1745.54	$n=75$	Em.Mean	1723.25	1770.39	1746.82
$\lambda=0.50$	RB	-0.15	2.60	1.22	$\lambda=0.50$	RB	-0.07	2.66	1.30
	REMSE	0.48	2.57	1.00		REMSE	0.42	2.73	1.00
$n=45$	Em.Mean	1721.84	1769.24	1733.69	$n=75$	Em.Mean	1723.25	1770.39	1735.04
$\lambda=0.75$	RB	-0.15	2.60	0.54	$\lambda=0.75$	RB	-0.07	2.66	0.61
	REMSE	0.78	4.22	1.00		REMSE	0.74	4.81	1.00
$n=45$	Em.Mean	1721.84	1769.24	1721.84	$n=75$	Em.Mean	1723.25	1770.39	1723.25
$\lambda \to 1$	RB	-0.15	2.60	-0.15	$\lambda \to 1$	RB	-0.07	2.66	-0.07
	REMSE	1.00	5.39	1.00		REMSE	1.00	6.47	1.00

Table 1: Computations are based on Eq.(28). The REMSE's are computed by using Eq. (32) for $\hat{t}^{*}_{y} = \hat{t}_{y.sin}$, $\widetilde{t}_{y\pi}$, and $\hat{t}_{y.new}$.

Estimator	$n=2$		$n=3$		$n=4$	
	t_y	S_y^2	t_y	S_y^2	t_y	S_y^2
$\hat{T}_R$	294.3058	2545.752	292.291	1549.1841	291.322	1081.5572
	(5.3059)	(**2573.904**)	(3.292)	(**1560.0213**)	(2.322)	(**1086.9488**)
$\hat{T}_{R(1)}$	289	1684.9349	289	1111.1305	289	834.9348
$\hat{T}_{R(2)}$	289	1650.5674	289	1087.7975	289	821.8695
$\hat{t}_{y\pi}$	289	2613.375	289	1666.5	289	1193.0625
$\widetilde{t}_{y\pi(1)}$	289	1689.2317	289	1115.3438	289	845.6788
$\widetilde{t}_{y\pi(2)}$	289	1650.5674	289	1087.7975	289	821.8695
$\hat{T}_R$ Family	$\frac{MSE(\hat{T}_{R(1)})}{MSE(\hat{T}_R)}=0.6546$		$\frac{MSE(\hat{T}_{R(1)})}{MSE(\hat{T}_R)}=0.7123$		$\frac{MSE(\hat{T}_{R(1)})}{MSE(\hat{T}_R)}=0.7682$	
	$\frac{MSE(\hat{T}_{R(2)})}{MSE(\hat{T}_R)}=0.6413$		$\frac{MSE(\hat{T}_{R(2)})}{MSE(\hat{T}_R)}=0.6973$		$\frac{MSE(\hat{T}_{R(2)})}{MSE(\hat{T}_R)}=0.7561$	
	$\frac{MSE(\hat{T}_{R(1)})}{MSE(\hat{T}_{R(2)})}=1.0208$		$\frac{MSE(\hat{T}_{R(1)})}{MSE(\hat{T}_{R(2)})}=1.0215$		$\frac{MSE(\hat{T}_{R(1)})}{MSE(\hat{T}_{R(2)})}=1.0159$	
Horvtiz-Thopson	$\frac{MSE(\widetilde{t}_{y\pi(1)})}{MSE(\hat{t}_{y\pi})}=0.6464$		$\frac{MSE(\widetilde{t}_{y\pi(1)})}{MSE(\hat{t}_{y\pi})}=0.6693$		$\frac{MSE(\widetilde{t}_{y\pi(1)})}{MSE(\hat{t}_{y\pi})}=0.7088$	
family	$\frac{MSE(\widetilde{t}_{y\pi(2)})}{MSE(\hat{t}_{y\pi})}=0.6316$		$\frac{MSE(\widetilde{t}_{y\pi(2)})}{MSE(\hat{t}_{y\pi})}=0.6527$		$\frac{MSE(\widetilde{t}_{y\pi(2)})}{MSE(\hat{t}_{y\pi})}=0.6889$	
	$\frac{MSE(\widetilde{t}_{y\pi(1)})}{MSE(\widetilde{t}_{y\pi(2)})}=1.0234$		$\frac{MSE(\widetilde{t}_{y\pi(1)})}{MSE(\widetilde{t}_{y\pi(2)})}=1.0253$		$\frac{MSE(\widetilde{t}_{y\pi(1)})}{MSE(\widetilde{t}_{y\pi(2)})}=1.029$	

Table 2: Empirical results based on real data set. For the estimator $\hat{T}_R$: the number between brackets under the mean is the bias and the bold one under the variance is the MSE of $\hat{T}_R$.

