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# Power Size Biased Two-Parameter Akash Distribution

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## ABSTRACT

In this paper, the two-parameter Akash distribution is generalized to size-biased two-parameter Akash distribution (SBTPAD). A further modification to SBTPAD is introduced, creating the power size-biased two-parameter Akash distribution (PSBTPAD). Several statistical properties of PSBTPAD distribution are proved. These properties include the following: moments, coefficient of variation, coefficient of skewness, coefficient of kurtosis, the maximum likelihood estimation of the distribution parameters, and finally order statistics. Moreover, plots of the density and distribution functions of PSBTPAD are presented and a reliability analysis is considered. The Rényi entropy of PSBTPAD is proved and the application of real data is discussed.

Mathematics Subject Classification: 62E10, 62F15.

**Key words:** Akash distribution, two-parameter Akash distribution, size-biased distribution, moments, coefficient of variation, coefficient of skewness, coefficient of kurtosis, maximum likelihood estimation, entropy.

## 1. Introduction

Recently, it has been noted that there has been an increasing interest in suggesting new flexible distributions for explaining and fitting data in different fields of science such as medicine, pharmacy, environment and so on. Many authors have introduced several types of new flexible distributions such as weighted distributions. The weighted distributions are quite flexible for model specification and data interpretation.

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Fisher (1934) was the first who introduced the concept of weighted distributions. He studied how the verification methods can affect the form of the distribution of recorded observations. Also, see Rao (1965), Patil and Rao (1978), Gupta and Keating (1986), Gupta and Kirmani (1990), and (Oluyede 1999).

For a non-negative continuous random variable $Y$ with probability density function (pdf) $f(y)$, the pdf of the weighted random variable $Y_w$ is defined as

$$f_w(y) = \frac{w(y)f(y)}{E[w(y)]} = \frac{w(y)f(y)}{\mu_w}, \tag{1}$$

where $w(y)$ is a non-negative weight function. A special case of Equation (1) arises when the weight function is $w(y) = y^{\beta}$. In this case the distribution is known as a size-biased distribution of order $\beta$ with pdf given by

$$f_{\beta}(y) = \frac{y^{\beta} f(y)}{\int y^{\beta} f(y)dy},$$

where for $\beta = 1$ or 2, the resulting are known as the length-biased and area-biased distributions, respectively.

Saghir et al. (2017) proposed several weighted distributions. A size biased Ishita distribution is introduced by Al-Omari et al. (2019) as a generalization of the Ishita distribution. Haq et al. (2017) proposed Marshall-Olkin length-biased exponential distribution. Al-Omari and Alsmairan (2019) suggested a length-biased Suja distribution as a modification of the Suja distribution, which is suggested by Shanker (2017).

Shanker (2015) suggested a one-parameter Akash distribution (AD). Then, Shanker and Shukla (2017) generalized the AD to suggest a two-parameter Akash distribution (TPAD) with pdf given by

$$f(y;\theta,\alpha) = \frac{\theta^3}{\alpha\theta^2 + 2}\left(\alpha + y^2\right)e^{-\theta y},\ y > 0, \theta, \alpha > 0, \tag{2}$$

and a cumulative distribution function (cdf) defined as

$$F(y;\theta,\alpha) = 1 - \left[1 + \frac{\theta y(\theta y + 2)}{\alpha\theta^2 + 2}\right]e^{-\theta y},\ y > 0, \theta, \alpha > 0. \tag{3}$$

The mean of TPAD is given by $E(Y) = \mu = \dfrac{\alpha\theta^2 + 6}{\theta(\alpha\theta^2 + 2)}$.

Abebe and Shanker (2018) suggested a discrete Akash distribution. Shanker et al. (2018) proposed a two-parameter Poisson-Akash distribution. Shanker et al. (2016) considered Poisson-Akash distribution. Shanker et al. (2018) proposed a generalized Akash distribution. Tesfalem et al. (2019) suggested a weighted Quasi Akash distribution. Shanker (2016) suggested Qausi Akash distribution. Shanker and Shukla (2017) introduced the power Akash distribution.

The main objective of this study is to add a more flexibility distribution for fitting real data in the field. This paper is organized as follows: in Section 2, the pdf and the cdf of SBTPAD and PSBTPAD are presented as well as the shapes of the distribution are illustrated for various parameters. In Section 3 we present some statistical properties of the PSBTPAD, including the $rth$ moment, mean, variance, coefficients of variation, skewness and kurtosis. Also, some simulations results are presented to illustrate these properties. The maximum likelihood estimators of the distribution parameters are derived in Section 4. The distributions of order statistics and reliability analysis are introduced in Section 5. An application of real data set is presented in Section 6 for illustration. Finally, the main results and some conclusions are provided in Section 7.

## 2. Suggested distributions

This section presents the pdf and cdf of the suggested distributions. A random variable $Y$ is said to have a size biased two-parameter Akash distribution (SBTPAD) if its probability density function is given by

$$f_{SBTPAD}(y;\theta,\alpha)=\frac{\theta^4 y\left(\alpha+y^2\right)}{\alpha\theta^2+6}e^{-\theta y}, y>0, \alpha,\theta>0, \tag{4}$$

and a cumulative distribution function is in the form

$$F_{SBTPAD}(y;\theta,\alpha)=1-\frac{6+\theta\left[\theta^2 y(\alpha+y^2)+\theta(\alpha+3y^2)+6y\right]}{\alpha\theta^2+6}e^{-\theta y}. \tag{5}$$

It is easy to derive the pdf given in Equation (4) by utilizing Equations (1) and the pdf of the TPAD given in (2), with the mean of the TPAD.

In this paper we modified the SBTPAD to a power size biased two-parameter Akash distribution (PSBTPAD) Taking the power transformation $X=Y^{1/\beta}$ in (4) a pdf of a random variable $X$ can be defined as

$$f_{PSBTPAD}(x;\theta,\alpha,\beta)=\frac{\beta\theta^4}{\alpha\theta^2+6}x^{2\beta-1}\left(\alpha+x^{2\beta}\right)e^{-\theta x^{\beta}}, x>0, \alpha,\theta,\beta>0. \tag{6}$$

We would call the density in (6) as the power size biased two-parameter Akash distribution (PSBTPAD). It is easy to prove that $\int_0^{\infty} f(x;\theta,\alpha,\beta)dx=1$.

