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A New Quasi Sujatha Distribution

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ABSTRACT

The aim of this paper is to introduce a new quasi Sujatha distribution (NQSD), of which the following are particular cases: the Sujatha distribution devised by Shanker (2016 a), the size-biased Lindley distribution, and the exponential distribution. Its moments and moments-based measures are derived and discussed. Statistical properties, including the hazard rate and mean residual life functions, stochastic ordering, mean deviations, Bonferroni and Lorenz curves and stress-strength reliability are also analysed. The method of moments and the method of maximum likelihood estimations is discussed for estimating parameters of the proposed distribution. A numerical example is presented to test its goodness of fit, which is then compared with other one-parameter and two-parameter lifetime distributions.

Key words: Sujatha distribution, quasi Sujatha distribution, moments, reliability properties, stochastic ordering, stress-strength reliability, estimation of parameters, goodness of fit.

1. Introduction

The Sujatha distribution, introduced by Shanker (2016 a), is defined by its probability density function (pdf) and cumulative distribution function

$$f_1(x;\theta) = \frac{\theta^3}{\theta^2+\theta+2}\left(1+x+x^2\right)e^{-\theta x} \quad ; x>0,\ \theta>0. \tag{1.1}$$

$$F_1(x;\theta) = 1-\left[1+\frac{\theta x(\theta x+\theta+2)}{\theta^2+\theta+2}\right]e^{-\theta x}; \quad x>0, \theta>0 \tag{1.2}$$

This distribution has been introduced for modelling lifetime data from engineering and biomedical science and it has been shown by Shanker (2016a) that it gives better fit

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than both exponential and Lindley (1958) distributions. It is a convex combination of exponential (θ), gamma $(2,\theta)$ and gamma $(3,\theta)$ distributions.

The first four moments about origin of the Sujatha distribution (1.1) are obtained as

$$\mu_1' = \frac{\theta^2+2\theta+6}{\theta\left(\theta^2+\theta+2\right)}\ , \qquad \mu_2' = \frac{2\left(\theta^2+3\theta+12\right)}{\theta^2\left(\theta^2+\theta+2\right)},$$

$$\mu_3' = \frac{6\left(\theta^2+4\theta+20\right)}{\theta^3\left(\theta^2+\theta+2\right)}, \qquad \mu_4' = \frac{24\left(\theta^2+5\theta+30\right)}{\theta^4\left(\theta^2+\theta+2\right)}$$

The central moments of the Sujatha distribution (1.1) are obtained as

$$\mu_2 = \frac{\theta^4+4\theta^3+18\theta^2+12\theta+12}{\theta^2\left(\theta^2+\theta+2\right)^2}$$

$$\mu_3 = \frac{2\left(\theta^6+6\theta^5+36\theta^4+44\theta^3+54\theta^2+36\theta+24\right)}{\theta^3\left(\theta^2+\theta+2\right)^3}$$

$$\mu_4 = \frac{3\left(3\theta^8+24\theta^7+172\theta^6+376\theta^5+736\theta^4+864\theta^3+912\theta^2+480\theta+240\right)}{\theta^4\left(\theta^2+\theta+2\right)^4}.$$

Shanker (2016a) studied some of its important properties including skewness, kurtosis, index of dispersion, hazard rate function, mean residual life function, stochastic ordering, mean deviations, Bonferroni and Lorenz curves and stress-strength reliability. Shanker (2016a) discussed the estimation of parameter using maximum likelihood estimation and discussed the applications of the Sujatha distribution for modelling lifetime data from engineering and biomedical sciences. Shanker (2016b) has also obtained a Poisson mixture of the Sujatha distribution named "Poisson-Sujatha distribution (PSD)" and discussed its various properties, estimation of parameter and applications for counts data. Further, Shanker and Hagos (2016a, 2015) have obtained the size-biased and zero-truncated version of PSD, discussed their statistical properties, estimation of their parameter, and applications for modelling data which structurally excludes zero-counts. Shanker and Hagos (2016b) have a detailed and critical study on applications of zero-truncated Poisson, Poisson-Lindley and Poisson-Sujatha distributions.

Recently Shanker (2016 c) has introduced a quasi Sujatha distribution (QSD) having pdf and cdf

$$f_2\left(x;\theta,\alpha\right) = \frac{\theta^2}{\alpha\theta+\theta+2}\left(\alpha+\theta x+\theta x^2\right)e^{-\theta x}\ ; x>0,\ \theta>0\ \alpha>0 \tag{1.3}$$

$$F_2\left(x;\theta,\alpha\right)=1-\left[1+\frac{\theta x\left(\theta x+\theta+2\right)}{\alpha\theta+\theta+2}\right]e^{-\theta x};\quad x>0,\theta>0,\alpha>0 \tag{1.4}$$

It can be easily verified that the Sujatha distribution and the size-biased Lindley distribution (SBLD) are particular cases of QSD at $\alpha=\theta$ and $\alpha=0$, respectively. Also, if $\alpha\to\infty$, QSD reduces to exponential distribution. Shanker (2016c) has studied its various mathematical and statistical properties including coefficient of variation, skewness, kurtosis, index of dispersion, hazard rate function, mean residual life function, stochastic ordering, mean deviations, Bonferroni and Lorenz curves and stress-strength reliability. The estimation of the parameters using both the maximum likelihood estimation and the method of moments has also been discussed and the goodness of fit has been discussed with a real lifetime data and compared with several well-known distributions.

The main motivation for searching a new two-parameter quasi Sujatha distribution (NQSD) is that the Sujatha distribution is a particular case of QSD whereas both Sujatha and exponential distributions are particular cases of NQSD and hence it is expected and hoped that NQSD will provide a better fit than QSD, Sujatha and exponential distributions.

In this paper, a new two-parameter quasi Sujatha distribution (NQSD) of which one-parameter Sujatha distribution introduced by Shanker (2016a) and exponential distribution are particular cases, has been proposed. Its raw moments and central moments have been obtained and coefficients of variation, skewness, kurtosis and index of dispersion have been discussed. Some of its important statistical properties including hazard rate function, mean residual life function, stochastic ordering, mean deviations, Bonferroni and Lorenz curves, and stress-strength reliability have also been discussed. The estimation of the parameters has been discussed using both the method of moments and the maximum likelihood estimation. A numerical example has been given to test the goodness of fit of the distribution and the fit has been compared with other well-known one-parameter and two-parameter lifetime distributions.