Estimator	$n_h=2$		$n_h=3$		$n_h=4$	
	t_{yh}	S_{yh}^2	t_{yh}	S_{yh}^2	t_{yh}	S_{yh}^2
$\hat{T}_R$	715.7886	1533.6014	714.4398	975.7520	713.7416	691.1576
	(3.7886)	(**1547.9549**)	(2.4398)	(**981.7046**)	(1.741556)	(**694.1906**)
$\hat{T}_{R(1)}$	712	1363.2611	712	522.6314	712	358.48553
$\hat{T}_{R(2)}$	712	1405.6602	712	523.4136	712	353.7434
$\hat{t}_{y\pi}$	712	5900.2105	712	3714.9474	712	2622.3158
$\widetilde{t}_{y\pi(1)}$	712	1483.1003	712	578.5657	712	391.3314
$\widetilde{t}_{y\pi(2)}$	712	1405.6602	712	523.4136	712	353.7434
$\hat{T}_R$ Family	$\frac{MSE(\hat{T}_{R(1)})}{MSE(\hat{T}_R)}=0.8807$		$\frac{MSE(\hat{T}_{R(1)})}{MSE(\hat{T}_R)}=0.5324$		$\frac{MSE(\hat{T}_{R(1)})}{MSE(\hat{T}_R)}=0.5164$	
	$\frac{MSE(\hat{T}_{R(2)})}{MSE(\hat{T}_R)}=0.9081$		$\frac{MSE(\hat{T}_{R(2)})}{MSE(\hat{T}_R)}=0.5332$		$\frac{MSE(\hat{T}_{R(2)})}{MSE(\hat{T}_R)}=0.5096$	
	$\frac{MSE(\hat{T}_{R(1)})}{MSE(\hat{T}_{R(2)})}=0.9698$		$\frac{MSE(\hat{T}_{R(1)})}{MSE(\hat{T}_{R(2)})}=0.9985$		$\frac{MSE(\hat{T}_{R(1)})}{MSE(\hat{T}_{R(2)})}=1.0134$	
Hort-Thom	$\frac{MSE(\widetilde{t}_{y\pi(1)})}{MSE(\hat{t}_{y\pi})}=0.2514$		$\frac{MSE(\widetilde{t}_{y\pi(1)})}{MSE(\hat{t}_{y\pi})}=0.1557$		$\frac{MSE(\widetilde{t}_{y\pi(1)})}{MSE(\hat{t}_{y\pi})}=0.1492$	
family	$\frac{MSE(\widetilde{t}_{y\pi(2)})}{MSE(\hat{t}_{y\pi})}=0.2382$		$\frac{MSE(\widetilde{t}_{y\pi(2)})}{MSE(\hat{t}_{y\pi})}=0.1409$		$\frac{MSE(\widetilde{t}_{y\pi(2)})}{MSE(\hat{t}_{y\pi})}=0.1349$	
	$\frac{MSE(\widetilde{t}_{y\pi(1)})}{MSE(\widetilde{t}_{y\pi(2)})}=1.0551$		$\frac{MSE(\widetilde{t}_{y\pi(1)})}{MSE(\widetilde{t}_{y\pi(2)})}=1.10537$		$\frac{MSE(\widetilde{t}_{y\pi(1)})}{MSE(\widetilde{t}_{y\pi(2)})}=1.1063$	

