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Statistical Properties and Estimation of Power-Transmuted Inverse Rayleigh Distribution

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ABSTRACT

A three-parameter continuous distribution is constructed, using a power transformation related to the transmuted inverse Rayleigh (TIR) distribution. A comprehensive account of the statistical properties is provided, including the following: the quantile function, moments, incomplete moments, mean residual life function and Rényi entropy. Three classical procedures for estimating population parameters are analysed. A simulation study is provided to compare the performance of different estimates. Finally, a real data application is used to illustrate the usefulness of the recommended distribution in modelling real data.

Key words: transmuted inverse Rayleigh, mean residual life function, maximum likelihood, percentiles.

1. Introduction

Trayer (1964) introduced an important model for lifetime analysis, known as the inverse Rayleigh **(IR)** distribution. The probability density function **(pdf)** and the cumulative distribution function **(cdf)** of a random variable *Y* have the IR distribution with scale parameter θ and are defined by:

$$f_{IR}(y;\theta) = 2\theta y^{-3} e^{-\theta y^{-2}}, \; ; y > 0, \; \theta > 0.$$

and

$$F_{IR}(y;\theta) = e^{-\theta y^{-2}}; \quad y, \theta > 0.$$

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Voda (1972) studied some properties of the maximum likelihood (ML) of its scale parameter. Gharraph (1993) provided closed-form expressions for the mean, harmonic mean, geometric mean, mode and the median of the IR distribution. A lot of works have been done in the literature upon estimation of the IR distribution; the reader can refer to Mohsin and Shahbaz (2005), Soliman et al. (2010), Dey (2012), Sindhu et al. (2013), Fan (2015), Rasheed et al. (2015), Panwar et al. (2015), Rasheed and Aref (2016).

In recent years, a number of extensions for the IR distribution have been developed using different methods of generalization by several authors, see, for example, beta IR distribution (Leao et al.; 2013), transmuted IR **(TIR)** distribution (Ahmed et al. 2014), modified IR **(MIR)** distribution (Khan; 2014), transmuted modified IR **(TMIR)** distribution (Khan and King; 2015) transmuted exponentiated IR **(TEIR)** distribution (Haq; 2015), Kumaraswamy exponentiated IR **(KEIR)** distribution (Haq; 2016), weighted IR distribution (Fatima and Ahmad; 2017) and odd Fréchet IR distribution (Elgarhy and Alrajhi; 2018).

The power transformation **(PT)** methodology has been used in many statistical aspects, although PT has been first proposed by Box and Cox (1964). One of the most important uses of the PT methodology is developing new distributions out of well-known distributions by adding an additional parameter, which gives several desirable properties and more flexibility in the form of the hazard rate and density functions. Also, it offers a more flexible model that can describe different types of real data. So, our objective in this study is developing a power transmuted inverse Rayleigh **(PTIR)** distribution out of the TIR distribution via the PT technique. Several statistical properties and different methods of estimation are discussed to obtain the point estimators regarding the proposed distribution.

This paper is organized as follows. Section 2 introduces the formation of the PTIR model. The structural characteristics of the PTIR distribution are studied in Section 3. Section 4 discusses parameter estimators for the PTIR distribution based on ML, least squares and percentile methods. Simulation schemes are performed in Section 5. A real life data application illustrates the potential of the PTIR distribution compared with some other distributions in Section 6. The article ends with some concluding remarks.

2. Model Formulation

The TIR distribution is a generalization of the IR distribution using the quadratic rank transmutation map (see Ahmed et al. 2014). The cdf of the TIR distribution is given by:

$$F_{TIR}(y;\theta,\lambda)=e^{-\theta y^{-2}}(1+\lambda-\lambda e^{-\theta y^{-2}}),\ \ ;\theta>0,|\lambda|\le 1,\ \ y>0.$$

Here, we propose a new extension of the TIR distribution by considering $X = Y^{1/\beta}$, where the random variable Y follows the TIR distribution with parameters θ and λ. The distribution function of a random variable X has the PTIR distribution and is defined as follows:

$$F_{PTIR}(x;\theta,\lambda,\beta) = e^{-\theta x^{-2\beta}}(1+\lambda-\lambda e^{-\theta x^{-2\beta}});\ \theta,\beta>0, |\lambda|\le 1,\ x>0. \tag{1}$$