2. A New Quasi Sujatha Distribution

A two-parameter new quasi Sujatha distribution (NQSD) having parameters θ and α is defined by its pdf and cdf

$$f_3\left(x;\theta,\alpha\right)=\frac{\theta^3}{\theta^3+\alpha\theta+2\alpha}\left(\theta+\alpha x+\alpha x^2\right)e^{-\theta x}\ ;x>0,\ \theta>0\ \alpha>0. \tag{2.1}$$

$$F_3\left(x;\theta,\alpha\right)=1-\left[1+\frac{\alpha\theta x\left(\theta x+\theta+2\right)}{\theta^3+\alpha\theta+2\alpha}\right]e^{-\theta x};\quad x>0,\theta>0,\alpha>0 \tag{2.2}$$

It can be easily verified that the Sujatha distribution and exponential distribution are particular cases of NQSD at $\alpha = \theta$ and $\alpha = 0$ respectively. Like QSD, if $\alpha \to \infty$, NQSD reduces to exponential distribution. Further, it can be easily shown that NQSD (2.1) is a convex combination of exponential (θ) gamma $(2,\theta)$ and gamma $(3,\theta)$ distributions. We have

$$f_3(x;\theta,\alpha) = p_1\, g_1(x;\theta) + p_2\, g_2(x;\theta,2) + (1 - p_1 - p_2)\, g_3(x;\theta,3) \tag{2.3}$$

where

$$p_1 = \frac{\theta^3}{\theta^3 + \alpha\theta + 2\alpha} \quad \text{and} \quad p_2 = \frac{\alpha\theta}{\theta^3 + \alpha\theta + 2\alpha}$$

$$g_1(x;\theta) = \theta e^{-\theta x}; x > 0, \theta > 0$$

$$g_2(x;\theta,2) = \frac{\theta^2}{\Gamma(2)} e^{-\theta x} x^{2-1}; x > 0, \theta > 0$$

$$g_3(x;\theta,3) = \frac{\theta^3}{\Gamma(3)} e^{-\theta x} x^{3-1}; x > 0, \theta > 0.$$

Graphs of the pdf of NQSD for varying values of parameters θ and α have been presented in Figure 1. Graphs of the cdf of NQSD for varying values of parameters θ and α have also been presented in Figure 2.

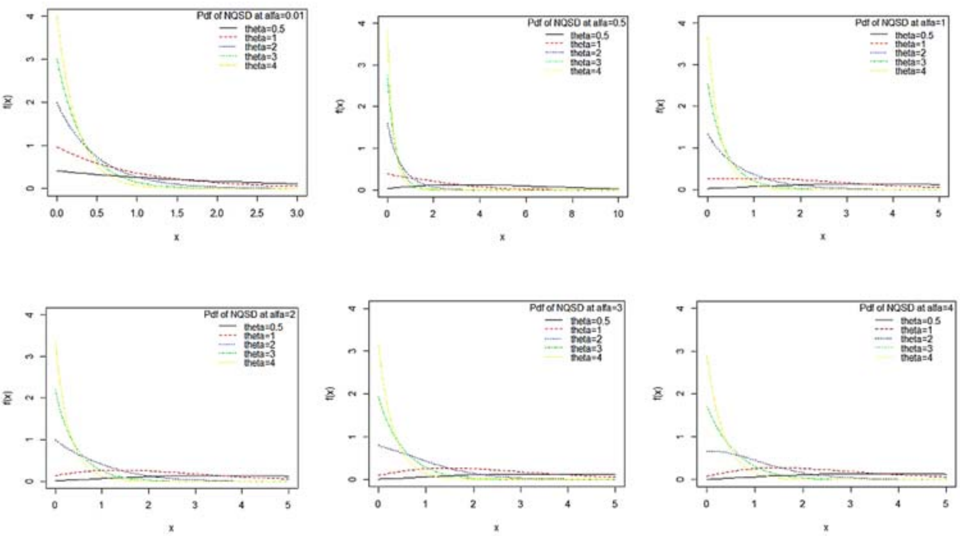

Figure 1. Graphs of the pdf of NQSD for varying values of parameters θ and α

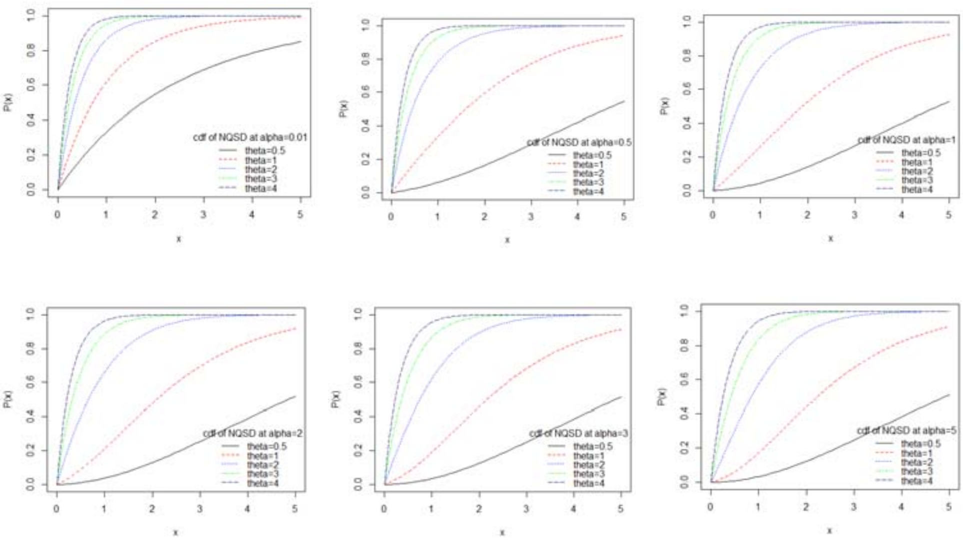

Figure 2. Graphs of the cdf of NQSD for varying values of parameters θ and α

3. Moments and Related Measures

Using the mixture representation (2.3), the r th moment about origin of NQSD (2.1) can be obtained as

$$\mu_r' = \frac{r!\left[\theta^3 + (r+1)\alpha\theta + (r+1)(r+2)\alpha\right]}{\theta^r\left(\theta^3 + \alpha\theta + 2\alpha\right)} \; ; r = 1,2,3,... \tag{3.1}$$