Table 3: Empirical results from cars93 for Stratum (1): When typecode=1

Estimator	$n_h=2$		$n_h=3$		$n_h=4$	
	t_{yh}	S^2_{yh}	t_{yh}	S^2_{yh}	t_{yh}	S^2_{yh}
$\hat{T}_R$	478.0681	306.3392	478.0383	188.7059	478.0252	130.3291
	(0.0681)	(**306.3439**)	(0.0383)	(**188.7074**)	(0.0252)	(**130.3297**)
$\hat{T}_{R(1)}$	478	443.4048	478	219.6018	478	139.9649
$\hat{T}_{R(2)}$	478	444.9974	478	220.1128	478	140.1862
$\hat{t}_{y\pi}$	478	968.8000	478	599.7333	478	415.2000
$\widetilde{t}_{y\pi(1)}$	478	446.9986	478	221.5335	478	141.1973
$\widetilde{t}_{y\pi(2)}$	478	444.9974	478	220.1128	478	140.1862
$\hat{T}_R$ Family	$\frac{MSE(\hat{T}_{R(1)})}{MSE(\hat{T}_R)}=1.4474$		$\frac{MSE(\hat{T}_{R(1)})}{MSE(\hat{T}_R)}=1.1637$		$\frac{MSE(\hat{T}_{R(1)})}{MSE(\hat{T}_R)}=1.0739$	
	$\frac{MSE(\hat{T}_{R(2)})}{MSE(\hat{T}_R)}=1.4526$		$\frac{MSE(\hat{T}_{R(2)})}{MSE(\hat{T}_R)}=1.1664$		$\frac{MSE(\hat{T}_{R(2)})}{MSE(\hat{T}_R)}=1.0756$	
	$\frac{MSE(\hat{T}_{R(1)})}{MSE(\hat{T}_{R(2)})}=0.9964$		$\frac{MSE(\hat{T}_{R(1)})}{MSE(\hat{T}_{R(2)})}=0.9977$		$\frac{MSE(\hat{T}_{R(1)})}{MSE(\hat{T}_{R(2)})}=0.9984$	
Hort-Thom family	$\frac{MSE(\widetilde{t}_{y\pi(1)})}{MSE(\hat{t}_{y\pi})}=0.4614$		$\frac{MSE(\widetilde{t}_{y\pi(1)})}{MSE(\hat{t}_{y\pi})}=0.3694$		$\frac{MSE(\widetilde{t}_{y\pi(1)})}{MSE(\hat{t}_{y\pi})}=0.3401$	
	$\frac{MSE(\widetilde{t}_{y\pi(2)})}{MSE(\hat{t}_{y\pi})}=0.4593$		$\frac{MSE(\widetilde{t}_{y\pi(2)})}{MSE(\hat{t}_{y\pi})}=0.3671$		$\frac{MSE(\widetilde{t}_{y\pi(2)})}{MSE(\hat{t}_{y\pi})}=0.3376$	
	$\frac{MSE(\widetilde{t}_{y\pi(1)})}{MSE(\widetilde{t}_{y\pi(2)})}=1.0045$		$\frac{MSE(\widetilde{t}_{y\pi(1)})}{MSE(\widetilde{t}_{y\pi(2)})}=1.0065$		$\frac{MSE(\widetilde{t}_{y\pi(1)})}{MSE(\widetilde{t}_{y\pi(2)})}=1.0072$	