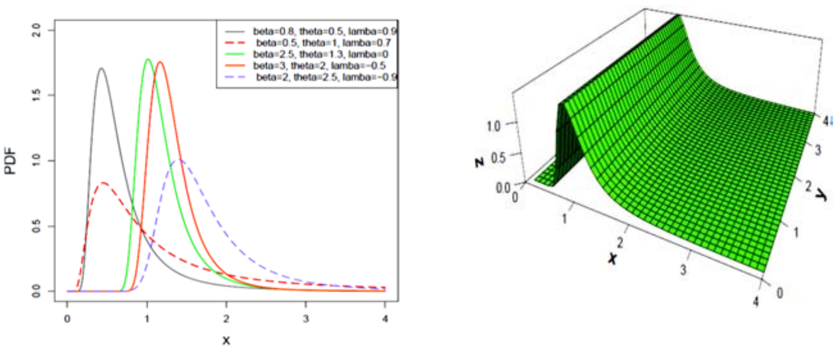
The pdf of the PTIR distribution corresponding to (1) is given by

$$f_{PTIR}(x;\theta,\lambda,\beta) = \frac{2\theta\beta}{x^{2\beta+1}} e^{-\theta x^{-2\beta}}(1+\lambda-2\lambda e^{-\theta x^{-2\beta}});\ \theta,\beta>0, |\lambda|\le 1,\ x>0. \tag{2}$$

A random variable X that follows the distribution (2) is denoted by $X \sim (\theta,\lambda,\beta)$. Two special sub models can be obtained from (2) as follows.

- For $\lambda = 0$, the pdf (2) reduces to a power IR **(PIR)** distribution as a new model.
- For $\lambda = 0$ and $\beta = 1$, the pdf (2) reduces to the IR distribution.

Some descriptive pdf plots of X have the PTIR distribution, which is illustrated in Figure 1 for some specific values of parameters.



(a) (b)

Figure 1. The pdf plots of the PTIR distribution (a) for some choices of parameters (b) for $\beta=1.5,\ \theta=1.0,\ \lambda=0.5$

From Figure 1, it can be shown that the shape of the PTIR distribution is unimodal. It can also be said that the distribution is positively skewed.

Furthermore, the survival function and the hazard rate function **(hrf)** are given, respectively, by

$$S_{PTIR}(x;\theta,\lambda,\beta) = 1 - e^{-\theta x^{-2\beta}}(1+\lambda-\lambda e^{-\theta x^{-2\beta}}),$$

and

$$h_{PTIR}(x;\theta,\lambda,\beta)=2\theta\beta e^{-\theta x^{-2\beta}}x^{-(2\beta+1)}(1+\lambda-2\lambda e^{-\theta x^{-2\beta}})\left(1-e^{-\theta x^{-2\beta}}(1+\lambda-\lambda e^{-\theta x^{-2\beta}})\right)^{-1}.$$

Some descriptive hrf plots of X are illustrated in Figure 2 for some specific values of parameters.

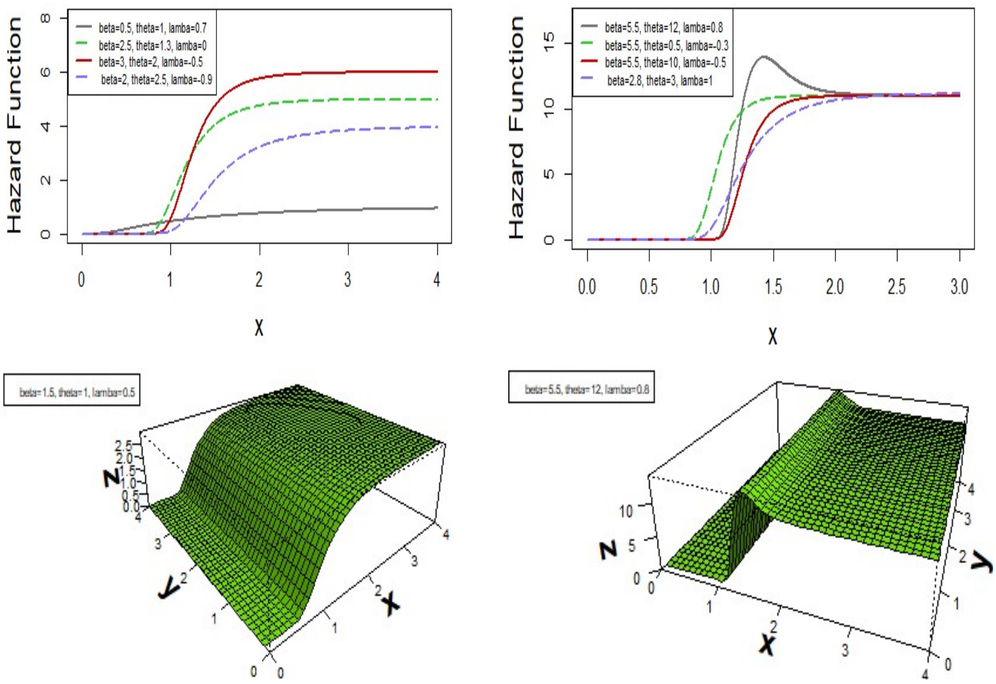


Figure 2. The hrf plots of the PTIR distribution for some choices of parameters

From Figure 2, it can be shown that the plots at several selected values of the parameters of hrfs have an increasing tendency.

The reversed hrf and cumulative hrf are given, respectively, by:

$$r_{PTIR}(x;\theta,\lambda,\beta)=2\theta\beta x^{-2\beta-1},$$

and

$$H_{PTIR}(x;\theta,\lambda,\beta)=-\ln\left(1-e^{-\theta x^{-2\beta}}(1+\lambda-\lambda e^{-\theta x^{-2\beta}})\right).$$

3. Some Structural Properties

In this section some structural properties are provided.