The first four moments about origin of NQSD are thus obtained as

$$\mu_1' = \frac{\theta^3 + 2\alpha\theta + 6\alpha}{\theta\left(\theta^3 + \alpha\theta + 2\alpha\right)}, \qquad \mu_2' = \frac{2\left(\theta^3 + 3\alpha\theta + 12\alpha\right)}{\theta^2\left(\theta^3 + \alpha\theta + 2\alpha\right)},$$

$$\mu_3' = \frac{6\left(\theta^3 + 4\alpha\theta + 20\alpha\right)}{\theta^3\left(\theta^3 + \alpha\theta + 2\alpha\right)}, \qquad \mu_4' = \frac{24\left(\theta^3 + 5\alpha\theta + 30\alpha\right)}{\theta^4\left(\theta^3 + \alpha\theta + 2\alpha\right)}$$

Using the relationship between central moments and moments about origin, central moments of NQSD are obtained as

$$\mu_2 = \frac{\theta^2\left(\alpha^2 + 4\alpha + 2\right) + 16\theta\alpha + 12(\theta + 1)}{\theta^2\left(\alpha\theta + \theta + 2\right)^2}$$

$$\mu_3 = \frac{2\left[\theta^3\left(\alpha^3+6\alpha^2+6\alpha+2\right)+6\theta^2\left(5\alpha^2+7\alpha+3\right)+36\theta\alpha+12\left(3\theta+2\right)\right]}{\theta^3\left(\alpha\theta+\theta+2\right)^3}$$

$$\mu_4 = \frac{3\begin{bmatrix}\theta^4\left(3\alpha^4+24\alpha^3+44\alpha^2+32\alpha+8\right)+8\theta^3\left(16\alpha^3+43\alpha^2+40\alpha+12\right)\\ +24\theta^2\left(17\alpha^2+32\alpha+14\right)+576\theta\alpha+240\left(2\theta+1\right)\end{bmatrix}}{\theta^4\left(\alpha\theta+\theta+2\right)^4}$$

The coefficient of variation $(C.V)$, coefficient of skewness $\left(\sqrt{\beta_1}\right)$, coefficient of kurtosis (β_2) and index of dispersion (γ) of NQSD are given by

$$C.V = \frac{\sigma}{\mu_1'} = \frac{\sqrt{\theta^2\left(\alpha^2+4\alpha+2\right)+16\theta\alpha+12\left(\theta+1\right)}}{\alpha\theta+2\theta+6}$$

$$\sqrt{\beta_1} = \frac{\mu_3}{\mu_2^{3/2}} = \frac{2\left[\theta^3\left(\alpha^3+6\alpha^2+6\alpha+2\right)+6\theta^2\left(5\alpha^2+7\alpha+3\right)+36\theta\alpha+12\left(3\theta+2\right)\right]}{\left[\theta^2\left(\alpha^2+4\alpha+2\right)+16\theta\alpha+12\left(\theta+1\right)\right]^{3/2}}$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{3\begin{bmatrix}\theta^4\left(3\alpha^4+24\alpha^3+44\alpha^2+32\alpha+8\right)+8\theta^3\left(16\alpha^3+43\alpha^2+40\alpha+12\right)\\ +24\theta^2\left(17\alpha^2+32\alpha+14\right)+576\theta\alpha+240\left(2\theta+1\right)\end{bmatrix}}{\left[\theta^2\left(\alpha^2+4\alpha+2\right)+16\theta\alpha+12\left(\theta+1\right)\right]^2}$$

$$\gamma = \frac{\sigma^2}{\mu_1'} = \frac{\theta^2\left(\alpha^2+4\alpha+2\right)+16\theta\alpha+12\left(\theta+1\right)}{\theta\left(\alpha\theta+\theta+2\right)\left(\alpha\theta+2\theta+6\right)}.$$

Note that at $\alpha=\theta$ and $\alpha=0$, these statistical constants reduce to the corresponding statistical constants of Sujatha and exponential distributions. Graphs for C.V, $\sqrt{\beta_1}$, β_2 and γ for varying values of parameters θ and α have been drawn and presented in Figure 3.

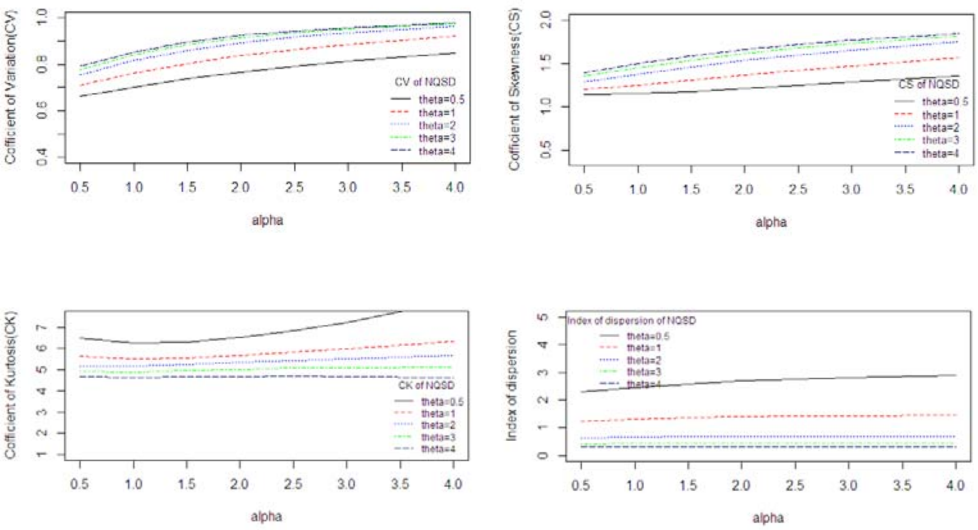

Figure 3. Graphs of C.V, C.S, C.K and I.D of NQSD for varying values of parameters θ and α

