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Generalised Odd Frechet Family of Distributions: Properties and Applications

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ABSTRACT

A new distribution called Generalized Odd Fréchet (GOF) distribution is presented and its properties explored. Some structural properties of the proposed distribution, including the shapes of the hazard rate function, moments, conditional moments, moment generating function, skewness, and kurtosis are presented. Mean deviations, Lorenz and Bonferroni curves, Rényi entropy, and the distribution of order statistics are given. The maximum likelihood estimation technique is used to estimate the model parameters, and finally applications of the model to a real data set are presented to illustrate the usefulness of the proposed distribution.

Key words: Fréchet distribution, Wiebull distribution, structural properties, failure-time, maximum likelihood estimation.

1. Introduction

Recently, some attempts have been made to define new families of distributions to extend well-known models and at the same time provide great flexibility in modelling data in practice. Several techniques could be employed to form a larger family from an existing distribution by incorporating extra parameters. These generalized distributions give more flexibility by adding one "or more" parameters to the baseline model. For example, Gupta et al. (1998) proposed the exponentiated-G class, which consists of raising the cumulative distribution function (cdf) to a positive power parameter. Many other classes can be cited such as the Marshall-Olkin-G family by Marshall and Olkin (1997), beta generalized-G family by Eugene et al. (2002), the gamma-generated family by Zografos and Balakrishnan (2009), Kumaraswamy G family by Cordeiro and de Castro (2011), Generalized beta generated distributions by Alexanderet et al. (2015a), exponentiated generalized-G family by Cordeiro et al. (2013), a new method for generating families of continuous distributions by Alzaatreh et al. (2013), exponentiated T-X family of distributions by Alzaghal et al. (2013), the Lomax generator of distributions by Cordeiro et al. (2014), the WeibullG family of probability distributions by Bourguignon et al. (2014), beta Marshall-Olkin by Alizadeh et al. (2015a), Kumaraswamy odd log-logistic by Alizadeh et al. (2015b), beta odd log-logistic by Cordeiro et al. (2015), Kumaraswamy Marshall-Olkin by Alizadeh et

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al. (2015c), transmuted exponentiated generalized-G family by Yousof et al. (2015), generalized transmuted-G by Nofal et al. (2015), generalized transmuted family by Alizadeh et al. (Alizadeh2015a), another generalized transmuted family by Merovci et al. (2015), Kumaraswamy transmuted-G by Afify et al. (2016a), transmuted geometric-G by Affify et al. (2016b), beta transmuted-H by Afify et al. (2016c), Burr X-G by Yousof et al. (2016), the odd Lindley-G family of distributions by Silva et al. (2016), exponentiated transmuted-G family by Merovci et al. (2016), odd-Burr generalized family by Alizadeh et al. (2016a) the complementary generalized transmuted Poisson family by Alizadeh et al. (2016b), logistic-X by Tahir et al. (2016a), a new Weibull-G by Tahir et al. (2016b), the two-sided power-G class by Korkmaz and Genc (2016), the type I half-logistic family by Cordeiro et al. (2016a), the Zografos-Balakrishnan odd log-logistic family of distributions by Cordeiro et al. (2016b), the generalized odd log-logistic family by Cordeiro et al.(2016c), the beta odd log-logistic generalized family of distributions by Cordeiro et al. (2016d), the Kumaraswamy odd log-logistic family of distributions by Alizadeh et al. (2016d) and a new generalized odd log-logistic family of distributions by Haghbin et al. (2017), the generalized odd log-logistic family of distributions: properties, regression models and applications by Cordeiro et al. (2017), the odd power Cauchy family of distributions by Alizadeh et al. (2018), a new family of the continuous distributions: the extended Weibull-family by Korkmaz (2018a), the Marshll-Olkin generalized G Piosson of distributions by Korkmaz et al. (2018b) and a new family of distributions with properties, regression models and applications by Yousof et al. (2018), among others.

The article is outlined as follows: in Section 2, we introduce the GOF distribution and provide plots of the density and hazard rate functions. Shapes, quantile function, moments, and moment generating function are also obtained. Moreover, mean deviation, order statistics, Lorenz and Bonferroni curves and finally asymptotic properties are presented in this section. Estimation by the method of maximum likelihood and an explicit expression for the observed information matrix are presented in Section 3. The simulation study is presented in Section 4. The applications to real data sets are considered in Section 5. Finally, Section 6 offers some concluding remarks.

2. Generalized Odd Frechet Family of distribution

The cdf of the Generalized Odd Frechet (GOF) Family of distributions is given by

$$F(x;a,b,\xi) = \exp\left\{-(G(x,\xi)^{-a}-1)^{b}\right\} \tag{1}$$

where $\xi = (\xi_1;\xi_2;...)$ is a parameter vector, and a and b are positive parameters. The corresponding probability density function (pdf) is

$$f(x;a,b,\xi) = a\,b\,g(x,\xi)G(x,\xi)^{-a-1}[G(x,\xi)^{-a}-1]^{b-1}\exp\left\{-(G(x,\xi)^{-a}-1)^{b}\right\} \tag{2}$$

For $a=1$ we obtain Odd Frechet family. Some of the possible shapes of the density function (2) of generalized odd Frechet Wiebull distribution (GOFW), for the selected parameter values are illustrated in Figure 1. As seen in Figure 1, the density function can take various forms depending on the parameter values.