3.1. Quantile Function

The quantile function of the PTIR distribution, say $Q(u) = F^{-1}(u)$ of X can be obtained by inverting (1) as follows:

$$\lambda\left(e^{-\theta/(Q(u))^{2\beta}}\right)^2 - (1+\lambda)e^{-\theta/(Q(u))^{2\beta}} + u = 0, \tag{3}$$

Factorizing (3) leads to

$$Q(u) = \left[-\theta\Big/\ln\left[\frac{(1+\lambda) - \sqrt{(1+\lambda)^2 - 4\lambda u}}{2\lambda}\right]\right]^{\frac{1}{2\beta}}, \tag{4}$$

where u has a uniform random variable on (0, 1). Also, (4) can be used in simulating PTIR random variables when the parameters θ, λ and β are known. Median (***m***) of the distribution is obtained by setting u = 0.5 in (4). Also, the first and third quantiles can be obtained by setting u = 0.25 and u = 0.75 in (4).

3.2. Moments of the PTIR Distribution

Moments are used to understand various characteristics of a frequency distribution. They have been applied in order to obtain mean, variance, in addition to some measures, such as skewness and kurtosis.

The r^{th} moment of X has the PTIR distribution and is derived by using (2) as follows:

$$E(X^r) = \int_0^\infty x^r \left[\frac{2\theta\beta}{x^{2\beta+1}} e^{\frac{-\theta}{x^{2\beta}}} (1 + \lambda - 2\lambda e^{\frac{-\theta}{x^{2\beta}}})\right] dx. \tag{5}$$

Let $z = \theta x^{-2\beta}$, then the r^{th} moment of the PTIR distribution is given by

$$E(X^r) = \theta^{\frac{r}{2\beta}} \left[\left(\int_0^\infty z^{\frac{-r}{2\beta}} e^{-z} + \lambda \int_0^\infty z^{\frac{-r}{2\beta}} e^{-z} - 2\lambda \int_0^\infty z^{\frac{-r}{2\beta}} e^{-2z}\right) dz\right],$$

which is the gamma function, so the r^{th} moment can be formed as follows:

$$E(X^r) = \theta^{\frac{r}{2\beta}} \Gamma\left(1 - \frac{r}{2\beta}\right)\left[1 + \lambda - 2^{\frac{r}{2\beta}} \lambda\right], \; r < 2\beta, \; r = 1, 2, 3, \ldots$$

Hence, the mean and variance of the PTIR distribution are given, respectively, by

$$\mu = \theta^{\frac{1}{2\beta}} \Gamma\left(1 - \frac{1}{2\beta}\right)\left[1 + \lambda - 2^{\frac{1}{2\beta}} \lambda\right], \; \beta > 0.5,$$

and

$$Var(X) = \theta^{\frac{1}{\beta}}\Gamma\left(1-\frac{1}{\beta}\right)\left[1+\lambda-2^{\frac{1}{\beta}}\lambda\right] - \left(\theta^{\frac{1}{2\beta}}\Gamma\left(1-\frac{1}{2\beta}\right)\left[1+\lambda-2^{\frac{1}{2\beta}}\lambda\right]\right)^2, \ \beta > 1.$$

Measurement of skewness and kurtosis of the distribution is obtained from complete moments using the well-known relationship. Plots of the PTIR skewness and kurtosis for some selected values are displayed in Figure 3.

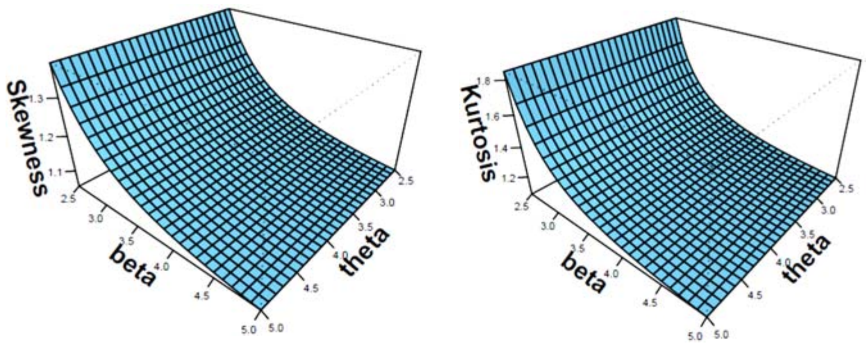


Figure 3. The skewness and kurtosis of the PTIR for $\lambda = 0.5$ and different values of θ and β

From Figure 3, it can be seen that both the skewness and the kurtosis are decreasing functions of θ, λ and β.

3.3. Incomplete Moments

The answer to many important questions in economics requires more than just knowing the mean of a distribution, but its shape as well. This is obvious not only in the study of econometrics and income distribution, but in other areas as well (see Butler and McDonald; 1989).