Table 4: Empirical results from cars93 for Stratum (2): When typecode=2

Estimator	$n_h=2$		$n_h=3$		$n_h=4$	
	t_{yh}	S^2_{yh}	t_{yh}	S^2_{yh}	t_{yh}	S^2_{yh}
$\hat{T}_R$	588.4592	449.2140	588.2849	277.1965	588.2004	194.4815
	(0.4592)	(**449.4248**)	(0.2849)	(**277.2777**)	(0.2004)	(**194.5216**)
$\hat{T}_{R(1)}$	588	525.4486	588	260.7300	588	174.2959
$\hat{T}_{R(2)}$	588	527.9447	588	261.3186	588	174.4998
$\hat{t}_{y\pi}$	588	1386.6667	588	878.2222	588	624.0000
$\widetilde{t}_{y\pi(1)}$	588	532.0924	588	264.2138	588	176.5988
$\widetilde{t}_{y\pi(2)}$	588	527.9447	588	261.3186	588	174.4998
$\hat{T}_R$ Family	$\frac{MSE(\hat{T}_{R(1)})}{MSE(\hat{T}_R)}=1.1692$		$\frac{MSE(\hat{T}_{R(1)})}{MSE(\hat{T}_R)}=0.9403$		$\frac{MSE(\hat{T}_{R(1)})}{MSE(\hat{T}_R)}=0.8960$	
	$\frac{MSE(\hat{T}_{R(2)})}{MSE(\hat{T}_R)}=1.1747$		$\frac{MSE(\hat{T}_{R(2)})}{MSE(\hat{T}_R)}=0.9424$		$\frac{MSE(\hat{T}_{R(2)})}{MSE(\hat{T}_R)}=0.8971$	
	$\frac{MSE(\hat{T}_{R(1)})}{MSE(\hat{T}_{R(2)})}=0.9953$		$\frac{MSE(\hat{T}_{R(1)})}{MSE(\hat{T}_{R(2)})}=0.9978$		$\frac{MSE(\hat{T}_{R(1)})}{MSE(\hat{T}_{R(2)})}=0.9988$	
Hort-Thom family	$\frac{MSE(\widetilde{t}_{y\pi(1)})}{MSE(\hat{t}_{y\pi})}=0.3837$		$\frac{MSE(\widetilde{t}_{y\pi(1)})}{MSE(\hat{t}_{y\pi})}=0.3009$		$\frac{MSE(\widetilde{t}_{y\pi(1)})}{MSE(\hat{t}_{y\pi})}=0.2830$	
	$\frac{MSE(\widetilde{t}_{y\pi(2)})}{MSE(\hat{t}_{y\pi})}=0.3807$		$\frac{MSE(\widetilde{t}_{y\pi(2)})}{MSE(\hat{t}_{y\pi})}=0.2976$		$\frac{MSE(\widetilde{t}_{y\pi(2)})}{MSE(\hat{t}_{y\pi})}=0.2797$	
	$\frac{MSE(\widetilde{t}_{y\pi(1)})}{MSE(\widetilde{t}_{y\pi(2)})}=1.0079$		$\frac{MSE(\widetilde{t}_{y\pi(1)})}{MSE(\widetilde{t}_{y\pi(2)})}=1.0111$		$\frac{MSE(\widetilde{t}_{y\pi(1)})}{MSE(\widetilde{t}_{y\pi(2)})}=1.0120$	

Table 5: Empirical results from cars93 for Stratum (3): When typecode=3

Estimator	$n_h = 2$		$n_h = 3$		$n_h = 4$	
	t_{yh}	S^2_{yh}	t_{yh}	S^2_{yh}	t_{yh}	S^2_{yh}
$\hat{T}_R$	294.4412	99.6516	294.2596	58.9851	294.1697	38.6799
	(0.4412)	(**99.8463**)	(0.2596)	(**59.0524**)	(0.1697)	(**38.7087**)
$\hat{T}_{R(1)}$	294	32.2542	294	28.9585	294	24.42012
$\hat{T}_{R(2)}$	294	31.9504	294	28.8365	294	24.3733
$\hat{t}_{y\pi}$	294	80.1000	294	47.4667	294	31.1500
$\tilde{t}_{y\pi(1)}$	294	32.0873	294	29.0126	294	24.5711
$\tilde{t}_{y\pi(2)}$	294	31.9504	294	28.8365	294	24.3733
$\hat{T}_R$ Family	$\frac{MSE\left(\hat{T}_{R(1)}\right)}{MSE\left(\hat{T}_R\right)} = 0.3230$		$\frac{MSE\left(\hat{T}_{R(1)}\right)}{MSE\left(\hat{T}_R\right)} = 0.4904$		$\frac{MSE\left(\hat{T}_{R(1)}\right)}{MSE\left(\hat{T}_R\right)} = 0.6309$	
	$\frac{MSE\left(\hat{T}_{R(2)}\right)}{MSE\left(\hat{T}_R\right)} = 0.3199$		$\frac{MSE\left(\hat{T}_{R(2)}\right)}{MSE\left(\hat{T}_R\right)} = 0.4883$		$\frac{MSE\left(\hat{T}_{R(2)}\right)}{MSE\left(\hat{T}_R\right)} = 0.6297$	
	$\frac{MSE\left(\hat{T}_{R(1)}\right)}{MSE\left(\hat{T}_{R(2)}\right)} = 1.0095$		$\frac{MSE\left(\hat{T}_{R(1)}\right)}{MSE\left(\hat{T}_{R(2)}\right)} = 1.0042$		$\frac{MSE\left(\hat{T}_{R(1)}\right)}{MSE\left(\hat{T}_{R(2)}\right)} = 1.0019$	
Hort-Thom	$\frac{MSE\left(\tilde{t}_{y\pi(1)}\right)}{MSE\left(\hat{t}_{y\pi}\right)} = 0.4006$		$\frac{MSE\left(\tilde{t}_{y\pi(1)}\right)}{MSE\left(\hat{t}_{y\pi}\right)} = 0.6112$		$\frac{MSE\left(\tilde{t}_{y\pi(1)}\right)}{MSE\left(\hat{t}_{y\pi}\right)} = 0.7888$	
family	$\frac{MSE\left(\tilde{t}_{y\pi(2)}\right)}{MSE\left(\hat{t}_{y\pi}\right)} = 0.3989$		$\frac{MSE\left(\tilde{t}_{y\pi(2)}\right)}{MSE\left(\hat{t}_{y\pi}\right)} = 0.6075$		$\frac{MSE\left(\tilde{t}_{y\pi(2)}\right)}{MSE\left(\hat{t}_{y\pi}\right)} = 0.7825$	
	$\frac{MSE\left(\tilde{t}_{y\pi(1)}\right)}{MSE\left(\tilde{t}_{y\pi(2)}\right)} = 1.0043$		$\frac{MSE\left(\tilde{t}_{y\pi(1)}\right)}{MSE\left(\tilde{t}_{y\pi(2)}\right)} = 1.0061$		$\frac{MSE\left(\tilde{t}_{y\pi(1)}\right)}{MSE\left(\tilde{t}_{y\pi(2)}\right)} = 1.0081$	