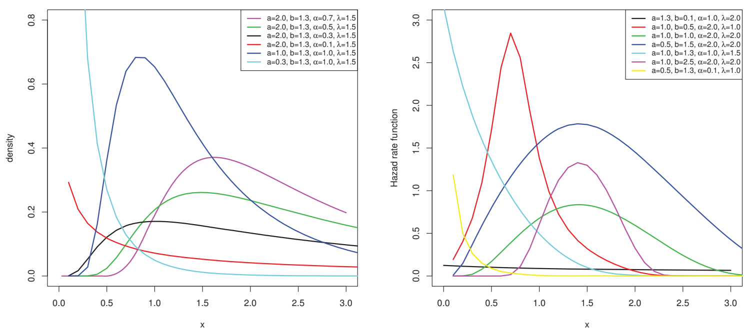

Figure 1: Different shapes of GOFW pdf (left) and Hazard function (right)

2.1. Survival and Hazard Rate Functions

A central role is played in the reliability theory by the quotient of the pdf and survival function. We obtain the survival function corresponding to (1) as

$$R(x) = 1 - \exp\left\{-(G(x,\xi)^{-a} - 1)^b\right\}$$

In reliability studies, The hazard rate [h(x)], reversed-hazard rate function [r(x)] and cumulative hazard rate function [H(X)] are important characteristics and fundamental to the design of safe systems in a wide variety of applications. Therefore, we discuss these properties of the GOF distribution. The $h(x), r(x)$ and $H(x)$ of X take the form

$$h(x) = \frac{a\,b\,g(x)\,G(x)^{-a-1}\,[G(x)^{-a} - 1]^{b-1}\,\mathrm{e}^{-[\mathrm{G(x)^{-a}-1}]^{\mathrm{b}}}}{1 - \mathrm{e}^{-[\mathrm{G(x)^{-a}-1}]^{\mathrm{b}}}}$$

$$r(x) = a\,b\,g(x,\xi)G(x,\xi)^{-a-1}[G(x,\xi)^{-a} - 1]^{b-1}$$

and

$$H(x) = -\log\left(1 - \exp\left\{-(G(x,\xi)^{-a} - 1)^b\right\}\right)$$

Plots of the hrf of the GOFW distribution for several parameter values are displayed in Figure 1.

2.2. Mixture representations for the pdf and cdf

Several structural properties of the extended distributions may be easily explored using mixture forms of Exp-G models. Therefore, we obtain mixture forms of exponentiated-G ("Exp-G") for F (x) and f (x). In this subsection, we provide alternative mixture representations for the pdf and cdf of X. Some useful expansions for (1) can be derived by using the concept of power series and generalized binomial expansion. We have

$$\begin{aligned} F(x) &= \exp\left(-(G(x)^{-a}-1)^{b}\right) = \exp\left(-\left(\frac{1-G(x)^{a}}{G(x)^{a}}\right)^{b}\right) \\ &= \sum_{i=0}^{\infty}\frac{(-1)^{i}}{i!}\left(\frac{1-G(x)^{a}}{G(x)^{a}}\right)^{bi} = \sum_{i,j=0}^{\infty}\frac{(-1)^{i+j}}{i!}\binom{bi}{j}G(x)^{aj}\,G(x)^{-abi} \quad (3) \\ &= \sum_{j,k=0}^{\infty}\sum_{l=0}^{k} w_{j,k,l}\,G(x)^{aj+l} \quad (4) \end{aligned}$$

where

$$w_{j,k,l} = \sum_{i=0}^{\infty}\frac{(-1)^{i+j+k+l}}{i!}\binom{bi}{j}\binom{-abi}{k}\binom{k}{l}$$

Furthermore, the corresponding GOF density function is obtained by differentiating (4)

$$f(x) = \sum_{j,k=0}^{\infty}\sum_{l=0}^{k} w_{j,k,l}\,(aj+l)\,g(x)\,G(x)^{aj+l-1} \quad (5)$$

Using relation (3) we obtain another form of expansions for (1) as bellow, which is used in rest of the paper,

$$F(x) = \sum_{i,j=0}^{\infty}\frac{(-1)^{i+j}}{i!}\binom{bi}{j}G(x)^{a(j-bi)} = \sum_{k=0}^{\infty} e_k H_k(x) \quad (6)$$

where $\bar{G}(x) = 1-G(x)$,

$$e_k = \sum_{i,j=0}^{\infty}\frac{(-1)^{i+j+k}}{i!}\binom{bi}{j}\binom{a(j-bi)}{k} \quad (7)$$

and $H_\delta(x) = (1-G(x))^{\delta}$ is the survival function of the Exp-G distribution with power parameter δ. Then the corresponding GOF density function is obtained by differentiating (6)

$$f(x) = \sum_{k=0}^{\infty} e_k\,h_k(x) \quad (8)$$

where $h_\delta(x) = \delta\,g(x)\overline{G}(x)^{\delta-1}$.