4. Hazard Rate Function and Mean Residual Life Function

The hazard rate function (also known as the failure rate function) and the mean residual life function of a continuous random variable X having pdf and cdf $f(x)$ and $F(x)$ are respectively defined as

$$h(x) = \lim_{\Delta x \to 0} \frac{P(X < x + \Delta x \mid X > x)}{\Delta x} = \frac{f(x)}{1 - F(x)} \tag{4.1}$$

and

$$m(x) = E\left[X - x \middle| X > x\right] = \frac{1}{1 - F(x)} \int_x^\infty \left[1 - F(t)\right]\, dt \tag{4.2}$$

The corresponding $h(x)$ and $m(x)$ of NQSD (2.1) are obtained as

$$h(x) = \frac{\theta^3\left(\theta + \alpha x + \alpha x^2\right)}{\alpha\,\theta x(\theta x + \theta + 2) + \left(\theta^3 + \alpha\,\theta + 2\alpha\right)} \tag{4.3}$$

and

$$m(x)=\frac{1}{\left[\alpha\theta x(\theta x+\theta+2)+\left(\theta^3+\alpha\theta+2\alpha\right)\right]e^{-\theta x}}\int_x^\infty\left[\alpha\theta t(\theta t+\theta+2)+\left(\theta^3+\alpha\theta+2\alpha\right)\right]e^{-\theta t}dt$$

$$=\frac{\alpha\theta x(\theta x+\theta+4)+\left(\theta^3+2\alpha\theta+6\alpha\right)}{\theta\left[\alpha\theta x(\theta x+\theta+2)+\left(\theta^3+\alpha\theta+2\alpha\right)\right]} \tag{4.4}$$

It is obvious that $h(0)=\dfrac{\theta^4}{\theta^3+\alpha\theta+2\alpha}=f(0)$ and $m(0)=\dfrac{\theta^3+2\alpha\theta+6\alpha}{\theta\left(\theta^3+\alpha\theta+2\alpha\right)}=\mu_1'$.

Graphs of $h(x)$ of NQSD for varying values of parameters θ and α are presented in Figure 4, whereas graphs of $m(x)$ of NQSD for varying values of parameters θ and α are presented in Figure 5. Graphs of $h(x)$ are either monotonically increasing or decreasing for varying values of parameters. Graphs of $m(x)$ are monotonically decreasing for varying values of parameters.

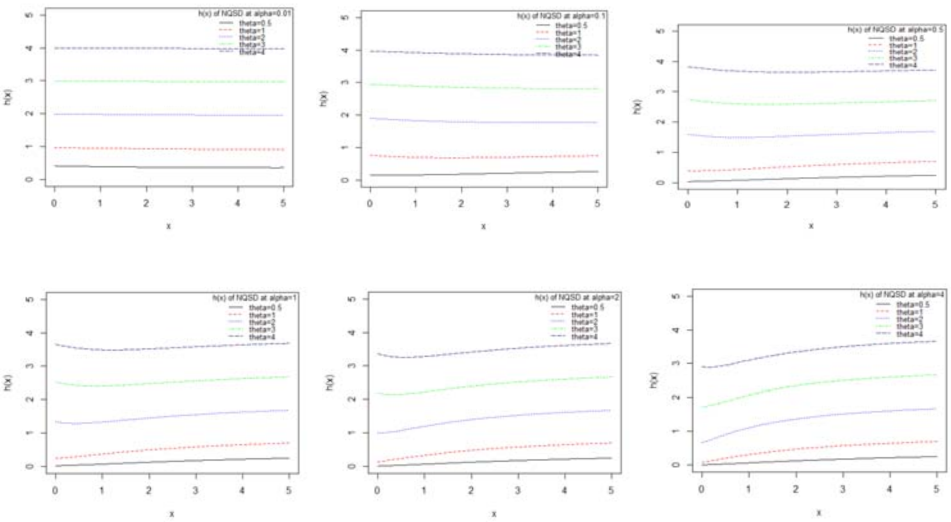

Figure 4. Graphs of $h(x)$ of NQSD for varying values of parameters θ and α

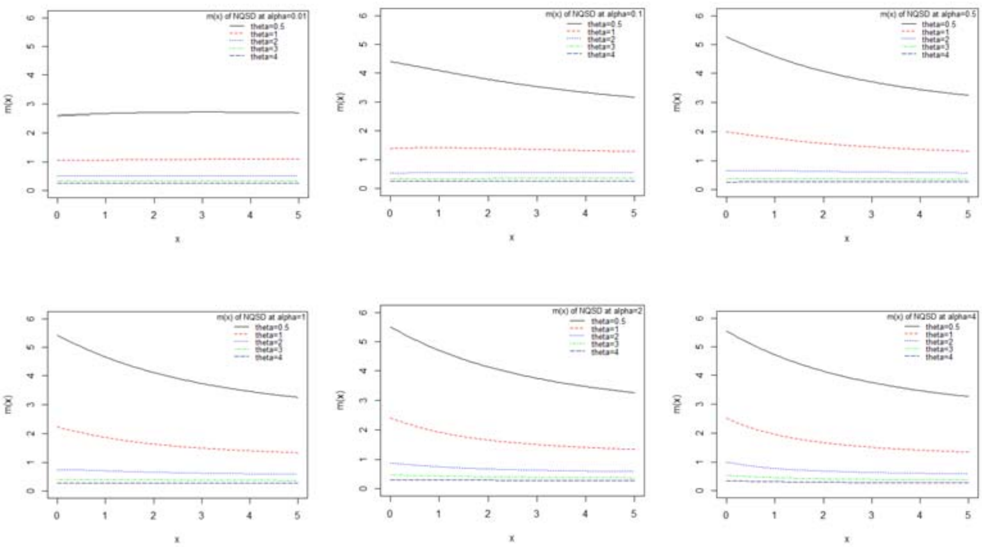

Figure 5. Graphs of $m(x)$ of NQSD for varying values of parameters θ and α

5. Stochastic Orderings

Stochastic ordering of positive continuous random variables is an important tool for judging their comparative behaviour. A random variable X is said to be smaller than a random variable Y in the:

(i) stochastic order $\left(X \leq_{st} Y\right)$ if $F_X(x) \geq F_Y(x)$ for all x

(ii) hazard rate order $\left(X \leq_{hr} Y\right)$ if $h_X(x) \geq h_Y(x)$ for all x

(iii) mean residual life order $\left(X \leq_{mrl} Y\right)$ if $m_X(x) \leq m_Y(x)$ for all x

(iv) likelihood ratio order $\left(X \leq_{lr} Y\right)$ if $\dfrac{f_X(x)}{f_Y(x)}$ decreases in x.