The s^{th} incomplete moment of a random variable X has the PTIR distribution and is obtained as follows:

$$£_{(s)}(t) = \int_0^t x^s f_{PTIR}(x;\theta,\lambda,\beta)dx = \int_0^t x^s \left[2\theta\beta x^{-(2\beta+1)} e^{\frac{-\theta}{x^{2\beta}}}(1+\lambda-2\lambda e^{\frac{-\theta}{x^{2\beta}}})\right]dx.$$

Let $z = \theta x^{-2\beta}$, then the s^{th} incomplete moment of the PTIR distribution is given by:

$$£_{(s)}(t) = \theta^{\frac{s}{2\beta}} \int_{\frac{\theta}{t^{2\beta}}}^{\infty} \left(z^{\frac{-s}{2\beta}} e^{-z} + \lambda z^{\frac{-s}{2\beta}} e^{-z} - 2\lambda z^{-\frac{s}{2\beta}} e^{-2z} \right) dz \,,$$

which is the upper incomplete moments, so

$$£_{(s)}(t) = \theta^{\frac{s}{2\beta}} \left[\Gamma\left(1 - \frac{s}{2\beta}, \frac{\theta}{t^{2\beta}}\right) + \lambda\Gamma\left(1 - \frac{s}{2\beta}, \frac{\theta}{t^{2\beta}}\right) - 2^{\frac{s}{2\beta}} \lambda\Gamma\left(1 - \frac{s}{2\beta}, \frac{2\theta}{t^{2\beta}}\right) \right], \tag{6}$$

where $\Gamma(.,x)$ is the upper incomplete moments. The first incomplete moment can be obtained by setting $s = 1$ in (6). The mean deviation about the mean (μ), denoted by δ_1, and the mean deviation about the median, denoted by δ_2, can be obtained, respectively, as follows:

$$\begin{aligned}\delta_1 &= 2\mu F_{PTIR}(\mu) - 2£_{(1)}(\mu)\\ &= 2\left(\theta^{\frac{1}{2\beta}}\Gamma\left(1-\frac{1}{2\beta}\right)\left[1+\lambda-2^{\frac{1}{2\beta}}\lambda\right]\right)\left(e^{\frac{-\theta}{\mu^{2\beta}}}(1+\lambda-\lambda e^{\frac{-\theta}{\mu^{2\beta}}})\right)\\ &\quad -2\theta^{\frac{1}{2\beta}}\left[\Gamma\left(1-\frac{1}{2\beta},\frac{\theta}{\mu^{2\beta}}\right)+\lambda\Gamma\left(1-\frac{1}{2\beta},\frac{\theta}{\mu^{2\beta}}\right)-2^{\frac{1}{2\beta}}\lambda\Gamma\left(1-\frac{1}{2\beta},\frac{2\theta}{\mu^{2\beta}}\right)\right].\end{aligned}$$

$$\begin{aligned}\delta_2 &= \mu - 2£_{(1)}(m).\\ &= \left(\theta^{\frac{1}{2\beta}}\Gamma\left(1-\frac{1}{2\beta}\right)\left[1+\lambda-2^{\frac{1}{2\beta}}\lambda\right]\right) - 2\theta^{\frac{1}{2\beta}}\left[\Gamma\left(1-\frac{1}{2\beta},\frac{\theta}{m^{2\beta}}\right) + \lambda\Gamma\left(1-\frac{1}{2\beta},\frac{\theta}{m^{2\beta}}\right) - 2^{\frac{1}{2\beta}}\lambda\Gamma\left(1-\frac{1}{2\beta},\frac{2\theta}{m^{2\beta}}\right)\right].\end{aligned}$$

Lorenz curve of the PTIR distribution is obtained as follows:

$$L_F(t) = \frac{£_{(1)}(t)}{E(T)} = \frac{\left[\Gamma\left(1-\frac{1}{2\beta},\frac{\theta}{t^{2\beta}}\right)+\lambda\Gamma\left(1-\frac{1}{2\beta},\frac{\theta}{t^{2\beta}}\right)-2^{\frac{1}{2\beta}}\lambda\Gamma\left(1-\frac{1}{2\beta},\frac{2\theta}{t^{2\beta}}\right)\right]}{\Gamma\left(1-\frac{1}{2\beta}\right)\left[1+\lambda-2^{\frac{1}{2\beta}}\lambda\right]}.$$

Bonferroni curve is obtained as follows:

$$B_F(t) = \frac{L_F(t)}{F(t)} = \frac{\left[\Gamma\left(1-\frac{1}{2\beta},\frac{\theta}{t^{2\beta}}\right)+\lambda\Gamma\left(1-\frac{1}{2\beta},\frac{\theta}{t^{2\beta}}\right)-2^{\frac{1}{2\beta}}\lambda\Gamma\left(1-\frac{1}{2\beta},\frac{2\theta}{t^{2\beta}}\right)\right]}{\Gamma\left(1-\frac{1}{2\beta}\right)\left[1+\lambda-2^{\frac{1}{2\beta}}\lambda\right]\left(e^{\frac{-\theta}{t^{2\beta}}}(1+\lambda-\lambda e^{\frac{-\theta}{t^{2\beta}}})\right)}.$$