Shukla and Shanker (2018) proposed a power Ishita distribution. Ghitany et al. (2013) introduced power Lindley distribution. Al-Omari et al. (2019) proposed a power length-biased Suja distribution. The corresponding pdf of the PSBTPAD is

$$F_{PSBTPAD}(x;\theta,\alpha,\beta)=1-\frac{\alpha\theta^2\Gamma\left(2,x^{\beta}\theta\right)+\Gamma\left(4,x^{\beta}\theta\right)}{\alpha\theta^2+6},\quad x>0,\theta,\alpha,\beta>0, \tag{7}$$

where $\Gamma(n+1,z)=n!e^{-z}\sum_{r=0}^{n}\frac{z^r}{r!}$ is the incomplete Gamma function. The lower incomplete gamma function is $\Gamma(\alpha,x)=\int_0^x t^{\alpha-1}e^{-t}dt$.

Figures 1 and 2 illustrate the shape of the pdf and cdf of the PSBTPAD for various values of the distribution parameters.

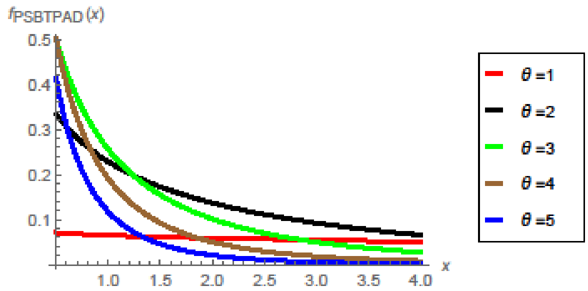


**Figure 1.** The pdf of PSBTPAD random variable *X* for $\theta=1,2,3,4,5$, $\alpha=1.7$ and $\beta=0.5$

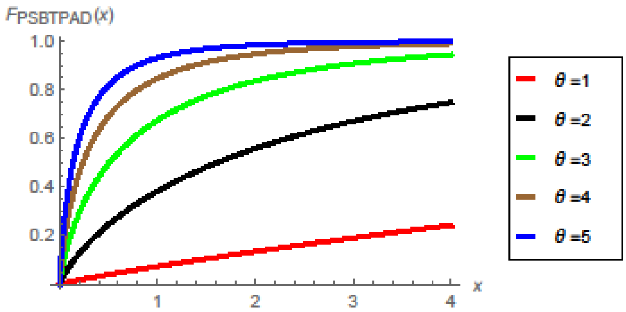


**Figure 2.** The cdf of PSBTPAD random variable *X* for $\theta=1,2,3,4,5$, $\alpha=1.7$ and $\beta=0.5$

Based on Figure 1, it can be seen that the PSBTPAD is asymmetric and skewed to the right.

## 3. Statistical properties

This section presents the *r*th moment, mean, variance, coefficients of variation, skewness and kurtosis of the PSBTPAD. Also, some simulations for these properties are provided.

### 3.1. Moments of the PSBTPAD

**Theorem 2:** Let $X \sim f_{PSBTPAD}(x;\theta,\alpha,\beta)$, then the $rth$ moment of $X$ about the origin is

$$E(X^r) = \frac{\theta^{-\frac{r}{\beta}}\Gamma\left(\frac{r}{\beta}+2\right)\left[\beta^2\left(\alpha\theta^2+6\right)+r^2+5\beta r\right]}{\beta^2\left(\alpha\theta^2+6\right)}, \tag{8}$$

for $2\beta + r > 0, \theta > 0, \beta > 0, r = 1,2,3,..$

**Proof:** By the expectation definition of the *r*th moment we have

$$\begin{aligned}
\mu^r_{PSBTPAD} &= E(X^r) = \int_0^\infty x^r \frac{\beta\theta^4}{\alpha\theta^2+6} x^{2\beta-1}\left(\alpha + x^{2\beta}\right)e^{-\theta x^\beta}dx \\
&= \frac{\beta\theta^4}{\alpha\theta^2+6}\left[\alpha\int_0^\infty x^{2\beta+r-1}e^{-\theta x^\beta}dx + \int_0^\infty x^{4\beta+r-1}e^{-\theta x^\beta}dx\right] \\
&= \frac{\beta\theta^4}{\alpha\theta^2+6}\left[\frac{\alpha\theta^{-\frac{r}{\beta}-2}\Gamma\left(\frac{r}{\beta}+2\right)}{\beta} + \frac{\theta^{-\frac{r}{\beta}-4}\Gamma\left(\frac{r}{\beta}+4\right)}{\beta}\right] \\
&= \theta^{-\frac{r}{\beta}}\Gamma\left(\frac{r}{\beta}+2\right)\left[\frac{\beta^2(\alpha\theta^2+6)+r^2+5r\beta}{\beta^2(\alpha\theta^2+6)}\right].
\end{aligned}$$

Based on Equation (8), it is simple to deduce the first, second, third and fourth moments of the BTPAD, respectively, as

$$E(X) = \frac{\theta^{-1/\beta}\Gamma\left(2+\frac{1}{\beta}\right)\left(\beta\left(\Psi+5\right)+1\right)}{\beta^2\left(\alpha\theta^2+6\right)},$$

$$E\left(X^2\right) = \frac{\theta^{-2/\beta}\Gamma\left(2+\frac{2}{\beta}\right)\left(\beta\left(\Psi+10\right)+4\right)}{\beta^2\left(\alpha\theta^2+6\right)},$$

$$E\left(X^3\right) = \frac{\theta^{-3/\beta}\Gamma\left(2+\frac{3}{\beta}\right)\left(\beta\left(\Psi+15\right)+9\right)}{\beta^2\left(\alpha\theta^2+6\right)},$$

$$E\left(X^4\right) = \frac{\theta^{-4/\beta}\Gamma\left(2+\frac{4}{\beta}\right)\left(\beta\left(\Psi+20\right)+16\right)}{\beta^2\left(\alpha\theta^2+6\right)},$$

where $\Psi=\beta\left(\alpha\theta^2+6\right)$. Hence, the variance of PSBTPAD is given by

$$Var(X)=E\left(X^2\right)-[E(X)]^2$$
$$=\frac{\theta^{-2/\beta}\left[\beta\Psi\Gamma\left(2+\frac{2}{\beta}\right)\left(\beta\left(\Psi+10\right)+4\right)-\Gamma\left(2+\frac{1}{\beta}\right)^2\left(\beta\left(\Psi+5\right)+1\right)^2\right]}{\beta^2\Psi^2} \tag{9}$$