The following results due to Shaked and Shanthikumar (1994) are well known for establishing stochastic ordering of continuous distributions

$$\begin{array}{c} X \leq_{lr} Y \Rightarrow X \leq_{hr} Y \Rightarrow X \leq_{mrl} Y \\ \Downarrow \\ X \leq_{st} Y \end{array}$$

The NQSD is ordered with respect to the strongest 'likelihood ratio ordering' as established in the following theorem:

Theorem: Let $X \sim \text{NQSD}(\theta_1, \alpha_1)$ and $Y \sim \text{NQSD}(\theta_2, \alpha_2)$. If $\alpha_1 = \alpha_2$ and $\theta_1 > \theta_2$ or $\theta_1 = \theta_2$ and $\alpha_1 < \alpha_2$, then $X \le_{lr} Y$ and hence $X \le_{hr} Y$, $X \le_{mrl} Y$ and $X \le_{st} Y$.

Proof: We have

$$\frac{f_X(x;\theta_1,\alpha_1)}{f_Y(x;\theta_2,\alpha_2)} = \frac{\theta_1^3\left(\theta_2^3+\alpha_2\,\theta_2+2\alpha_2\right)}{\theta_2^3\left(\theta_1^3+\alpha_1\,\theta_1+2\alpha_1\right)}\left(\frac{\theta_1+\alpha_1 x+\alpha_1 x^2}{\theta_2+\alpha_2 x+\alpha_2 x^2}\right)e^{-(\theta_1-\theta_2)x} \quad ; x>0$$

Now

$$\ln\frac{f_X(x;\theta_1,\alpha_1)}{f_Y(x;\theta_2,\alpha_2)} = \ln\left[\frac{\theta_1^3\left(\theta_2^3+\alpha_2\,\theta_2+2\alpha_2\right)}{\theta_2^3\left(\theta_1^3+\alpha_1\,\theta_1+2\alpha_1\right)}\right]+\ln\left(\frac{\theta_1+\alpha_1 x+\alpha_1 x^2}{\theta_2+\alpha_2 x+\alpha_2 x^2}\right)-(\theta_1-\theta_2)x.$$

This gives $\displaystyle \frac{d}{dx}\ln\frac{f_X(x;\theta_1,\alpha_1)}{f_Y(x;\theta_2,\alpha_2)} = \frac{(\alpha_1\theta_2-\alpha_2\theta_1)+2(\alpha_1\theta_2-\alpha_2\theta_1)x}{\left(\theta_1+\alpha_1 x+\alpha_1 x^2\right)\left(\theta_2+\alpha_2 x+\alpha_2 x^2\right)}-(\theta_1-\theta_2)$.

Thus, if $\alpha_1 = \alpha_2$ and $\theta_1 > \theta_2$ or $\theta_1 = \theta_2$ and $\alpha_1 < \alpha_2$, $\displaystyle \frac{d}{dx}\ln\frac{f_X(x;\theta_1,\alpha_1)}{f_Y(x;\theta_2,\alpha_2)} < 0$. This means that $X \le_{lr} Y$ and hence $X \le_{hr} Y$, $X \le_{mrl} Y$ and $X \le_{st} Y$. This shows flexibility of NQSD over the Sujatha distribution introduced by Shanker (2016 a) and exponential distributions.

6. Mean Deviations

The amount of scatter in a population is measured to some extent by the totality of deviations usually from the mean and the median, known as the mean deviation about the mean and the mean deviation about the median, and is defined by

$\delta_1(X) = \int_0^\infty |x-\mu|\, f(x)\,dx$ and $\delta_2(X) = \int_0^\infty |x-M|\, f(x)\,dx$, respectively, where $\mu = E(X)$ and $M = \text{Median}\,(X)$. The measures $\delta_1(X)$ and $\delta_2(X)$ can be calculated using the following simplified relationships

$$\delta_1(X) = \int_0^\mu (\mu-x) f(x)\,dx + \int_\mu^\infty (x-\mu) f(x)\,dx$$

$$= \mu F(\mu) - \int_0^{\mu} x\, f(x)\, dx - \mu\left[1 - F(\mu)\right] + \int_{\mu}^{\infty} x\, f(x)\, dx$$

$$= 2\mu F(\mu) - 2\mu + 2\int_{\mu}^{\infty} x\, f(x)\, dx$$

$$= 2\mu F(\mu) - 2\int_0^{\mu} x\, f(x)\, dx \tag{6.1}$$

and

$$\delta_2(X) = \int_0^{M} (M - x) f(x)\, dx + \int_M^{\infty} (x - M) f(x)\, dx$$

$$= M\, F(M) - \int_0^{M} x\, f(x)\, dx - M\left[1 - F(M)\right] + \int_M^{\infty} x\, f(x)\, dx$$

$$= -\mu + 2\int_M^{\infty} x\, f(x)\, dx$$

$$= \mu - 2\int_0^{M} x\, f(x)\, dx \tag{6.2}$$

Using pdf (2.1) and expression for the mean of NQSD, we get

$$\int_0^{\mu} x\, f_3(x;\theta,\alpha)\, dx = \mu - \frac{\left\{\begin{matrix} \mu\theta^4 + \left(\alpha\mu^3 + \alpha\mu^2 + 1\right)\theta^3 + \left(3\alpha\mu^2 + 2\alpha\mu\right)\theta^2 \\ +\left(6\alpha\mu + 2\alpha\right)\theta + 6\alpha \end{matrix}\right\} e^{-\theta\mu}}{\theta\left(\theta^3 + \alpha\theta + 2\alpha\right)} \tag{6.3}$$

$$\int_0^{M} x\, f_3(x;\theta,\alpha)\, dx = \mu - \frac{\left\{\begin{matrix} M\theta^4 + \left(\alpha M^3 + \alpha M^2 + 1\right)\theta^3 + \left(3\alpha M^2 + 2\alpha M\right)\theta^2 \\ +\left(6\alpha M + 2\alpha\right)\theta + 6\alpha \end{matrix}\right\} e^{-\theta M}}{\theta\left(\theta^3 + \alpha\theta + 2\alpha\right)} \tag{6.4}$$