3.4. Mean Residual Life Function

Mean residual life **(MRL)** function has been used in estimating time to failure for one or more existing and future failure modes. For an example nowadays MRL or remaining useful life is recognized as a key feature in maintenance strategies, while the real prognostic systems are rare in industry, even in mining industry. The n^{th} moment of the residual life of X is given by

$$m_n(t) = E\left((X - t)^n \left|X > t\right.\right) = \frac{1}{S(t)} \int_t^{\infty} (X - t)^n f(x) dx$$

Using the binomial expansion, for the term $(X\text{-}t)^n$, then $m_n(t)$ will be

$$m_n(t) = E\left((X - t)^n \left|X > t\right.\right) = \frac{\sum_{j=0}^{n} \binom{n}{j} (-t)^{n-j}}{S(t)} \int_t^{\infty} x^j f(x) \, d\, x. \tag{7}$$

The n^{th} moment of the residual life is obtained by substituting (2) in (7) and using $z = \theta x^{-2\beta}$, which leads to

$$m_n(t) = \frac{1}{S_{PTIR}(t)} \sum_{j=0}^{n} \binom{n}{j} (-t)^{n-j} (\theta)^{\frac{j}{2\beta}} \int_0^{\frac{\theta}{t^{2\beta}}} z^{\frac{-j}{2\beta}} \left[e^{-z}(1 + \lambda - 2\lambda e^{-z})\right] dz,$$

which is the lower incomplete gamma function, so the n^{th} moment of the PTIR distribution takes the following form:

$$m_n(t) = \frac{1}{S_{PTIR}(t)} \sum_{j=0}^{n} \binom{n}{j} (-t)^{n-j} (\theta)^{\frac{j}{2\beta}} \left[\gamma\left(1 - \frac{j}{2\beta}, \frac{\theta}{t^{2\beta}}\right) \lambda\gamma\left(1 - \frac{j}{2\beta}, \frac{\theta}{t^{2\beta}}\right) - 2^{\frac{j}{2\beta}} \lambda\gamma\left(1 - \frac{j}{2\beta}, \frac{2\theta}{t^{2\beta}}\right)\right],$$

where $\gamma(.,x)$ is the lower incomplete moments.

3.5. Rényi Entropy

The Rényi entropy is used to quantify the diversity, uncertainty or randomness of a system; it has various fields of application such as ecology, statistics. Also, it is important in quantum information, where it can be used as a measure of entanglement.

$$I_R(X) = \frac{1}{1-\rho} \ln\left\{\int_R f^{\rho}(x) dx\right\},$$

where for some real values $\rho > 0$ and $\rho \neq 1$, the entropy of the PTIR random variable X has the pdf (2) and is given by

$$I_R(X) = \frac{1}{1-\rho} \ln\left\{\int_0^{\infty} \frac{(2\theta\beta)^{\rho}}{x^{\rho(2\beta+1)}} e^{\frac{-\rho\theta}{x^{2\beta}}} \left(1 + \lambda - 2\lambda e^{\frac{-\theta}{x^{2\beta}}}\right)^{\rho}\right\}.$$

So,

$$I_R(X) = \frac{1}{1-\rho} \ln\left\{\int_0^{\infty} \frac{(2\theta\beta)^{\rho}}{x^{\rho(2\beta+1)}} e^{\frac{-\rho\theta}{x^{2\beta}}} (1 + \lambda)^{\rho} \left(1 - \frac{2\lambda}{1+\lambda} e^{\frac{-\theta}{x^{2\beta}}}\right)^{\rho}\right\}.$$

By using binomial expansion and after simplification, the Rényi entropy is

$$I_R(X) = \frac{1}{1-\rho} \ln\left\{ \theta^{\rho} (2\beta)^{\rho-1} (1+\lambda)^{\rho} \sum_{j=0}^{\infty} \binom{\rho}{j} (-1)^j \left(\frac{2\lambda}{1+\lambda}\right)^j \Gamma\left(\frac{\frac{\rho(2\beta+1)-1}{2\beta}}{\left[\theta(\rho+j)\right]^{\frac{\rho(2\beta+1)-1}{2\beta}}} \right) \right\}.$$

4. Parameter Estimation

In this section, parameter estimators are obtained for the PTIR distribution based on **ML**, least squares **(LS)** and percentiles **(PR)** methods.