2.3. Moments and Moment Generating Function

Some of the most important features and characteristics of a distribution can be studied through moments (e.g. tendency, dispersion, skewness and kurtosis). Now we obtain ordinary moments and the moment generating function (mgf) of the GOF distribution. The rth ordinary moment of X is given by

$$\mu_r' = E(X^r) = \int x^r f(x)dx = \sum_{k=0}^{\infty} e_k E(Y_k^r) \tag{9}$$

where $E(Y_k^r) = \int x^r k\, g(x)\overline{G}(x)^{k-1}dx$; which can be computed numerically for most parent distributions. The skewness and kurtosis measures can be calculated from the ordinary moments using well-known relationships. One can also find the kth central moment of the GOF distribution through the following well-known equation

$$\mu_k = E(X-\mu)^k = \sum_{r=0}^{k} \binom{k}{r} \mu_r' (-\mu)^{k-r}. \tag{10}$$

Using (10), the variance, skewness and kurtosis measures can be obtained. Skewness measures the degree of the long tail and kurtosis is a measure of the degree of tail heaviness. The skewness can be computed as

$$S = \frac{\mu_3}{\mu_2^{3/2}} = \frac{\mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3}{(\mu_2' - \mu_1'^2)^{3/2}}$$

and the kurtosis is based on octiles as

$$K = \frac{\mu_4}{\mu_2^2} = \frac{\mu_4' - 4\mu_1'\mu_3' + 6\mu_1'^2\mu_2' - 3\mu_1'^4}{\mu_2' - \mu_1'^2}.$$

When the distribution is symmetric $S = 0$, and when the distribution is right (or left) skewed $S > 0 (or\ S < 0)$. As K increases, the tail of the distribution becomes heavier. These measures are less sensitive to outliers and they exist even for distributions without moments.

The rth moment of generalized odd Frechet Weibull (GOFW) distribution using relation (8) is given by

$$\begin{aligned} \mu_r' &= \int_0^{\infty} x^r f(x)dx = \sum_{k=0}^{\infty} k\, e_k \int_0^{\infty} x^r \frac{\alpha}{\lambda}(\frac{x}{\lambda})^{\alpha-1} e^{-(\frac{x}{\lambda})^{\alpha}} (e^{-(\frac{x}{\lambda})^{\alpha}})^{k-1} dx \\ &= \sum_{k=0}^{\infty} k\, e_k \int_0^{\infty} x^r \frac{\alpha x^{\alpha-1}}{\lambda^{\alpha}} e^{-k(\frac{x}{\lambda})^{\alpha}} dx = \lambda^r A(\lambda,\alpha,r) \end{aligned} \tag{11}$$

where $\Gamma(a) = \int_0^{\infty} x^{a-1}e^{-x}dx$ is gamma function and

$$A(\lambda,\alpha,r) = \sum_{k=0}^{\infty} (\frac{e_k}{k^{r/\alpha}})\Gamma(1 + \frac{k^{1/\alpha} r}{\lambda}).$$

Using power series, the moment generating function of GOFW is as bellow

$$M_X(t) = E(e^{tX}) = \sum_{n=0}^{\infty} \frac{t^n}{n!} E(X^n) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \lambda^n A(\lambda,\alpha,n)$$

It is to be highlighted that the equation (11) can be easily computed numerically using mathematical or statistical software. For this purpose, one can compute this equation for a large natural number, say N, instead of infinity in the sums. Therefore, several quantities of X such as moments, skewness and kurtosis can be computed numerically using (11). Plots for skewness and kurtosis are presented in Figure 2.

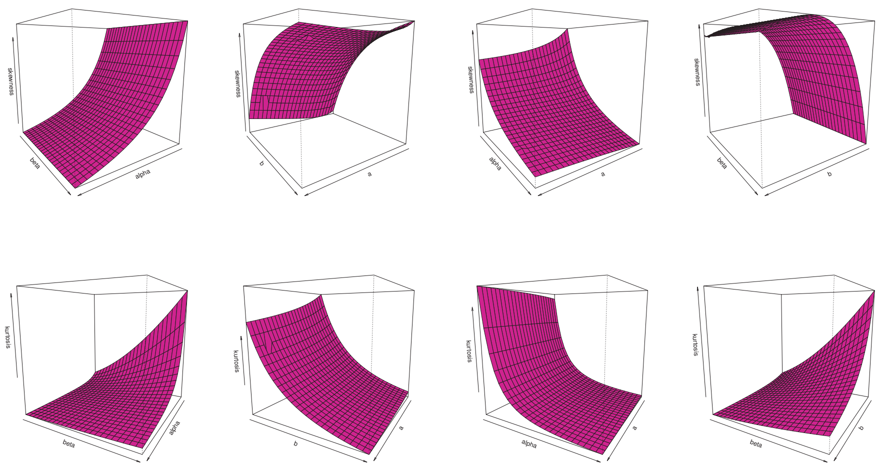

Figure 2: The skewness and kurtosis plots of GOF distribution for selected a, b, α, β.