Table 6: Empirical results from cars93 for Stratum (4): When typecode=4

Estimator	$n_h = 2$		$n_h = 3$		$n_h = 4$	
	t_{yh}	S^2_{yh}	t_{yh}	S^2_{yh}	t_{yh}	S^2_{yh}
$\hat{T}_R$	404.9682	464.2094	404.2003	279.8831	403.8170	189.1534
	(1.9682)	(**468.0833**)	(1.2003)	(**281.3238**)	(0.8170)	(**189.8209**)
$\hat{T}_{R(1)}$	403	219.7774	403	117.0185	403	96.4114
$\hat{T}_{R(2)}$	403	221.1506	403	114.8667	403	94.8695
$\hat{t}_{y\pi}$	403	1113.6923	403	680.5897	403	464.0385
$\tilde{t}_{y\pi(1)}$	403	234.9529	403	123.6658	403	100.9741
$\tilde{t}_{y\pi(2)}$	403	221.1506	403	114.8667	403	94.8695
$\hat{T}_R$ Family	$\frac{MSE\left(\hat{T}_{R(1)}\right)}{MSE\left(\hat{T}_R\right)} = 0.4695$		$\frac{MSE\left(\hat{T}_{R(1)}\right)}{MSE\left(\hat{T}_R\right)} = 0.4160$		$\frac{MSE\left(\hat{T}_{R(1)}\right)}{MSE\left(\hat{T}_R\right)} = 0.5079$	
	$\frac{MSE\left(\hat{T}_{R(2)}\right)}{MSE\left(\hat{T}_R\right)} = 0.4725$		$\frac{MSE\left(\hat{T}_{R(2)}\right)}{MSE\left(\hat{T}_R\right)} = 0.4083$		$\frac{MSE\left(\hat{T}_{R(2)}\right)}{MSE\left(\hat{T}_R\right)} = 0.4998$	
	$\frac{MSE\left(\hat{T}_{R(1)}\right)}{MSE\left(\hat{T}_{R(2)}\right)} = 0.9938$		$\frac{MSE\left(\hat{T}_{R(1)}\right)}{MSE\left(\hat{T}_{R(2)}\right)} = 1.0187$		$\frac{MSE\left(\hat{T}_{R(1)}\right)}{MSE\left(\hat{T}_{R(2)}\right)} = 1.0163$	
Hort-Thom	$\frac{MSE\left(\tilde{t}_{y\pi(1)}\right)}{MSE\left(\hat{t}_{y\pi}\right)} = 0.2110$		$\frac{MSE\left(\tilde{t}_{y\pi(1)}\right)}{MSE\left(\hat{t}_{y\pi}\right)} = 0.1817$		$\frac{MSE\left(\tilde{t}_{y\pi(1)}\right)}{MSE\left(\hat{t}_{y\pi}\right)} = 0.2176$	
family	$\frac{MSE\left(\tilde{t}_{y\pi(2)}\right)}{MSE\left(\hat{t}_{y\pi}\right)} = 0.1986$		$\frac{MSE\left(\tilde{t}_{y\pi(2)}\right)}{MSE\left(\hat{t}_{y\pi}\right)} = 0.1688$		$\frac{MSE\left(\tilde{t}_{y\pi(2)}\right)}{MSE\left(\hat{t}_{y\pi}\right)} = 0.2044$	
	$\frac{MSE\left(\tilde{t}_{y\pi(1)}\right)}{MSE\left(\tilde{t}_{y\pi(2)}\right)} = 1.0624$		$\frac{MSE\left(\tilde{t}_{y\pi(1)}\right)}{MSE\left(\tilde{t}_{y\pi(2)}\right)} = 1.0766$		$\frac{MSE\left(\tilde{t}_{y\pi(1)}\right)}{MSE\left(\tilde{t}_{y\pi(2)}\right)} = 1.0644$	