### 3.2. The coefficient of skewness

The coefficient of skewness determines the degree of skewness of SBTPAD. It is given by:

$$Sk_{PSBTPAD}=\frac{\theta^{-3/\beta}\left[\beta^2\Psi^2\Gamma\left(2+\frac{3}{\beta}\right)\left(\beta\left(\Psi+15\right)+9\right)-3\beta\Psi\Phi\Gamma\left(2+\frac{2}{\beta}\right)\left(\beta\left(\Psi+10\right)+4\right)+2\Phi^3\right]}{\beta^3\Psi^3\left[\frac{\theta^{-2/\beta}\left(\beta\Psi\Gamma\left(2+\frac{2}{\beta}\right)\left(\beta\left(\Psi+10\right)+4\right)-\Phi^2\right)}{\beta^2\Psi^2}\right]^{3/2}}, \tag{10}$$

where $\Psi=\beta\left(\alpha\theta^2+6\right)$ and $\Phi=\Gamma\left(2+\frac{1}{\beta}\right)\left(\beta\left(\Psi+5\right)+1\right)$

### 3.3. The coefficient of kurtosis

The coefficient of kurtosis measures the flatness of the distribution. The coefficient of kurtosis for PSBTPAD is defined as

$$Ku_{PSBTPAD}=\frac{\left\{\begin{array}{l}\beta^3\Psi^3\Gamma\left(2+\frac{4}{\beta}\right)\left(\beta\left(\Psi+20\right)+16\right)+6\beta\,\Phi^2\Lambda\\-3\Phi^4-4\beta^2\Psi^2\Phi\Gamma\left(2+\frac{3}{\beta}\right)\left(\beta\left(\Psi+15\right)+9\right)\end{array}\right\}}{\left(\Phi^2-\beta\Lambda\right)^2} \tag{11}$$

where $\Psi=\beta\left(\alpha\theta^2+6\right)$, and $\Phi=\Gamma\left(2+\frac{1}{\beta}\right)\left(\beta\left(\Psi+5\right)+1\right)$,

$\Lambda=\Psi\Gamma\left(2+\frac{2}{\beta}\right)\left(\beta\left(\Psi+10\right)+4\right)$.

### 3.4. The coefficient of variation

The coefficient of variation of the PSBTPAD is given by

$$Cv_{PSBTPAD} = \frac{\beta\Psi\theta^{1/\beta}\sqrt{\dfrac{\theta^{-2/\beta}\left(\beta\Psi\Gamma\left(2+\dfrac{2}{\beta}\right)\left(\beta\left(\Psi+10\right)+4\right)-\Phi^2\right)}{\beta^4\left(\alpha\theta^2+6\right)^2}}}{\Phi}, \quad (12)$$

where $\theta>0,\beta>0.$

**Theorem 2:** Let $X \sim f_{PSBTPAD}(x;\theta,\alpha,\beta)$, then the harmonic mean of *X* is

$$H(\theta,\alpha,\beta) = \frac{\beta\Psi}{\theta^{1/\beta}\Gamma\left(2-\dfrac{1}{\beta}\right)\left[\beta\left(\Psi-5\right)+1\right]}, \theta>0, \beta>\frac{1}{2}. \quad (13)$$

To investigate the behaviour of these measures, we calculate some values of $\mu_{PSBTPAD}$, $\sigma_{PSBTPAD}$, $Cv_{PSBTPAD}$, $Sk_{PSBTPAD}$ and $Ku_{PSBTPAD}$ of the PSBTPAD for ($\theta=5$, $\beta=3$), ($\theta=5$, $\beta=7$), for various values of $\alpha$ and the results are presented in Tables 1 and 2, respectively.

**Table 1.** The mean, variance, coefficients of variation, skewness and kurtosis for the SBTPAD distribution for some values of $\alpha$ with $\theta=5$ and $\beta=3$

| $\alpha$ | $\mu_{PSBTPAD}$ | $\sigma_{PSBTPAD}$ | $Cv_{PSBTPAD}$ | $Sk_{PSBTPAD}$ | $Ku_{PSBTPAD}$ |
|---|---|---|---|---|---|
| 1 | 0.736221 | 0.187730 | 0.254991 | 0.080619 | 2.79987 |
| 1.1 | 0.733241 | 0.186927 | 0.254933 | 0.085470 | 2.80863 |
| 1.2 | 0.730675 | 0.186195 | 0.254826 | 0.089115 | 2.81627 |
| 1.3 | 0.728442 | 0.185527 | 0.254690 | 0.091850 | 2.82290 |
| 1.4 | 0.726482 | 0.184916 | 0.254536 | 0.093889 | 2.82868 |
| 1.5 | 0.724746 | 0.184356 | 0.254373 | 0.095393 | 2.83371 |
| 1.6 | 0.723200 | 0.183842 | 0.254206 | 0.096480 | 2.83809 |
| 1.7 | 0.721813 | 0.183368 | 0.254039 | 0.097241 | 2.84192 |
| 1.8 | 0.720562 | 0.182931 | 0.253873 | 0.097745 | 2.84528 |
| 1.9 | 0.719427 | 0.182527 | 0.253711 | 0.098047 | 2.84822 |
| 2 | 0.718395 | 0.182151 | 0.253553 | 0.098188 | 2.85080 |
| 2.1 | 0.717450 | 0.181802 | 0.253400 | 0.098202 | 2.85308 |
| 2.2 | 0.716583 | 0.181477 | 0.253253 | 0.098113 | 2.85509 |
| 2.3 | 0.715784 | 0.181173 | 0.253111 | 0.097944 | 2.85687 |
| 2.4 | 0.715045 | 0.180888 | 0.252974 | 0.097711 | 2.85844 |
| 2.5 | 0.714361 | 0.180621 | 0.252843 | 0.097426 | 2.85984 |
| 2.6 | 0.713725 | 0.180370 | 0.252717 | 0.097102 | 2.86107 |
| 2.7 | 0.713132 | 0.180134 | 0.252596 | 0.096747 | 2.86218 |
| 2.8 | 0.712578 | 0.179912 | 0.252480 | 0.096368 | 2.86315 |