Using expressions from (6.1), (6.2), (6.3) and (6.4), the mean deviation about mean $\delta_1(X)$ and the mean deviation about median $\delta_2(X)$ of NQSD are obtained as

$$\delta_1(X) = \frac{2\left\{\theta^3 + \mu(\mu+1)\alpha\theta^2 + 2(2\mu+1)\alpha\theta + 6\alpha\right\} e^{-\theta\mu}}{\theta\left(\theta^3 + \alpha\theta + 2\alpha\right)} \tag{6.5}$$

$$\delta_2\left(X\right)=\frac{2\left\{\begin{array}{l}M\,\theta^4+\left(\alpha\,M^3+\alpha\,M^2+1\right)\theta^3+\left(3\,\alpha\,M^2+2\,\alpha\,M\right)\theta^2\\+\left(6\alpha\,M+2\alpha\right)\theta+6\alpha\end{array}\right\}e^{-\theta M}}{\theta\left(\theta^3+\alpha\,\theta+2\alpha\right)}-\mu \tag{6.6}$$

7. Bonferroni and Lorenz Curves

The Bonferroni and Lorenz curves (Bonferroni, 1930) and Bonferroni and Gini indices have applications not only in economics to study income and poverty, but also in other fields like reliability, demography, insurance and medicine. The Bonferroni and Lorenz curves are defined as

$$B\left(p\right)=\frac{1}{p\mu}\int_0^q x\,f\left(x\right)dx=\frac{1}{p\mu}\left[\int_0^\infty x\,f\left(x\right)dx-\int_q^\infty x\,f\left(x\right)dx\right]=\frac{1}{p\mu}\left[\mu-\int_q^\infty x\,f\left(x\right)dx\right] \tag{7.1}$$

and

$$L\left(p\right)=\frac{1}{\mu}\int_0^q x\,f\left(x\right)dx=\frac{1}{\mu}\left[\int_0^\infty x\,f\left(x\right)dx-\int_q^\infty x\,f\left(x\right)dx\right]=\frac{1}{\mu}\left[\mu-\int_q^\infty x\,f\left(x\right)dx\right] \tag{7.2}$$

respectively or equivalently

$$B\left(p\right)=\frac{1}{p\mu}\int_0^p F^{-1}\left(x\right)dx \tag{7.3}$$

and

$$L\left(p\right)=\frac{1}{\mu}\int_0^p F^{-1}\left(x\right)dx \tag{7.4}$$

respectively, where $\mu=E\left(X\right)$ and $q=F^{-1}\left(p\right)$.

The Bonferroni and Gini indices are thus defined as

$$B=1-\int_0^1 B\left(p\right)dp \tag{7.5}$$

and

$$G=1-2\int_0^1 L\left(p\right)dp \tag{7.6}$$

respectively.

Using pdf of NQSD (2.1), we get

$$\int_q^\infty x\, f_3\left(x;\theta,\alpha\right)dx = \frac{\left\{q\,\theta^4+\left(\alpha\, q^3+\alpha\, q^2+1\right)\theta^3+\left(3\alpha\, q^3+2\alpha\, q\right)\theta^2+\left(6\alpha\, q+2\alpha\right)\theta+6\alpha\right\}e^{-\theta q}}{\theta\left(\theta^3+\alpha\,\theta+2\alpha\right)} \tag{7.7}$$

Now, using equation (7.7) in (7.1) and (7.2), we get

$$B\left(p\right)=\frac{1}{p}\left[1-\frac{\left\{\begin{array}{l} q\,\theta^4+\left(\alpha\, q^3+\alpha\, q^2+1\right)\theta^3+\left(3\alpha\, q^3+2\alpha\, q\right)\theta^2 \\ +\left(6\alpha\, q+2\alpha\right)\theta+6\alpha \end{array}\right\}e^{-\theta q}}{\theta^3+2\alpha\,\theta+6\alpha}\right] \tag{7.8}$$

and $$L\left(p\right)=1-\frac{\left\{\begin{array}{l} q\,\theta^4+\left(\alpha\, q^3+\alpha\, q^2+1\right)\theta^3+\left(3\alpha\, q^3+2\alpha\, q\right)\theta^2 \\ +\left(6\alpha\, q+2\alpha\right)\theta+6\alpha \end{array}\right\}e^{-\theta q}}{\theta^3+2\alpha\,\theta+6\alpha} \tag{7.9}$$

Now, using equations (7.8) and (7.9) in (7.5) and (7.6), the Bonferroni and Gini indices of QSD are thus obtained as

$$B=1-\frac{\left\{\begin{array}{l} q\,\theta^4+\left(\alpha\, q^3+\alpha\, q^2+1\right)\theta^3+\left(3\alpha\, q^3+2\alpha\, q\right)\theta^2 \\ +\left(6\alpha\, q+2\alpha\right)\theta+6\alpha \end{array}\right\}e^{-\theta q}}{\theta^3+2\alpha\,\theta+6\alpha} \tag{7.10}$$

$$G=\frac{2\left\{\begin{array}{l} q\,\theta^4+\left(\alpha\, q^3+\alpha\, q^2+1\right)\theta^3+\left(3\alpha\, q^3+2\alpha\, q\right)\theta^2 \\ +\left(6\alpha\, q+2\alpha\right)\theta+6\alpha \end{array}\right\}e^{-\theta q}}{\theta^3+2\alpha\,\theta+6\alpha}-1 \tag{8.11}$$

8. Stress-Strength Reliability

The stress-strength reliability describes the life of a component which has random strength X that is subjected to a random stress Y . When the stress applied to it exceeds the strength, the component fails instantly and the component will function satisfactorily until $X > Y$. Therefore, $R = P\left(Y < X\right)$ is a measure of component reliability and in the statistical literature it is known as stress-strength parameter. It has wide applications in almost all areas of knowledge especially in biomedical sciences and engineering.