Table 7: Empirical results from cars93 for Stratum (5): When typecode=5

Estimator	$n_h = 2$		$n_h = 3$		$n_h = 4$	
	t_{yh}	S^2_{yh}	t_{yh}	S^2_{yh}	t_{yh}	S^2_{yh}
$\hat{T}_R$	197.0919	23.2645	197.0515	13.2744	197.0319	8.2891
	(0.0919)	(**23.2730**)	(0.0515)	(**13.2770**)	(0.0319)	(**8.2901**)
$\hat{T}_{R(1)}$	197	22.6543	197	11.1726	197	7.1120
$\hat{T}_{R(2)}$	197	22.6484	197	11.1820	197	7.1201
$\hat{t}_{y\pi}$	197	66.5000	197	38.0000	197	23.75
$\widetilde{t}_{y\pi(1)}$	197	22.6953	197	11.2614	197	7.2046
$\widetilde{t}_{y\pi(2)}$	197	22.6484	197	11.1820	197	7.1201
$\hat{T}_R$ Family	$\frac{MSE(\hat{T}_{R(1)})}{MSE(\hat{T}_R)} = 0.9734$		$\frac{MSE(\hat{T}_{R(1)})}{MSE(\hat{T}_R)} = 0.8415$		$\frac{MSE(\hat{T}_{R(1)})}{MSE(\hat{T}_R)} = 0.8579$	
	$\frac{MSE(\hat{T}_{R(2)})}{MSE(\hat{T}_R)} = 0.9732$		$\frac{MSE(\hat{T}_{R(2)})}{MSE(\hat{T}_R)} = 0.8422$		$\frac{MSE(\hat{T}_{R(2)})}{MSE(\hat{T}_R)} = 0.8589$	
	$\frac{MSE(\hat{T}_{R(1)})}{MSE(\hat{T}_{R(2)})} = 1.0003$		$\frac{MSE(\hat{T}_{R(1)})}{MSE(\hat{T}_{R(2)})} = 0.9992$		$\frac{MSE(\hat{T}_{R(1)})}{MSE(\hat{T}_{R(2)})} = 0.9989$	
Hort-Thom	$\frac{MSE(\widetilde{t}_{y\pi(1)})}{MSE(\hat{t}_{y\pi})} = 0.3413$		$\frac{MSE(\widetilde{t}_{y\pi(1)})}{MSE(\hat{t}_{y\pi})} = 0.2964$		$\frac{MSE(\widetilde{t}_{y\pi(1)})}{MSE(\hat{t}_{y\pi})} = 0.3034$	
family	$\frac{MSE(\widetilde{t}_{y\pi(2)})}{MSE(\hat{t}_{y\pi})} = 0.3406$		$\frac{MSE(\widetilde{t}_{y\pi(2)})}{MSE(\hat{t}_{y\pi})} = 0.2943$		$\frac{MSE(\widetilde{t}_{y\pi(2)})}{MSE(\hat{t}_{y\pi})} = 0.2998$	
	$\frac{MSE(\widetilde{t}_{y\pi(1)})}{MSE(\widetilde{t}_{y\pi(2)})} = 1.0021$		$\frac{MSE(\widetilde{t}_{y\pi(1)})}{MSE(\widetilde{t}_{y\pi(2)})} = 1.0071$		$\frac{MSE(\widetilde{t}_{y\pi(1)})}{MSE(\widetilde{t}_{y\pi(2)})} = 1.0119$	