**Table 2.** The mean, variance, coefficients of variation, skewness and kurtosis for the SBTPAD distribution for some values of $\alpha$ with $\theta=5$ and $\beta=7$

| $\alpha$ | $\mu_{PSBTPAD}$ | $\sigma_{PSBTPAD}$ | $Cv_{PSBTPAD}$ | $Sk_{PSBTPAD}$ | $Ku_{PSBTPAD}$ |
|---|---|---|---|---|---|
| 1 | 0.869614 | 0.098748 | 0.113554 | -0.35893 | 3.12901 |
| 1.1 | 0.868112 | 0.098522 | 0.113490 | -0.35521 | 3.13229 |
| 1.2 | 0.866818 | 0.098309 | 0.113413 | -0.35250 | 3.13574 |
| 1.3 | 0.865692 | 0.098109 | 0.113330 | -0.35056 | 3.13918 |
| 1.4 | 0.864704 | 0.097922 | 0.113244 | -0.34918 | 3.14253 |
| 1.5 | 0.863829 | 0.097748 | 0.113157 | -0.34824 | 3.14572 |
| 1.6 | 0.863049 | 0.097587 | 0.113072 | -0.34761 | 3.14874 |
| 1.7 | 0.862350 | 0.097436 | 0.112989 | -0.34725 | 3.15158 |
| 1.8 | 0.861719 | 0.097296 | 0.112909 | -0.34707 | 3.15424 |
| 1.9 | 0.861147 | 0.097165 | 0.112832 | -0.34705 | 3.15673 |
| 2 | 0.860626 | 0.097042 | 0.112758 | -0.34714 | 3.15906 |
| 2.1 | 0.860150 | 0.096928 | 0.112687 | -0.34732 | 3.16123 |
| 2.2 | 0.859713 | 0.096821 | 0.112620 | -0.34757 | 3.16326 |
| 2.3 | 0.859310 | 0.096720 | 0.112555 | -0.34787 | 3.16515 |
| 2.4 | 0.858938 | 0.096625 | 0.112494 | -0.34821 | 3.16693 |
| 2.5 | 0.858593 | 0.096536 | 0.112435 | -0.34859 | 3.16859 |
| 2.6 | 0.858272 | 0.096452 | 0.112380 | -0.34899 | 3.17015 |
| 2.7 | 0.857973 | 0.096373 | 0.112326 | -0.34940 | 3.17162 |
| 2.8 | 0.857693 | 0.096298 | 0.112275 | -0.34982 | 3.17300 |

From Tables 1- 3 we can conclude the following:

1. For fixed values of $\alpha$, the values of $\mu_{PSBTPAD}$ and $Ku_{PSBTPAD}$ of the PSBTPAD decrease as the values of $\beta$ increase.
2. The $Cv_{PSBTPAD}$ values are about 0.25 when $\theta=5$ and $\beta=3$, and it is about 0.11 when $\theta=5$ and $\beta=7$.
3. The $Sk_{PSBTPAD}$ values are about 0.098 for all the parameter values in Table 1 and about -0.35 in for the parameters in Table 2. This indicates that the shape of the PSBTPAD depends on the parameter values.

## 4. Maximum likelihood estimation

Let $X_1, X_2, ..., X_n$ be a random sample of size $n$ from PSBTPAD with parameters $\alpha>0$, $\beta>0$ and $\theta>0$. The maximum likelihood estimators for the parameters of PSBTPAD can be derived based on the likelihood function as

$$L(\theta,\alpha,\beta)=\prod_{i=1}^{n}\frac{\beta\theta^{4}}{\alpha\theta^{2}+6}x_i^{2\beta-1}\left(\alpha+x_i^{2\beta}\right)e^{-\theta x_i^{\beta}}$$
$$=\left(\frac{\beta\theta^{4}}{\alpha\theta^{2}+6}\right)^{n}\prod_{i=1}^{n}x_i^{2\beta-1}\left(\alpha+x_i^{2\beta}\right)e^{\sum_{i=1}^{n}-\theta x_i^{\beta}}.$$

Then, the log likelihood function is given by

$$\ln L(\theta,\alpha,\beta)=\ln\left[\left(\frac{\beta\theta^{4}}{\alpha\theta^{2}+6}\right)^{n}\prod_{i=1}^{n}x_i^{2\beta-1}\left(\alpha+x_i^{2\beta}\right)e^{\sum_{i=1}^{n}\theta x_i^{\beta}}\right]$$
$$=4n\ln(\theta)+n\ln(\beta)-n\ln\left(\alpha\theta^{2}+6\right)+\sum_{i=1}^{n}\ln\left(x_i^{2\beta-1}\right)+\sum_{i=1}^{n}\ln\left(\alpha+x_i^{2\beta}\right)-\sum_{i=1}^{n}\theta x_i^{\beta}. \tag{14}$$