4.1. Maximum Likelihood Estimators

The ML estimator procedure is considered to estimate the population parameters of the PTIR distribution. The likelihood function is given by

$$L = (2\theta\beta)^n \prod_{i=1}^{n} x_i^{-(2\beta+1)} e^{\frac{-\theta}{x_i^{2\beta}}} (1+\lambda-2\lambda e^{\frac{-\theta}{x_i^{2\beta}}}).$$

The log likelihood function is given by

$$\ln L = n\ln 2\beta + n\ln\theta - (2\beta+1)\sum_{i=1}^{n}\ln x_i - \sum_{i=1}^{n}\frac{\theta}{x_i^{2\beta}} + \sum_{i=1}^{n}\ln(1+\lambda-2\lambda e^{\frac{-\theta}{x_i^{2\beta}}}). \tag{8}$$

Therefore, the ML estimators of θ, λ and β, which maximizes (8), satisfy the following normal equations.

$$\frac{\partial \ln L}{\partial\theta} = \frac{n}{\theta} - \sum_{i=1}^{n} x_i^{-2\beta} + 2\lambda\sum_{i=1}^{n}\frac{e^{-\theta x_i^{-2\beta}}}{x_i^{2\beta}(1+\lambda-2\lambda e^{-\theta x_i^{-2\beta}})}, \tag{9}$$

$$\frac{\partial \ln L}{\partial\lambda} = \sum_{i=1}^{n}\frac{1-2e^{-\theta x_i^{-2\beta}}}{(1+\lambda-2\lambda e^{-\theta x_i^{-2\beta}})}, \tag{10}$$

and

$$\frac{\partial \ln L}{\partial\beta} = \frac{n}{\beta} - 2\sum_{i=1}^{n}\ln x_i + 2\theta\sum_{i=1}^{n} x_i^{-2\beta}\ln x_i + 4\lambda\theta\sum_{i=1}^{n}\frac{x_i^{-2\beta} e^{\frac{-\theta}{x_i^{2\beta}}}\ln x_i}{(1+\lambda-2\lambda e^{\frac{-\theta}{x^{2\beta}}})}. \tag{11}$$

Then ML estimators of the parameters θ, λ and β denoted by $\hat{\theta}, \hat{\lambda}$ and $\hat{\beta}$ are determined by solving numerically the non-linear Equations (9), (10) and (11) after setting them equal to zeros simultaneously.

4.2. Least Squares Estimators

Let $X_1,...,X_n$ be a random sample of size n from the PTIR distribution. Suppose that $X_{(1)},...,X_{(n)}$ denotes the corresponding ordered sample. Therefore, the LS estimators of θ, λ and β say, $\tilde{\theta}, \tilde{\lambda}$ and $\tilde{\beta}$ respectively, can be obtained by minimizing the following function with respect to θ, λ and β.

$$LS = \sum_{i=1}^{n}\left[\left(e^{\frac{-\theta}{x_{(i)}^{2\beta}}} + \lambda e^{\frac{-\theta}{x_{(i)}^{2\beta}}} - \lambda e^{\frac{-2\theta}{x_{(i)}^{2\beta}}}\right) - \frac{i}{n+1}\right]^2. \tag{12}$$

Differentiating (12) with respect to θ, λ and β respectively, and equating with zeros, allows the LS estimators $\tilde{\theta}, \tilde{\lambda}$ and $\tilde{\beta}$ to be obtained.

4.3. Percentiles Estimators

Let $X_1,\ldots,X_n$ be a random sample of size n from the PTIR distribution. Suppose that $X_{(1)},\ldots,X_{(n)}$ denotes some estimates of $F\left(x_{(i)};\theta,\lambda,\beta\right)$ then the estimates of θ,λ and β can be obtained by minimizing the following equation:

$$PR = \sum_{i=1}^{n}\left[x_{(i)} - \left[\frac{-1}{\theta}\ln\left[\frac{(1+\lambda)-\sqrt{(1+\lambda)^2-4\lambda p_i}}{2\lambda}\right]\right]^{\frac{-1}{2\beta}}\right]^2, \tag{13}$$

with respect to θ,λ and β. In percentiles method, we estimate the unknown parameters θ,λ and β by equating the sample percentile points with the corresponding population percentile points, where $p_i = i/n+1$ is the estimates for $F\left(x_{(i)};\theta,\lambda,\beta\right)$. Then the PR estimators of θ,λ and β say, $\bar{\theta},\bar{\lambda}$ and $\bar{\beta}$ respectively, can be obtained by minimizing (13) with respect to θ,λ and β.

5. Simulation Studies

A numerical study is performed to evaluate and compare the performance of the estimates with respect to their absolute biases **(ABs)**, and mean square errors **(MSEs)** for different sample sizes and for different parameter values. The numerical procedures are described as follows:

Step (1): A random sample $X_1,\ldots,X_n$ of sizes n=10,20,30,100 is selected. These random samples are generated from the PTIR distribution by using the transformation (4).

Step (2): Four different set values of the parameters are selected as:

Set 1 = ($\theta=1.0$, $\lambda=0.5$, $\beta=0.5$), **Set 2** =($\theta=1.0$, $\lambda=0.5$, $\beta=1.5$), **Set 3** = ($\theta=1.0$, $\lambda=0.5$, $\beta=2$) and **Set 4** = ($\theta=0.5$, $\lambda=-0.7$, $\beta=1$).