Let X and Y be independent strength and stress random variables having NQSD (2.1) with parameter (θ_1, α_1) and (θ_2, α_2) respectively. Then, the stress-strength reliability R of NQSD can be obtained as

$$R = P(Y < X) = \int_0^\infty P(Y < X \mid X = x) f_X(x)\, dx$$

$$= \int_0^\infty f_3(x; \theta_1, \alpha_1)\; F_3(x; \theta_2, \alpha_2)\, dx$$

$$= 1 - \frac{\theta_1^3 \begin{bmatrix} 24\alpha_1\alpha_2\theta_2^2 + 6\left\{\alpha_1\alpha_2\theta_2^2 + \alpha_1\alpha_2\theta_2(\theta_2+2)\right\}(\theta_1+\theta_2) \\ +2\left\{\alpha_2\theta_1\theta_2^2 + \alpha_1\alpha_2\theta_2(\theta_2+2) + \alpha_1\left(\theta_2^3+\alpha_2\theta_2+2\alpha_2\right)\right\}(\theta_1+\theta_2)^2 \\ +\left\{\alpha_2\theta_1\theta_2(\theta_2+2) + \alpha_1\left(\theta_2^3+\alpha_2\theta_2+2\alpha_2\right)\right\}(\theta_1+\theta_2)^3 \\ +\theta_1\left(\theta_2^3+\alpha_2\theta_2+2\alpha_2\right)(\theta_1+\theta_2)^4 \end{bmatrix}}{\left(\theta_1^3+\alpha_1\theta_1+2\alpha_1\right)\left(\theta_2^3+\alpha_2\theta_2+2\alpha_2\right)(\theta_1+\theta_2)^5}.$$

It can be easily verified that the above expression reduces to the corresponding expression for the Sujatha distribution and exponential distribution at $(\alpha_1 = \theta_1,\ \alpha_2 = \theta_2)$ and $(\alpha_1 = \alpha_2 = 0)$.

9. Estimation of Parameters

9.1. Method of Moments Estimates (MOME)

Since NQSD (2.1) has two parameters to be estimated, the first two moments about origin are required to estimate its parameters. Equating the population mean to the sample mean, we have

$$\overline{x} = \frac{\theta^3 + 2\alpha\theta + 6\alpha}{\theta\left(\theta^3 + \alpha\theta + 2\alpha\right)} = \frac{\theta^3 + \alpha\theta + 2\alpha}{\theta\left(\theta^3 + \alpha\theta + 2\alpha\right)} + \frac{\alpha(\theta+4)}{\theta\left(\theta^3 + \alpha\theta + 2\alpha\right)}$$

$$\overline{x} = \frac{1}{\theta} + \frac{\alpha(\theta+4)}{\theta\left(\theta^3 + \alpha\theta + 2\alpha\right)}$$

$$\frac{\theta\overline{x}-1}{\theta+4}=\frac{\alpha}{\theta^3+\alpha\theta+2\alpha} \tag{9.1.1}$$

Again, replacing the second population moment with the corresponding sample moment, we have

$$m_2'=\frac{2\left(\theta^3+3\alpha\theta+12\alpha\right)}{\theta^2\left(\theta^3+\alpha\theta+2\alpha\right)}=\frac{2\left(\theta^3+\alpha\theta+2\alpha\right)}{\theta^2\left(\theta^3+\alpha\theta+2\alpha\right)}+\frac{4\alpha\left(\theta+5\right)}{\theta^2\left(\theta^3+\alpha\theta+2\alpha\right)}$$

$$=\frac{2}{\theta^2}+\frac{4\alpha\left(\theta+5\right)}{\theta^2\left(\theta^3+\alpha\theta+2\alpha\right)}$$

$$\frac{m_2'\theta^2-2}{4\left(\theta+5\right)}=\frac{\alpha}{\theta^3+\alpha\theta+2\alpha} \tag{9.1.2}$$

Equations (9.1.1) and (9.1.2) give the following cubic equation in θ

$$m_2'\theta^3+4\left(m_2'-\overline{x}\right)\theta^2-2\left(10\overline{x}-1\right)\theta+12=0 \tag{9.1.3}$$

Solving equation (9.1.3) using any iterative methods such as the Newton-Raphson method, the Regula-Falsi method or the Bisection method, the method of moments estimate (MOME) $\tilde{\theta}$ of θ can be obtained, and substituting the value of $\tilde{\theta}$ in equation (9.1.1) MOME $\tilde{\alpha}$ of α can be obtained as

$$\tilde{\alpha}=\frac{\left(1-\tilde{\theta}\,\overline{x}\right)\left(\tilde{\theta}\right)^3}{\overline{x}\left(\tilde{\theta}\right)^2+2\left(\overline{x}-1\right)\tilde{\theta}-6} \tag{9.1.4}$$

9.2. Maximum Likelihood Estimates (MLE)

Let $\left(x_1,x_2,x_3,\dots,x_n\right)$ be a random sample from NQSD (2.1). The likelihood function L of (2.1) is given by

$$L=\left(\frac{\theta^3}{\theta^3+\alpha\theta+2\alpha}\right)^n\prod_{i=1}^{n}\left(\theta+\alpha x_i+\alpha x_i^2\right)e^{-n\theta\overline{x}}$$

The natural log likelihood function is thus obtained as

$$\ln L=n\ln\left(\frac{\theta^3}{\theta^3+\alpha\theta+2\alpha}\right)+\sum_{i=1}^{n}\ln\left(\theta+\alpha x_i+\alpha x_i^2\right)-n\theta\,\overline{x}$$

The maximum likelihood estimates (MLEs) $\hat{\theta}$ and $\hat{\alpha}$ of θ and α are then the solutions of the following log likelihood equations:

$$\frac{\partial \ln L}{\partial \theta}=\frac{3n}{\theta}-\frac{n\left(3\theta^2+\alpha\right)}{\theta^3+\alpha\theta+2\alpha}+\sum_{i=1}^{n}\frac{1}{\theta+\alpha x_i+\alpha x_i^2}-n\bar{x}=0$$

$$\frac{\partial \ln L}{\partial \alpha}=-\frac{n\left(\theta+2\right)}{\theta^3+\alpha\theta+2\alpha}+\sum_{i=1}^{n}\frac{x_i+x_i^2}{\theta+\alpha x_i+\alpha x_i^2}=0,$$

where $\bar{x}$ is the sample mean.