Take the derivative of Equation (14) with respect to $\theta$, $\alpha$ and $\beta$, respectively, as

$$\frac{\partial lnL(\theta,\alpha,\beta)}{\partial\theta}=\frac{4n}{\theta}-\frac{2n\alpha\theta}{\alpha\theta^{2}+6}-\sum_{i=1}^{n}x_i^{\beta}, \tag{15}$$

$$\frac{\partial\ln L(\theta,\alpha,\beta)}{\partial\alpha}=\frac{n}{\beta}-\frac{n\theta^{2}}{\alpha\theta^{2}+6}+\sum_{i=1}^{n}\frac{1}{\alpha+x_i^{2\beta}}, \tag{16}$$

and

$$\frac{\partial\ln L(\theta,\alpha,\beta)}{\partial\beta}=\frac{n}{\beta}+\sum_{i=1}^{n}\frac{2\ln(x_i)}{x_i^{2\beta-1}}+\sum_{i=1}^{n}\frac{2\ln(x_i)}{\alpha+x_i^{2\beta}}-\theta x_i^{\beta}\ln(x_i). \tag{17}$$

Since there is no closed form solutions for the above system of equations, the MLEs of the PSBTPAD parameters $\alpha$, $\theta$, and $\beta$ denoted as $\hat{\alpha},\hat{\theta}$ and $\hat{\beta}$, respectively, can be obtained by solving the equations $\frac{\partial lnL(\theta,\alpha,\beta)}{\partial\theta}=0$, $\frac{\partial lnL(\theta,\alpha,\beta)}{\partial\alpha}=0$, $\frac{\partial lnL(\theta,\alpha,\beta)}{\partial\beta}=0$ numerically.

## 5. Order statistics and reliability analysis

Let $X_1,X_2,\ldots X_m$ be a random sample of size $m$ from the power size biased two-parameter Akash distribution. Also, let $X_{(1:m)},X_{(2:m)},\ldots,X_{(m:m)}$ denote the corresponding order statistics of the sample. The probability density function of the *i*th order statistic $X_{(i:m)}$ for $1\le i\le m$ is

$$f_{(i:m)}(x)=\frac{m!}{(i-1)(m-i)}\left[F(x)\right]^{i-1}\left[1-F(x)\right]^{m-i}f(x). \tag{18}$$

By substituting the pdf and cdf of the PSBTPAD in Equation (18), the pdf of $X_{(i:m)}$ is given by

$$f_{(i:m)}(x;\alpha,\theta,\beta)=\frac{\beta\theta^4 m!x^{2\beta-1}\left(\alpha+x^{2\beta}\right)e^{-\theta x^\beta}}{\left(\alpha\theta^2+6\right)\Gamma(i)\Gamma(-i+m+1)}\mathrm{H}, \tag{19}$$

where $\mathrm{H}=\left(1-\frac{\alpha\theta^2\Gamma\left(2,x^\beta\theta\right)-\Gamma\left(4,x^\beta\theta\right)}{\alpha\theta^2+6}\right)^{i-1}\left(\frac{\alpha\theta^2\Gamma\left(2,x^\beta\theta\right)+\Gamma\left(4,x^\beta\theta\right)}{\alpha\theta^2+6}\right)^{m-i}$.

Based on Equation (19) the pdfs of smallest order statistic, $X_{(1:m)}$ and largest order statistic, $X_{(m:m)}$, are respectively, given by

$$f_{(1:m)}(x;\alpha,\theta,\beta)=\frac{\beta\theta^4 m x^{2\beta-1}\left(\alpha+x^{2\beta}\right)e^{-\theta x^\beta}\left[\alpha\theta^2\Gamma\left(2,x^\beta\theta\right)+\Gamma\left(4,x^\beta\theta\right)\right]^{m-1}}{\left(\alpha\theta^2+6\right)^m}, \tag{20}$$

and

$$f_{(m:m)}\left(x;\alpha,\theta,\beta\right)=\frac{\beta\theta^4 m x^{2\beta-1}\left(\alpha+x^{2\beta}\right)e^{-\theta x^\beta}\left[\alpha\theta^2\left(1-\Gamma\left(2,x^\beta\theta\right)\right)-\Gamma\left(4,x^\beta\theta\right)+6\right]^{m-1}}{\left(\alpha\theta^2+6\right)^m} .(21)$$

The reliability and hazard rate functions of the PSBTPAD random variable are given by

$$\begin{aligned}R_{PSBTPAD}(x;\alpha,\theta,\beta)&=1-F_{PSBTPAD}(x;\alpha,\theta,\beta)\\&=\frac{\alpha\theta^2\Gamma\left(2,x^\beta\theta\right)+\Gamma\left(4,x^\beta\theta\right)}{\alpha\theta^2+6},\end{aligned} \tag{22}$$

$$\begin{aligned}H_{PSBTPAD}(x;\alpha,\theta,\beta)&=\frac{f_{PSBTPAD}(x;\alpha,\theta,\beta)}{1-F_{PSBTPAD}(x;\alpha,\theta,\beta)}\\&=\frac{\beta\theta^4x^{2\beta-1}\left(\alpha+x^{2\beta}\right)Exp\left(-\theta x^\beta\right)}{\alpha\theta^2\Gamma\left(2,x^\beta\theta\right)+\Gamma\left(4,x^\beta\theta\right)}.\end{aligned} \tag{23}$$

Figure (3) shows the reliability and hazard rate functions of the PSBTPAD with $\theta=1,2,3,4,5$, $\alpha=1.7$ and $\beta=0.5$.

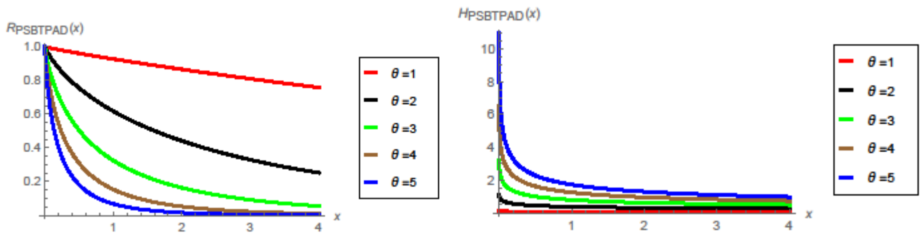


**Figure 3.** The reliability and hazard rate functions PSBTPAD for $\theta=1,2,3,4,5$, $\alpha=1.7$ and $\beta=0.5$.

Figure (3) shows that the plots of the reliability and hazard rate functions of the PSBTAD are decreasing functions.