Step (3): The ML, LS and PR estimates of θ,λ and β are computed for each set of parameters and for each sample size.

Step (4): Steps from 1 to 3 are repeated 5000 times for each sample size and for selected sets of parameters. Then, the ABs and MSEs of the ML, LS, PR estimates are computed.

Table 1. ABs and MSEs of the PTIR distribution for Set 1, Set 2, Set 3 and Set 4

n	Method	Properties	Set 1			Set 2		
			θ	λ	β	θ	λ	β
10	ML	MSE	0.0022	0.0055	0.0002	0.0017	0.0041	0.0013
		AB	0.0425	0.0667	0.0132	0.0365	0.0540	0.0319
	LS	MSE	0.0048	0.0122	0.0006	0.0005	0.0027	0.0007
		AB	0.0695	0.1106	0.0241	0.0226	0.0516	0.0259
	PR	MSE	0.0121	0.0397	0.0008	0.0060	0.0175	0.0038
		AB	0.1101	0.1993	0.0282	0.0774	0.1325	0.0619
20	ML	MSE	0.0003	0.0006	0.0000	0.0010	0.0025	0.0007
		AB	0.0043	0.0016	0.0013	0.0282	0.0442	0.0240
	LS	MSE	0.0015	0.0035	0.0002	0.0005	0.0016	0.0007
		AB	0.0386	0.0595	0.0128	0.0219	0.0396	0.0258
	PR	MSE	0.0013	0.0039	0.0001	0.0031	0.0088	0.0018
		AB	0.0367	0.0622	0.0095	0.0554	0.0939	0.0428
30	ML	MSE	0.0002	0.0004	0.0000	0.0005	0.0013	0.0005
		AB	0.0027	0.0010	0.0009	0.0177	0.0292	0.0181
	LS	MSE	0.0011	0.0022	0.0001	0.0003	0.0011	0.0002
		AB	0.0333	0.0470	0.0116	0.0160	0.0326	0.0155
	PR	MSE	0.0006	0.0014	0.0000	0.0013	0.0038	0.0008
		AB	0.0244	0.0375	0.0060	0.0360	0.0618	0.0275
100	ML	MSE	0.0001	0.0002	0.0000	0.0001	0.0002	0.0001
		AB	0.0020	0.0058	0.0005	0.0035	0.0058	0.0048
	LS	MSE	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
		AB	0.0068	0.0070	0.0029	0.0009	0.0050	0.0006
	PR	MSE	0.0001	0.0003	0.0000	0.0005	0.0014	0.0003
		AB	0.0098	0.0170	0.0040	0.0218	0.0373	0.0187

(cont.)

N	Method	Properties	Set 3			Set 4		
			θ	λ	β	θ	λ	β
10	ML	MSE	0.0006	0.0013	0.0002	0.0002	0.0018	0.0000
		AB	0.0065	0.0176	0.0091	0.0053	0.0226	0.0023
	LS	MSE	0.0156	0.0287	0.0044	0.0008	0.0071	0.0000
		AB	0.1248	0.1695	0.0663	0.0277	0.0844	0.0068
	PR	MSE	0.0129	0.0222	0.0025	0.0004	0.0051	0.0006
		AB	0.1135	0.1489	0.0500	0.0189	0.0712	0.0240
20	ML	MSE	0.0003	0.0006	0.0001	0.0001	0.0011	0.0000
		AB	0.0043	0.0017	0.0015	0.0040	0.0166	0.0007
	LS	MSE	0.0054	0.0112	0.0016	0.0007	0.0061	0.0000
		AB	0.0734	0.1060	0.0401	0.0266	0.0784	0.0060
	PR	MSE	0.0045	0.0093	0.0010	0.0001	0.0016	0.0001
		AB	0.0668	0.0967	0.0309	0.0115	0.0400	0.0108
30	ML	MSE	0.0002	0.0005	0.0001	0.0001	0.0007	0.0000
		AB	0.0057	0.0076	0.0005	0.0027	0.0105	0.0009
	LS	MSE	0.0040	0.0079	0.0012	0.0005	0.0041	0.0000
		AB	0.0632	0.0887	0.0340	0.0219	0.0637	0.0028
	PR	MSE	0.0030	0.0059	0.0005	0.0001	0.0008	0.0000
		AB	0.0550	0.0766	0.0232	0.0086	0.0291	0.0061
100	ML	MSE	0.0001	0.0002	0.0000	0.0000	0.0002	0.0000
		AB	0.0077	0.0101	0.0038	0.0009	0.0049	0.0015
	LS	MSE	0.0004	0.0007	0.0001	0.0000	0.0002	0.0000
		AB	0.0206	0.0270	0.0106	0.0045	0.0124	0.0007
	PR	MSE	0.0002	0.0003	0.0000	0.0000	0.0001	0.0000
		AB	0.0132	0.0170	0.0052	0.0046	0.0106	0.0002

The following conclusions can be observed on the properties of estimated parameters (see Table 1).