2.4. Order statistics

Order statistics make their appearance in many areas of statistical theory and practice. Suppose $X_1, \ldots, X_n$ is a random sample from any GOF distribution. Let $X_{i:n}$ denote the ith order statistic. The pdf of $X_{i:n}$ can be expressed as

$$f_{i:n}(x) = K f(x) F^{i-1}(x) \{1-F(x)\}^{n-i} = K \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} f(x) F(x)^{j+i-1},$$

where $K = 1/B(i, n-i+1)$. We use the result of Gradshteyn and Ryzhik (2000) for a power series raised to a positive integer n (for $n \geq 1$)

$$\left(\sum_{i=0}^{\infty} a_i u^i\right)^n = \sum_{i=0}^{\infty} d_{n,i} u^i, \tag{12}$$

where the coefficients $d_{n,i}$ (for $i = 1,2,\ldots$) are determined from the recurrence equation (with $d_{n,0} = a_0^n$)

$$d_{n,i} = (i\,a_0)^{-1} \sum_{m=1}^{i} [m(n+1) - i]\, a_m\, d_{n,i-m}. \tag{13}$$

We can show that the density function of the *i*th order statistic of any GOF distribution can be expressed as

$$f_{i:n}(x) = \sum_{r,k=0}^{\infty} m_{r,k}\, H_{r+k+1}(x), \tag{14}$$

where $H_{r+k+1}(x)$ stands for the the survival function of the Exp-G distribution with power parameter $r+k+1$.

$$m_{r,k} = \frac{n!\,(r+1)\,(i-1)!\,e_{r+1}}{(r+k+1)} \sum_{j=0}^{n-i} \frac{(-1)^j\, f_{j+i-1,k}}{(n-i-j)!\, j!}.$$

Here, e_r is given by (7) and the quantities $f_{j+i-1,k}$ can be determined given that $f_{j+i-1,0} = e_0^{j+i-1}$ and recursively we have:

$$f_{j+i-1,k} = (k\,e_0)^{-1} \sum_{m=1}^{k} [m\,(j+i) - k]\, e_m\, f_{j+i-1,k-m}, k \geq 1.$$

Equation (14) is the main result of this section. Therefore, several mathematical quantities of these order statistics like ordinary and incomplete moments, factorial moments, mgf, mean deviations and others can be derived using this result.

2.5. Mean Deviations, Lorenz and Bonferroni Curves

Mean deviation about the mean and mean deviation about the median as well as Lorenz and Bonferroni curves for the GOF distribution are presented in this section. Bonferroni and Lorenz curves are a widely used tool for analysing and visualizing income inequality. Lorenz curve, L(p) can be regarded as the proportion of total income volume accumulated by those units with income lower than or equal to the volume y, and Bonferroni curve, B(p) is the scaled conditional mean curve, that is, ratio of group mean income of the population.

2.5.1 Mean deviations

The amount of scatter in a population may be measured to some extent by deviations from the mean and median. These are known as the mean deviation about the mean and the mean deviation about the median, defined by

$$\delta_1(X) = \int_0^{\infty} |x-\mu|\, f(x)\,\mathrm{d}x, \quad \textit{and} \qquad \delta_2(X) = \int_0^{\infty} |x-M|\, f(x)\,\mathrm{d}x.$$

respectively, where $\mu = E(X)$ and $M = \text{Median}(X) = Q(0.5)$ denotes the median and $Q(p)$ is the quantile function. The measures $\delta_1(X)$ and $\delta_2(X)$ can be calculated using the relationships

$$\delta_1(X) = 2\mu F(\mu) - 2\int_0^{\mu} x f(x)\,\mathrm{d}x, \quad \textit{and} \quad \delta_2(X) = \mu - 2\int_0^{M} x f(x)\,\mathrm{d}x$$

Finally for GOFW distribution we have

$$\begin{aligned}\delta_1(X) &= 2\mu F(\mu) - 2\sum_{k=0}^{\infty} k e_k \int_0^{\mu} x \frac{\alpha x^{\alpha-1}}{\lambda^{\alpha}} e^{-k(\frac{x}{\lambda})^{\alpha}} dx \\ &= 2\mu F(\mu) - 2\lambda B(\lambda, \alpha, \mu)\end{aligned}$$

where $\gamma(s,x) = \int_0^x t^{s-1} e^{-t} dt$ is lower incomplete gamma function and

$$B(\lambda, \alpha, \mu) = \sum_{k=0}^{\infty} \frac{e_k}{k^{1/\alpha}} \gamma(2, \frac{\mu \lambda^{\alpha}}{k})$$