The reversed hazard rate and odds functions of the PSBTAD, respectively, are defined as

$$\begin{aligned} RH_{PSBTPAD}(x;\alpha,\theta,\beta) &= \frac{f_{PSBTPAD}(x;\alpha,\theta,\beta)}{F_{PSBTPAD}(x;\alpha,\theta,\beta)} \\ &= \frac{\beta\theta^4 x^{2\beta-1}\left(\alpha+x^{2\beta}\right)Exp\left(-\theta x^{\beta}\right)}{\alpha\theta^2+6-\alpha\theta^2\Gamma\left(2,x^{\beta}\theta\right)+\Gamma\left(4,x^{\beta}\theta\right)}, \end{aligned} \tag{23}$$

and

$$O_{PSBTPAD}(x;\alpha,\theta,\beta) = \frac{F_{PSBTPAD}(x;\alpha,\theta,\beta)}{1-F_{PSBTPAD}(x;\alpha,\theta,\beta)} = \frac{\alpha\theta^2+6-\alpha\theta^2\Gamma\left(2,x^{\beta}\theta\right)-\Gamma\left(4,x^{\beta}\theta\right)}{\alpha\theta^2\Gamma\left(2,x^{\beta}\theta\right)+\Gamma\left(4,x^{\beta}\theta\right)}. \tag{24}$$

Figure (4) represents the reversed hazard and odds functions of the PSBTPAD distribution with $\theta=1,2,3,4,5$, $\alpha=1.7$ and $\beta=0.5$.

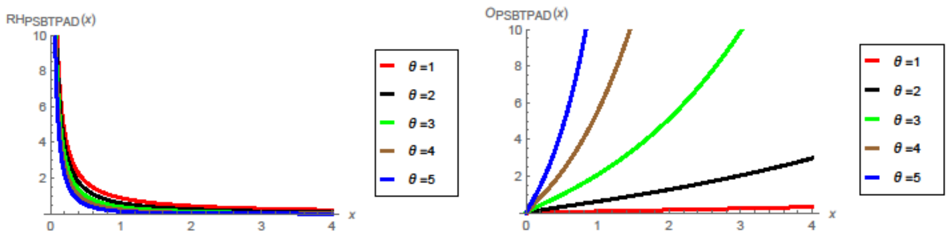


Figure **4.** The reversed hazard and odds functions of the PSBTPAD for $\theta=1,2,3,4,5$, $\alpha=1.7$ and $\beta=0.5$.

The mean residual life function is defined as

$$\begin{aligned} m_{PSBTPAD}(x;\theta,\alpha,\beta) &= E\left(X-x \mid X>x\right) \\ &= \frac{1}{1-F_{PSBTPAD}(x;\theta,\alpha,\beta)}\int_x^{\infty}\left(1-F_{PSBTPAD}(t;\theta,\alpha,\beta)\right)dt \\ &= \frac{1}{\alpha\theta^2\Gamma\left(2,x^{\beta}\theta\right)+\Gamma\left(4,x^{\beta}\theta\right)}\int_x^{\infty}\left(\alpha\theta^2\Gamma\left(2,x^{\beta}\theta\right)+\Gamma\left(4,x^{\beta}\theta\right)\right)dt. \end{aligned}$$

The Mills ratio of the PSBTAD is defined as

$$\begin{aligned} MR_{PSBTPAD}(x;\alpha,\theta,\beta) &= \frac{1}{RH_{PSBTPAD}(x;\alpha,\theta,\beta)} \\ &= \frac{F_{PSBTPAD}(x;\alpha,\theta,\beta)}{f_{PSBTPAD}(x;\alpha,\theta,\beta)} \\ &= \frac{\alpha\theta^2+6-\alpha\theta^2\Gamma\left(2,x^{\beta}\theta\right)+\Gamma\left(4,x^{\beta}\theta\right)}{\beta\theta^4x^{2\beta-1}\left(\alpha+x^{2\beta}\right)Exp\left(-\theta x^{\beta}\right)} \end{aligned}$$

Plots of the Mills ratio of the PSBTAD are given in Figure (4) for various parameters.

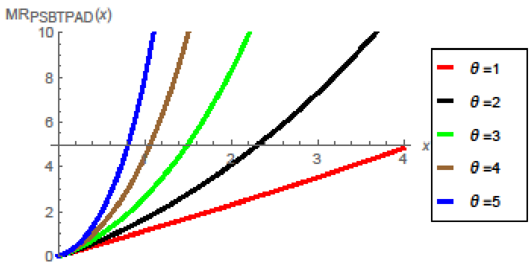


**Figure 4.** The Mills ratio of the PSBTPAD for $\theta=1,2,3,4,5$, $\alpha=1.7$ and $\beta=0.5$

## 6. Rényi Entropy

The Rényi entropy (RE) of a random variable $X$ is a measure of variation of the uncertainty. The RE is defined as $RE(\omega)=\frac{1}{1-\omega}\log\left(\int_0^{\infty} f(x)^{\omega}dx\right)$, $\omega>0$ and $\omega\neq 1$. The entropy can be used for performing a goodness fit test. For more about entropy see, for example, Al-Omari and Zamanzade (2017, 2018) for goodness of fit for Laplace and logistic distributions, respectively; Zamanzade and Mahdizadeh (2017) for entropy estimation using ranked set sampling; Zamanzade (2014) for testing uniformity using new entropy estimators, and Zamanzade and Arghami (2011) for goodness-of-fit test with correcting moments of modified entropy estimator; Al-Omari and Haq (2019) for novel entropy estimators of a continuous random variables.