- The MSEs of the ML, LS and PR estimates decrease as the sample sizes increase for selected sets of parameters.
- The MSEs for the ML estimates of θ, λ and β take the smallest values compared to the MSEs of the LS and PR estimates in almost all of the cases.
- The ABs of the ML estimates are smaller than the ABs of the PR and LS estimates in almost all of the cases especially at small and moderate sample sizes.
- The ABs and MSEs of the ML, PR and LS estimates of β are smaller than the corresponding estimates of θ and λ in almost all of the cases.

6. Applications to Real Data

In this section, a real data analysis is provided in order to assess the goodness-of-fit of the PTIR model comparing with some known distributions such as IR, TIR, PIR, MIR, TMIR, KEIR.

In order to compare the models, criteria like maximized likelihood ($-2\hat{\ell}$), Akaike information criterion (**AIC**), consistent AIC (**CAIC**), Bayesian information criterion (**BIC**) and Hannan-Quinn information criterion (**HQIC**) are applied. The model with the minimum values of AIC, BIC, CAIC and HQIC is considered to be the best model to fit the proposed data.

The data set represents the survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli, observed and reported by Bjerkedal (1960). Plots of the estimated PTIR density and cumulative functions in addition to that of the compared models (TIR – PIR – IR – KEIR – MIR - TMIR) for the data set are displayed in Figure 4.

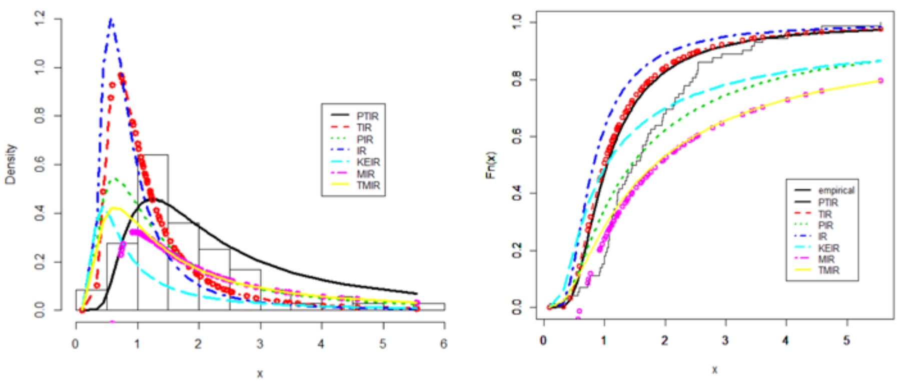


Figure 4. Estimated pdfs and cdfs of models for the data set

It can be observed from Figure 4 that the PTIR distribution is the most fitted distribution compared with the other models mentioned above concerning the Bjerkedal data.

The ML estimates and their standard errors (**SEs**) of the PTIR model compared with some known distributions such as IR,TIR, PIR,MIR, TMIR, KEIR are computed (see Table 2). Also, the corresponding measures of fit statistic using $-2\hat{\ell}$, AIC, BIC, CAIC, and HQIC, are provided in Table 3.

Table 2. ML estimates of the model parameters and the corresponding SEs

Model	θ	β	λ	α	a	b
PTIR	0.6056 (0.0808)	0.6577 (0.0463)	-0.9108 (0.0873)			
TIR	0.3525 (0.0434)		-0.9416 (0.0539)			
PIR	1.0691 (0.1325)	0.5865 (0.0421)				
IR	0.4629 (0.0546)					
KEIR	0.4001 (4.7575)			0.3657 (4.3316)	1.4444 (17.4921)	0.4045 (0.0581)
MIR	0.0465 (0.0187)			1.2500 (0.1537)		
TMIR	0.0105 (0.0278)		-0.9166 (0.0989)	0.6575 (0.0960)		

Table 3. The statistics $-2\hat{\ell}$, AIC, CAIC, BIC and HQIC

Distribution	PTIR	IR	TIR	PIR	MIR	TMIR	KEIR
$-2\hat{\ell}$	225.273	327.518	280.538	236.332	237.825	236.819	280.492
AIC	231.273	329.518	284.538	240.332	241.825	243.825	288.492
CAIC	231.625	329.575	284.712	240.506	241.999	244.178	289.089
BIC	238.103	331.795	289.092	244.885	246.378	250.655	297.599
HQIC	233.992	330.424	286.351	242.145	243.638	246.544	292.118

Also, it can be confirmed from Table 3 that the PTIR distribution is the most fitted distribution among other models for the data set as the PTIR distribution has the minimum values of AIC, BIC, CAIC and HQIC.

7. Concluding Remarks

In this article, a new model, called a power transmuted inverse Rayleigh distribution is introduced. Some statistical properties of the proposed distribution are derived and discussed. The estimation of the model parameters is discussed through the maximum likelihood, least squares and percentiles methods. A simulation study is carried out to compare the performance of different estimates. The simulation study revealed that the ML performs better than the LS and PR estimates, in approximately most of the situations. An application to a real data set indicates that the new model is superior to the fits than the other suggested distributions.

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