And

$$\delta_2(X) = \mu - 2\lambda B(\lambda, \alpha, M).$$

2.5.2 Bonferroni and Lorenz curves

The Bonferroni and Lorenz curves have applications in economics as well as other fields like reliability, medicine and insurance. Let $X \sim GOFW(a,b,\alpha,\lambda)$ and $F(x)$ be the cdf of X, then the Bonferroni curve of the GOFW distribution is given by

$$B(F(x)) = \frac{1}{\mu F(x)} \int_0^x t f(t)\mathrm{d}t,$$

where $\mu = E(X)$. Therefore, from (15), we have

$$B(F(x)) = \frac{1}{\mu F(x)} \times \lambda B(\lambda, \alpha, x).$$

The Lorenz curve of the GOFW distribution can be obtained using the relation

$$L(F(x)) = F(x)B(F(x)) = \frac{\lambda}{\mu} B(\lambda, \alpha, x).$$

2.6. Asymptotic Properties

One of the main usage of the idea of an asymptotic distribution is in providing approximations to the cumulative distribution functions of the statistical estimators.

The asymptotic of cdf, pdf and hrf of the GOF distribution as $x \to 0$ are, respectively, given by

$$\begin{aligned}
&F(x) \sim \exp(-G(x)^{-ab}) \quad as \quad x \to 0, \\
&f(x) \sim a\,b\,g(x)\,G(x)^{-ab-1}\exp(-G(x)^{-ab}) \quad as \quad x \to 0, \\
&h(x) \sim a\,b\,g(x)\,G(x)^{-ab-1} \quad as \quad x \to 0.
\end{aligned}$$

The asymptotic of cdf, pdf and hrf of the GOF distribution as $x \to \infty$ are, respectively, given by

$$\begin{aligned}
&1 - F(x) \sim (a\,\bar{G}(x))^{b} \quad as \quad x \to \infty, \\
&f(x) \sim b\,a^{b}\,g(x)\bar{G}(x)^{b-1} \quad as \quad x \to \infty, \\
&h(x) \sim \frac{b\,g(x)}{\bar{G}(x)} \quad as \quad x \to \infty.
\end{aligned}$$

These equations show the effect of parameters on the tails of the GOF distribution.

3. Estimation

Several approaches for parameter estimation have been proposed in the literature but the maximum likelihood method is the most commonly employed. Here, we consider estimation of the unknown parameters of the GOF distribution by the method of maximum likelihood. Let $x_1, x_2, \ldots, x_n$ be observed values from the GOF distribution with parameters a, b and ξ, where ξ is the parameter of based distribution function. The log-likelihood function for $(a;b;\xi)$ is given by

$$\begin{aligned}
\ell_n &= n\log(a) + n\log(b) + \sum_{i=1}^{n}\log(g(x_i,\xi)) - (a+1)\sum_{i=1}^{n}\log(G(x_i,\xi)) \\
&\quad + (b-1)\sum_{i=1}^{n}\log(G(x_i,\xi)^{-a} - 1) - \sum_{i=1}^{n}(G(x_i,\xi)^{-a} - 1)^{b}.
\end{aligned}$$

The derivatives of the log-likelihood function with respect to the parameters $(a;b;\xi)$ are given respectively, by

$$\begin{aligned}
\frac{\partial \ell_n}{\partial a} &= \frac{n}{a} - \sum_{i=1}^{n}\log(G(x_i,\xi)) + (b-1)\sum_{i=1}^{n}\frac{-\log(G(x_i,\xi))G(x_i)^{-a}}{G(x_i,\xi)^{-a} - 1} \\
&\quad + \sum_{i=1}^{n} b(G(x_i,\xi)^{-a} - 1)^{b-1}G(x_i,\xi)^{-a}\log(G(x_i,\xi))
\end{aligned}$$

$$\frac{\partial \ell_n}{\partial b} = \frac{n}{b} + \sum_{i=1}^{n}\log(G(x_i,\xi)^{-a} - 1) - \sum_{i=1}^{n}\log(-(G(x_i,\xi)^{-a} - 1))(G(x_i,\xi)^{-a} - 1)^{b}$$

and

$$\frac{\partial \ell_n}{\partial \xi} = \sum_{i=1}^{n} \frac{g'(x_i,\xi)}{g(x_i,\xi)} - (a+1)\sum_{i=1}^{n} \frac{G'(x_i,\xi)}{G(x_i,\xi)} - (b-1)\sum_{i=1}^{n} \frac{a\,G'(x_i,\xi)G(x_i,\xi)}{G(x_i,\xi)^{-a}-1} + \sum_{i=1}^{n} a\,b\,G'(x_i,\xi)(G^{-a}-1)^{b-1}$$

where

$$g'(x_i,\xi) = \frac{\partial g(x_i,\xi)}{\partial \xi}, \quad G'(x_i,\xi) = \frac{\partial G(x_i,\xi)}{\partial \xi}$$