These two natural log likelihood equations do not seem to be solved directly because they are not in closed forms. However, Fisher's scoring method can be applied to solve these equations. We have

$$\frac{\partial^2 \ln L}{\partial \theta^2}=-\frac{3n}{\theta^2}+\frac{n\left(3\theta^4+\alpha^2-12\alpha\theta\right)}{\left(\theta^3+\alpha\theta+2\alpha\right)^2}-\sum_{i=1}^{n}\frac{1}{\left(\theta+\alpha x_i+\alpha x_i^2\right)^2}$$

$$\frac{\partial^2 \ln L}{\partial \alpha^2}=\frac{n\left(\theta+2\right)^2}{\left(\theta^3+\alpha\theta+2\alpha\right)^2}-\sum_{i=1}^{n}\frac{\left(x_i+x_i^2\right)^2}{\left(\theta+\alpha x_i+\alpha x_i^2\right)^2}$$

$$\frac{\partial^2 \ln L}{\partial \theta\,\partial \alpha}=-\frac{2n\theta^2\left(\theta+3\right)}{\left(\theta^3+\alpha\theta+2\alpha\right)^2}-\sum_{i=1}^{n}\frac{\left(x_i+x_i^2\right)}{\left(\theta+\alpha x_i+\alpha x_i^2\right)^2}$$

The following equations can be solved for MLEs $\hat{\theta}$ and $\hat{\alpha}$ of θ and α of NQSD

$$\begin{bmatrix}\dfrac{\partial^2 \ln L}{\partial \theta^2} & \dfrac{\partial^2 \ln L}{\partial \theta\,\partial \alpha}\\ \dfrac{\partial^2 \ln L}{\partial \theta\,\partial \alpha} & \dfrac{\partial^2 \ln L}{\partial \alpha^2}\end{bmatrix}_{\substack{\hat{\theta}=\theta_0\\ \hat{\alpha}=\alpha_0}}\begin{bmatrix}\hat{\theta}-\theta_0\\ \hat{\alpha}-\alpha_0\end{bmatrix}=\begin{bmatrix}\dfrac{\partial \ln L}{\partial \theta}\\ \dfrac{\partial \ln L}{\partial \alpha}\end{bmatrix}_{\substack{\hat{\theta}=\theta_0\\ \hat{\alpha}=\alpha_0}}$$

where θ_0 and α_0 are the initial values of θ and α, respectively, as given by the method of moments. These equations are solved iteratively until sufficiently close values of $\hat{\theta}$ and $\hat{\alpha}$ are obtained.

10. An Illustrative Example

A numerical example of real lifetime data has been presented to test the goodness of fit of NQSD over other one-parameter and two-parameter lifetime distribution.

The following data represent the tensile strength, measured in GPa, of 69 carbon fibres tested under tension at gauge lengths of 20mm, available in Bader and Priest (1982)

1.312 1.314 1.479 1.552 1.700 1.803 1.861 1.865 1.944 1.958 1.966 1.997
2.006 2.021 2.027 2.055 2.063 2.098 2.140 2.179 2.224 2.240 2.253 2.270
2.272 2.274 2.301 2.301 2.359 2.382 2.382 2.426 2.434 2.435 2.478 2.490
2.511 2.514 2.535 2.554 2.566 2.570 2.586 2.629 2.633 2.642 2.648 2.684
2.697 2.726 2.770 2.773 2.800 2.809 2.818 2.821 2.848 2.880 2.954 3.012
3.067 3.084 3.090 3.096 3.128 3.233 3.433 3.585 3.585

For this data set, NQSD has been fitted along with one-parameter exponential, Lindley and Sujatha distributions and two-parameter QSD. The ML estimates of parameters, values of $-2\ln L$, AIC (Akaike Information Criterion), AICC (Akaike Information Criterion Corrected), BIC (Bayesian Information Criterion) and K-S Statistic (Kolmogorov-Smirnov Statistic) for the considered data set have been computed and presented in Table 1. The formulae for computing AIC, AICC, BIC and K-S Statistic (Kolmogorov-Smirnov Statistic) are as follows:

$AIC = -2\ln L + 2k$, $AICC = AIC + \dfrac{2k(k+1)}{(n-k-1)}$, $BIC = -2\ln L + k\ln n$, and

$K\text{-}S = \operatorname*{Sup}_x \left| F_n(x) - F_0(x) \right|$, where k is the number of parameters involved in the respective distributions, n is the sample size and $F_n(x)$ is the empirical distribution function. The distribution corresponding to the lower values of $-2\ln L$, AIC, AICC, BIC and K-S statistic is the best fit distribution.

Table 1: MLE's, - 2ln L, AIC, AICC, BIC, and K-S of the fitted distributions

Distributions	ML Estimates		$-2\ln L$	AIC	AICC	BIC	KS
	$\hat{\theta}$	$\hat{\alpha}$					
NQSD	1.0693	40.01604	199.36	205.36	205.54	205.36	0.332
QSD	0.44259	87.0494	264.72	268.72	268.90	270.72	0.448
Sujatha	0.93611	------	221.60	223.60	223.66	224.60	0.364
Lindley	0.65450	-------	238.38	240.38	240.44	241.37	0.401
Exponential	0.40794	--------	261.73	263.73	263.79	264.73	0.448

It is obvious from the above table that NQSD is the best distribution among the considered distributions for modelling the considered lifetime data from engineering. Therefore, NQSD can be one of the important lifetime distributions for lifetime data from engineering.

11. Concluding Remarks

A two-parameter new quasi Sujatha distribution (NQSD), which includes one-parameter Sujatha distribution and exponential distribution as particular cases, has been proposed and studied. Its mathematical properties including moments, coefficient of variation, skewness, kurtosis, index of dispersion, hazard rate function, mean residual life function, stochastic ordering, mean deviations, Bonferroni and Lorenz curves, and stress-strength reliability have been discussed. The method of moments and the method of maximum likelihood estimation have also been discussed for estimating its parameters. Finally, a numerical example of real lifetime data set has been presented to test the goodness of fit of NQSD over one-parameter exponential, Lindley, Sujatha distributions and two-parameter QSD.

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