Table 8: Empirical results from cars93 for Stratum (6): When typecode=6

Estimator	$n = 12$		$n = 18$		$n = 24$	
	t_y	S^2_y	t_y	S^2_y	t_y	S^2_y
$\hat{T}_R$	2678.8172	453844.48	2676.2744	178628.06	2674.9857	88260.276
	(6.8171677)	(**453863.13**)	(4.2743444)	(**178635.6**)	(2.9856758)	(**88264.047**)
$\hat{T}_{R(1)}$	2672	431418.52	2672	117843.17	2672	56576.944
$\hat{T}_{R(2)}$	2672	439877.98	2672	117935.4	2672	56173.604
$\hat{t}_{y\pi}$	2672	1575220	2672	621987.81	2672	308599.03
$\widetilde{t}_{y\pi(1)}$	2672	456121.5	2672	125146.15	2672	59655.399
$\widetilde{t}_{y\pi(2)}$	2672	439877.98	2672	117935.4	2672	56173.604
$\hat{T}_R$ Family	$\frac{MSE(\hat{T}_{R(1)})}{MSE(\hat{T}_R)} = 0.9506$		$\frac{MSE(\hat{T}_{R(1)})}{MSE(\hat{T}_R)} = 0.6597$		$\frac{MSE(\hat{T}_{R(1)})}{MSE(\hat{T}_R)} = 0.6410$	
	$\frac{MSE(\hat{T}_{R(2)})}{MSE(\hat{T}_R)} = 0.9692$		$\frac{MSE(\hat{T}_{R(2)})}{MSE(\hat{T}_R)} = 0.6602$		$\frac{MSE(\hat{T}_{R(2)})}{MSE(\hat{T}_R)} = 0.6364$	
	$\frac{MSE(\hat{T}_{R(1)})}{MSE(\hat{T}_{R(2)})} = 0.9808$		$\frac{MSE(\hat{T}_{R(1)})}{MSE(\hat{T}_{R(2)})} = 0.9992$		$\frac{MSE(\hat{T}_{R(1)})}{MSE(\hat{T}_{R(2)})} = 1.0072$	
Hort-Thom	$\frac{MSE(\widetilde{t}_{y\pi(1)})}{MSE(\hat{t}_{y\pi})} = 0.2896$		$\frac{MSE(\widetilde{t}_{y\pi(1)})}{MSE(\hat{t}_{y\pi})} = 0.2012$		$\frac{MSE(\widetilde{t}_{y\pi(1)})}{MSE(\hat{t}_{y\pi})} = 0.1933$	
family	$\frac{MSE(\widetilde{t}_{y\pi(2)})}{MSE(\hat{t}_{y\pi})} = 0.2793$		$\frac{MSE(\widetilde{t}_{y\pi(2)})}{MSE(\hat{t}_{y\pi})} = 0.1896$		$\frac{MSE(\widetilde{t}_{y\pi(2)})}{MSE(\hat{t}_{y\pi})} = 0.1820$	
	$\frac{MSE(\widetilde{t}_{y\pi(1)})}{MSE(\widetilde{t}_{y\pi(2)})} = 1.0369$		$\frac{MSE(\widetilde{t}_{y\pi(1)})}{MSE(\widetilde{t}_{y\pi(2)})} = 1.0611$		$\frac{MSE(\widetilde{t}_{y\pi(1)})}{MSE(\widetilde{t}_{y\pi(2)})} = 1.0620$	

Table 9: Empirical results from cars93 based on stratified sampling designs

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