The maximum likelihood estimates (MLEs) of $(a;b;\xi)$, say $(\widehat{a};\widehat{b};\widehat{\xi})$, are the simultaneous solution of the equations $\frac{\partial \ell_n}{\partial a}=0; \frac{\partial \ell_n}{\partial b}=0; \frac{\partial \ell_n}{\partial \xi}=0.$

For estimating the model parameters, numerical iterative techniques should be used to solve these equations. We can investigate the global maxima of the log-likelihood by setting different starting values for the parameters. The information matrix will be required for interval estimation. Let $\theta=(\alpha;\beta,\gamma,\lambda)^T$, then the asymptotic distribution of $\sqrt{n}(\theta-\hat{\theta})$ is $N_4(0,K(\theta)^{-1})$, under standard regularity conditions (see Lehmann and Casella, 1998, pp. 461-463), where $K(\theta)$ is the expected information matrix. The asymptotic behaviour remains valid if $K(\theta))$ is superseded by the observed information matrix multiplied by $1/n$, say $I(\theta)/n$, approximated by $\hat{\theta}$, i.e. $I(\hat{\theta})/n$. We have

$$I(\theta) = -\begin{bmatrix} I_{\alpha\alpha} & I_{\alpha\beta} & I_{\alpha\gamma} & I_{\alpha\lambda} \\ I_{\beta\alpha} & I_{\beta\beta} & I_{\beta\gamma} & I_{\beta\lambda} \\ I_{\gamma\alpha} & I_{\gamma\beta} & I_{\gamma\gamma} & I_{\gamma\lambda} \\ I_{\lambda\alpha} & I_{\lambda\beta} & I_{\lambda\gamma} & I_{\lambda\lambda} \end{bmatrix}$$

where

$$I_{\alpha\alpha} = \frac{\partial^2 \ell_n}{\partial \alpha^2}; \quad I_{\alpha\beta} = I_{\alpha\beta} = \frac{\partial^2 \ell_n}{\partial \alpha \partial \beta}; \quad I_{\alpha\gamma} = I_{\alpha\gamma} = \frac{\partial^2 \ell_n}{\partial \alpha \partial \gamma}; \quad I_{\alpha\lambda} = I_{\alpha\gamma} = \frac{\partial^2 \ell_n}{\partial \alpha \partial \lambda}$$

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4. Simulation study

In this section, we propose the inverse cdf method for generating random data from the GOF distribution. If $U \sim U(0,1)$ and if G has an inverse function, then

$$x = G^{-1}\left(\left[1+(-ln(u))^{\frac{1}{b}}\right]^{\frac{-1}{a}}\right)$$

has cdf (1). Particularly,

$$x = \lambda\left[-\ln\left(\left([1+(-\ln(u))^{\frac{1}{b}}]^{\frac{-1}{a}}\right)\right)\right]^{\frac{-1}{\alpha}}$$

is a random data with GOFW distribution.

Moreover, the performance of the maximum likelihood method is evaluated for estimating the GOFW parameters using a Monte Carlo simulation study. The mean square error (MSES) and the bias of the parameter estimates are calculated. We generate $N = 10,000$ samples of sizes $n = 50, 55, ..., 300$ from the GOFW distribution with $a = 2$, $b = 1.5$, $\alpha = 1.5, \lambda = 1$. Let $(\widehat{\alpha}, \widehat{\lambda}, \widehat{a}, \widehat{b})$ be the MLEs of the new model parameters and $(s_{\widehat{\alpha}}, s_{\widehat{\lambda}}, s_{\widehat{a}}, s_{\widehat{b}})$ be the standard errors of the MLEs. The estimated biases and MSEs are given by

$$\widehat{Bias}_{\varepsilon}(n) = \frac{1}{N}\sum_{i=1}^{N}(\hat{\varepsilon}_i - \varepsilon)$$

and

$$\widehat{MSE}_{\varepsilon}(n) = \frac{1}{N}\sum_{i=1}^{N}(\hat{\varepsilon}_i - \varepsilon)^2,$$

for $\varepsilon = \alpha, \lambda, a, b$. Figure 3 displays the numerical results for the above measures. We conclude below results from these plots:

- ✓ The estimated biases decrease when the sample size n increases,
- ✓ The estimated MSEs decay toward zero as n increases,

These results reveal the consistency property of the MLEs.