**Theorem 3:** If $X \sim f_{PSBTPAD}(x;\theta,\alpha,\beta)$, the Rényi entropy of *X* is defined as

$$RE_{\text{PSBTPAD}}(\omega)=\frac{1}{1-\omega}\log\left[\begin{array}{l}\left(\frac{\omega^{-4}}{\alpha\theta^2+6}\right)^{\omega}\left(\beta(\omega\theta)^{\frac{1}{\beta}}\right)^{\omega\text{-}1}\times\\ \sum_{j=0}^{\omega}\binom{\omega}{j}(\alpha\theta^2\omega^2)^j\,\Gamma\left(-2\text{j}+4\omega-\frac{\omega-1}{\beta}\right)\end{array}\right]. \tag{25}$$

**Proof:** The Rényi entropy of the PSBTPAD can be obtained as

$$\begin{aligned}RE_{PSBTPAD}(\omega)&=\frac{1}{1-\omega}\log\left[\int_0^{\infty}\left(f_{PSBTPAD}(x;\theta,\alpha,\beta)\right)^{\omega}dx\right]\\ &=\frac{1}{1-\omega}\log\left[\int_0^{\infty}\left(\frac{\beta\theta^4}{\alpha\theta^2+6}x^{2\beta-1}\left(\alpha+x^{2\beta}\right)Exp\left(-\theta x^{\beta}\right)\right)^{\omega}dx\right]\\ &=\frac{1}{1-\omega}\log\left[\left(\frac{\beta\theta^4}{\alpha\theta^2+6}\right)^{\omega}\int_0^{\infty}\left(\alpha x^{2\beta-1}+x^{4\beta-1}\right)^{\omega}e^{-\theta\omega x^{\beta}}dx\right]\\ &=\frac{1}{1-\omega}\log\left[\left(\frac{\beta\theta^4}{\alpha\theta^2+6}\right)^{\omega}\int_0^{\infty}\alpha^j\sum_{j=0}^{\omega}\binom{\omega}{j}x^{(2\beta-1)j}\left(x^{4\beta-1}\right)^{\omega-j}e^{-\theta\omega x^{\beta}}dx\right]\\ &=\frac{1}{1-\omega}\log\left[\begin{array}{l}\left(\frac{\omega^{-4}}{\alpha\theta^2+6}\right)^{\omega}\left(\beta(\omega\theta)^{\frac{1}{\beta}}\right)^{\omega\text{-}1}\sum_{j=0}^{\omega}\binom{\omega}{j}(\alpha\theta^2\omega^2)^j\\ \Gamma\left(-2\text{j}+4\omega-\frac{\omega-1}{\beta}\right)\end{array}\right].\end{aligned}$$

To investigate the behaviour of the PSBTPAD Rényi entropy, Tables 3 and 4 involve some Rényi entropy values of the PSBTPAD for some values of the distribution parameters.

**Table 3**. Rényi entropy values for the PSBTPAD with $\beta=\theta=2$, $\omega=9$ and $\alpha=2,3,...,46$

| $\alpha$ | $RE_{PSBTPAD}(\omega)$ | $\alpha$ | $RE_{PSBTPAD}(\omega)$ | $\alpha$ | $RE_{PSBTPAD}(\omega)$ |
|---|---|---|---|---|---|
| 1 | 0.191233 | 17 | 0.037523 | 32 | 0.014837 |
| 2 | 0.167737 | 18 | 0.034947 | 33 | 0.014018 |
| 3 | 0.143430 | 19 | 0.032617 | 34 | 0.013246 |
| 4 | 0.123293 | 20 | 0.030499 | 35 | 0.012515 |
| 5 | 0.107162 | 21 | 0.028566 | 36 | 0.011823 |
| 6 | 0.094186 | 22 | 0.026795 | 37 | 0.011167 |
| 7 | 0.083608 | 23 | 0.025166 | 38 | 0.010544 |
| 8 | 0.074857 | 24 | 0.023663 | 39 | 0.009951 |
| 9 | 0.067515 | 25 | 0.022272 | 40 | 0.009387 |
| 10 | 0.061276 | 26 | 0.020981 | 41 | 0.008849 |
| 11 | 0.055914 | 27 | 0.019779 | 42 | 0.008336 |
| 12 | 0.051260 | 28 | 0.018658 | 43 | 0.007845 |
| 13 | 0.047184 | 29 | 0.017610 | 44 | 0.007376 |
| 14 | 0.043585 | 30 | 0.016627 | 45 | 0.006928 |
| 15 | 0.040386 | 31 | 0.015705 | 46 | 0.006497 |

**Table 4**. Rényi entropy values for the PSBTPAD with $\beta=3$, $\alpha=4$, $\omega=1.1$ and $\theta=1,2,...,45$

| $\theta$ | $RE_{PSBTPAD}(\omega)$ | $\theta$ | $RE_{PSBTPAD}(\omega)$ | $\theta$ | $RE_{PSBTPAD}(\omega)$ |
|---|---|---|---|---|---|
| 1 | 0.56641 | 16 | 4.26337 | 31 | 5.36132 |
| 2 | 1.11634 | 17 | 4.36373 | 32 | 5.41414 |
| 3 | 1.62485 | 18 | 4.45843 | 33 | 5.46534 |
| 4 | 2.03848 | 19 | 4.54806 | 34 | 5.51501 |
| 5 | 2.37837 | 20 | 4.63313 | 35 | 5.56325 |
| 6 | 2.66442 | 21 | 4.71409 | 36 | 5.61013 |
| 7 | 2.91043 | 22 | 4.79132 | 37 | 5.65573 |
| 8 | 3.12584 | 23 | 4.86514 | 38 | 5.70012 |
| 9 | 3.31722 | 24 | 4.93583 | 39 | 5.74336 |
| 10 | 3.48928 | 25 | 5.00366 | 40 | 5.78551 |
| 11 | 3.64551 | 26 | 5.06884 | 41 | 5.82662 |
| 12 | 3.78853 | 27 | 5.13158 | 42 | 5.86674 |
| 13 | 3.92037 | 28 | 5.19205 | 43 | 5.90591 |
| 14 | 4.04264 | 29 | 5.25040 | 44 | 5.94419 |
| 15 | 4.15663 | 30 | 5.30678 | 45 | 5.98161 |

Based on Table 3, we can say that the RE values approach zero for $\beta=\theta=2$ and $\omega=9$ as $\alpha$ starts increasing from 2 up to 46. But from Table 4, the RE values are increasing as the values of $\theta$ are increasing for fixed values of $\beta=3$, $\alpha=4$ and $\omega=1.1$.