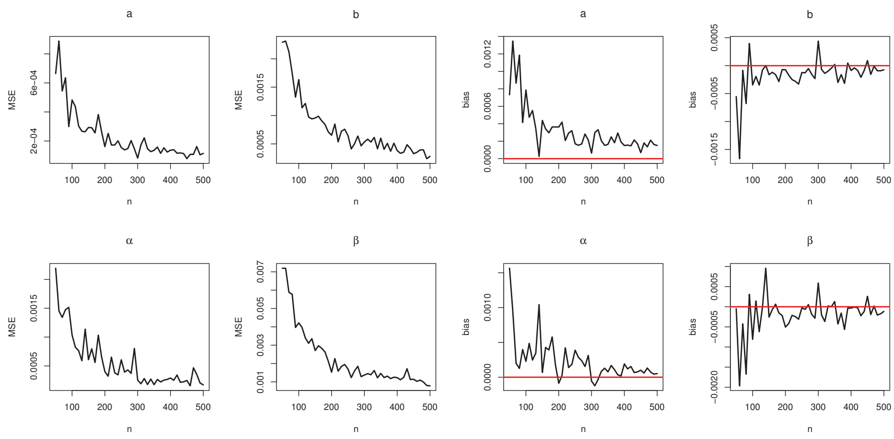

Figure 3: Estimated biases and MSEs for the selected parameter values.

5. Application

In this section, we illustrate the fitting performance of the GOFW distribution using two real data sets. For the purpose of comparison, we fitted the following models to show the fitting performance of GOFW distribution by means of real data set:
i) Weibull Distribution, $W(\alpha,\lambda)$.
ii) Exponentiated Weibull distribution, $EW(\alpha,\lambda,a)$, with distribution function given by

$$F_{ew}(x) = \left(1-e^{-(\frac{x}{\lambda})^{\alpha}}\right)^{a}.$$

iii) Kumaraswamy Weibull, $KwW(a,b,\alpha,\lambda)$

$$F_{kww}(x) = 1-[1-W(x,\alpha,\lambda)^{a}]^{b}.$$

iv) Beta Weibull, $BW(a,b,\alpha,\lambda)$, with distribution function given by

$$F_{bw}(x) = \int_0^{W(x,\alpha,\lambda)} t^{a-1}(1-t)^{b-1}dt.$$

v) Mc Weibull distribution $McW(a,b,\alpha\lambda,c)$, with distribution function given by

$$F_{mcw}(x) = \int_0^{(W(x,\alpha,\lambda))^{c}} t^{a-1}(1-t)^{b-1}dt.$$

vi) Generalized Odd Log-Logistic Weibull distribution $GOLLW(a,b,\alpha,\lambda)$, with distribution function given by

$$F_{gollw}(x) = \frac{W(x,\alpha,\lambda)^{ab}}{W(x,\alpha,\lambda)^{ab}+(1-W(x,\alpha,\lambda)^{a})^{b}}.$$

vii) Type I General Exponential Weibull distribution $TIGEW(a,b,\alpha,\lambda)$, with distribution function given by

$$F_{tigew}(x) = e^{b\left\{1-W(x,\alpha,\lambda)^{-a}\right\}}.$$

viii) Odd Frechet Weibull distribution $OFW(b,\alpha,\lambda)$, with distribution function given by

$$F(x;a,b,\xi) = \exp\left\{-(G(x,\xi)-1)^{b}\right\}$$

Estimates of the parameters of GOF distribution, Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), Cramer Von Mises and Anderson-Darling statistics (W^* and A^*) are presented for each data set. We have also considered the Kolmogorov-

Smirnov (K-S) statistic and its corresponding p-value and the minimum value of the minus log-likelihood function (-Log(L)) for the sake of comparison. Generally speaking, the smaller values of AIC, BIC, W^* and A^*, the better fit to a data set. All the computations were carried out using the software R.

In the rest of the paper, we model the lower discharge of at least seven consecutive days and return period (time) of ten years ($Q_{7,10}$) of the Cuiab*á* River, Cuiab*á*, Mato Grosso, Brazil. We consider the data set presented by Andrade et al. (2007). The calculation of the lower discharge for seven consecutive days and return period (time) of ten years ($Q_{7,10}$) is an important hydrological parameter with applications in the study planning and management of the use of water resources. This study aims to model the lower flood (discharge) of at least seven consecutive days and return period (time) of 10 years ($Q_{7,10}$) in Cuiab*á* River, part of the Brazilian Pantanal (Swamp), since the ecosystem is strongly influenced by the hydrological system. The calculations of $Q_{7,10}$ use a data series from 38 years (January 1962 to October 1999) relating to lower flows of n^o66260001 hydrological station, installed in the Cuiab*á* River in the city of Cuiab*á*, Mato Grosso, Brazil. The data, which have also been analysed by Cordeiro et al. (2012), are listed in Table 1.

Table 1: Data set.

43.86	44.97	46.27	51.29	61.19	61.20	67.80	69.00	71.84
77.31	85.39	86.59	86.66	88.16	96.03	102.00	108.29	113.00
115.14	116.71	126.86	127.00	127.14	127.29	128.00	134.14	136.14
140.43	146.43	146.43	148.00	148.43	150.86	151.29	151.43	156.14
163.00	186.43							

The ML estimates of the parameters and the goodness-of-fit test statistics for the real data set are presented in Table 3 and 4 respectively. As we can see, the smallest values of AIC, BIC, A^*, W^* and $-l$ statistics and the largest p-values belong to the GOFW distribution. Therefore, the GOFW distribution outperforms the other competitive considered distribution in the sense of this criteria.