## 7. Application and goodness of fit

In this section, the proposed PSBTPAD is applied to model data. We compare the fits of the PSBTPAD model with

1) Sushila distribution (SD) suggested Shanker et al. (2013):

$$f(x;\alpha,\delta)=\frac{\delta^2}{\alpha(\delta+1)}\left(1+\frac{x}{\alpha}\right)e^{-\frac{\delta}{\alpha}x};\quad x>0,\delta>0,\alpha>0.$$

2) Akash distribution (AD) Shanker (2015):

$$f\left(x,a\right)=\frac{a^3}{a^2+2}\left(1+x^2\right)e^{-ax},x>0,a>0.$$

3) Size biased Akash distribution (SBAD):

$$f\left(x,a\right)=\frac{a^3}{a^2+2}\left(1+x^2\right)e^{-ax},x>0,a>0.$$

4) Two-parameters Akash distribution (TPAD) Shanker and Shukla (2017):

$$f_{TPAD}(x;\theta,\alpha)=\frac{\theta^3}{\alpha\theta^2+2}\left(\alpha+x^2\right)e^{-\theta x};\ x>0,\ \theta>0.$$

5) Two-parameter quasi Akash distribution (TPQAD):

$$f_{TPQAD}(x;\theta,\alpha)=\frac{\theta^2}{\alpha\theta+2}\left(\alpha+\theta x^2\right)e^{-\theta x};\ x>0,\ \theta>0.$$

6) Marshall-Olkin Esscher Transformed Laplace distribution (MOETL), Georgea and Georgea (2013):

$$f(x)=\frac{\lambda k}{1+k^2}\begin{cases}Exp\left(\dfrac{\lambda}{k}x\right), & x<0\\ Exp\left(-k\lambda x\right), & x\geq 0.\end{cases}$$

We considered the negative maximized log-likelihood values (-MLL), Hannan-Quinn Information Criterion (HQIC), Bayesian Information Criterion (BIC), Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (CAIC) and Kolmogorov-Smirnov (K-S) test statistic. These measures are defined as

$$AIC=-2MLL+2i, CAIC=-2MLL+\frac{2in}{n-i-1},$$

$$BIC=-2MLL+iLog(n)\ \text{and}\ HQIC=2ln\left[ln(n)(i-2MLL)\right],$$

where $i$ is the number of parameters and $n$ is the sample size. Also, the Kolmogorov-Smirnov (KS) test is defined as $KS = Sup_n \left|F_n(x) - F(x)\right|$, where $F_n(x) = \frac{1}{n}\sum_{i=1}^{n} I_{x_i \le x}$ is the empirical distribution function and $F(x)$ is the cumulative distribution function. In general, lesser values of the above measures indicate a better fit of the model to the data set. The data set represent the strength data of glass of the aircraft window reported by Fuller et al. (1994). The data are as follows:

18.83, 20.80, 21.657, 23.03, 23.23, 24.05, 24.321, 25.5, 25.52, 25.80, 26.69, 26.77, 26.78, 27.05, 27.67, 29.90, 31.11, 33.2, 33.73, 33.76, 33.89, 34.76, 35.75, 35.91, 36.98, 37.08, 37.09, 39.58, 44.045, 45.29, 45.381.

**Table 5.** The -2LL, KS, P-value, AIC, CAIC, BIC, HQIC and the MLE based on the real data

| Model | AIC | CAIC | BIC | HQIC | KS | P-Value | -2LL | MLE |
|---|---|---|---|---|---|---|---|---|
| **AD** | 242.68 | 242.82 | 244.12 | 243.15 | 0.2987 | 0.0060 | 120.34 | $\hat{\alpha} = 0.0971$ |
| **SD** | 256.48 | 256.91 | 259.35 | 257.42 | 0.3616 | 0.0004 | 126.24 | $\hat{\alpha} = 0.1327$ |
| **SBAD** | 545.82 | 546.00 | 547.25 | 546.29 | 0.6472 | 3.4 e-13 | 271.91 | $\hat{\alpha} = 0.1298$ |
| | | | | | | | | $\hat{\delta} = 0.0086$ |
| **MOETL** | 278.57 | 279.00 | 281.44 | 279.51 | 0.4585 | 1.8 e-06 | 137.29 | $\hat{k} = -0.0262$ |
| | | | | | | | | $\hat{\lambda} = -1.2363$ |
| **TPAD** | 244.56 | 244.99 | 247.43 | 245.50 | 0.2902 | 0.0083 | 120.28 | $\hat{\theta} = 0.0959$ |
| | | | | | | | | $\hat{\alpha} = 0.3316$ |
| **TPQAD** | 238.77 | 239.20 | 241.64 | 239.70 | 0.4520 | 2.7 e-06 | 117.38 | $\hat{\theta} = 0.0904$ |
| | | | | | | | | $\hat{\alpha} = 11.7621$ |
| **PSBTPAD** | **215.84** | **216.72** | **220.14** | **217.24** | **0.1074** | **0.8295** | **104.92** | $\hat{\theta} = 0.0052$ |
| | | | | | | | | $\hat{\alpha} = 0.5914$ |
| | | | | | | | | $\hat{\beta} = 1.9242$ |

Accordingly, the PSBTPAD is the appropriate model for fitting the data since it has the smallest values of AIC, CAIC, BIC, HQIC and KS with larger P-value as compared to the competitive models considered in this study.

## 7. Conclusions

In this paper, we proposed a new continuous distribution which generalizes the size biased two-parameter Akash distribution. The distribution is named power size biased two-parameter Akash distribution. Various statistical properties of the PSBTPAD are derived and discussed such as the moments, coefficient of variation, coefficient of skewness, coefficient of kurtosis and the distribution of order statistics. The model parameters are estimated using the maximum likelihood estimation procedure. Finally, the distribution is fitted to real data. The new distribution is found to provide a better fit than its competitors used in this study.

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