Here, we also applied likelihood ratio (LR) tests. The LR tests can be used for comparing the GOFW distribution with its sub-models. For example, the test of $H_0 : \alpha = 1$ against $H_1 : \alpha \neq 1$ is equivalent to comparing the GOFW and OFW distributions with each other. For this test, the LR statistic can be calculated by the following relation:

$$LR = 2\left[l(\widehat{\alpha}, \widehat{\beta}, \widehat{\gamma}, \widehat{\lambda}) - l(\widehat{\alpha^*}, 1, \widehat{\gamma^*}, \widehat{\lambda^*})\right],$$

where $\widehat{\alpha^*}, \widehat{\gamma^*}$ and $\widehat{\lambda^*}$ are the ML estimators of α, γ and λ, respectively, obtained under H_0. Under the regularity conditions and if H_0 is assumed to be true, the LR test statistic converges in distribution to a chi square with r degrees of freedom, where r equals the difference between the number of parameters estimated under H_0 and the number of parameters estimated in general, (for $H_0 : \beta = 1$, we have $r = 1$). Table 4 gives the LR statistics and the corresponding p-value. From Table 4, we observe that the computed p-value is too small so

Table 2: Parameter ML estimates (standard errors in the parentheses).

Model	$\widehat{a}$	$\widehat{b}$	$\widehat{\alpha}$	$\widehat{\lambda}$	$\widehat{c}$
$Weibull(\alpha,\lambda)$	–	–	3.3298	123.2008	–
			(0.4430)	(6.3067)	–
$EW(a,\alpha,\lambda)$	0.3522	–	6.8679	150.7316	–
	(0.2271)		(3.3323)	(14.4243)	–
$KwW(a,b,\alpha,\lambda)$	42.6066	1964.352	0.2112	7.2576	–
	(33.5498)	(8567.8074)	(0.1208)	(18.8249)	–
$BL(a,b,\alpha,\lambda)$	0.4034	0.3105	5.7524	114.9745	–
	(0.2071)	(0.3381)	(2.1554)	(33.4861)	–
$McW(a,b,\alpha,\lambda,c)$	0.1293	868.3850	0.5352	24.3734	112.9874
	(0.1093)	(4921.221)	(0.8856)	(122.134)	(401.6450)
$GOLLW(a,b,\alpha,\lambda)$	0.1734	4.7498	5.2297	94.0411	–
	(0.0234)	(0.0093)	(0.0039)	(0.0039)	–
$TIGEW(a,b,\alpha,\lambda)$	1.9133	0.0787	9.8806	164.239	–
	(2.2559)	(0.0555)	(5.6555)	(15.3034)	–
$OFW(b,\alpha,\lambda)$	–	3.3892	0.8968	49.1821	–
	–	(6.624)	(0.8048)	(54.0428)	–
$GOFW(a,b,\alpha,\lambda)$	2.2737	0.1542	5.0860	92.4172	–
	(0.5557)	(0.0274)	(0.0034)	(0.0034)	–

Table 3: Goodness-of-fit test statistics.

Model	W^*	A^*	$p-value$	AIC	BIC	$-l$
$Weibull(\alpha,\lambda)$	0.1019	0.6238	0.4312	386.6742	389.9494	191.3371
$EW(a,\alpha,\lambda)$	0.0585	0.4091	0.8515	386.8977	391.8104	190.4488
$KwW(a,b,\alpha,\lambda)$	0.1210	0.7323	0.3251	391.7345	398.2848	191.8672
$BL(a,b,\alpha,\lambda)$	0.0540	0.3879	0.8466	388.6756	395.2260	190.3378
$McW(a,b,\alpha,\lambda,c)$	0.0616	0.4093	0.5995	389.6777	397.8656	189.8388
$GOLLW(a,b,\alpha,\lambda)$	0.0358	0.2887	0.7564	385.1893	391.7396	188.5946
$TIGEW(a,b,\alpha,\lambda)$	0.0615	0.4140	0.6144	387.4896	394.0399	189.7448
$OFW(b,\alpha,\lambda)$	0.2655	1.6203	0.1651	400.1903	405.1031	197.0951
$GOFW(a,b,\alpha,\lambda)$	0.0285	0.2391	0.9775	382.8198	389.3701	187.4099

we reject the null hypotheses and conclude that the GOFW fits the first data better than the considered sub-model according to the LR criterion.

Table 4: The LR test results.

	Hypotheses	LR	p-value
GOFW versus OFW	$H_0: a = 1$	18.8816	0.00001

In addition, PP plot of the GOFW distribution are plotted in Figure 4. We also plotted the fitted pdfs and cdfs of the considered models for the sake of visual comparison, in Figure 5. Figure 4 and 5 suggest that the GOFW fits the skewed data very well.

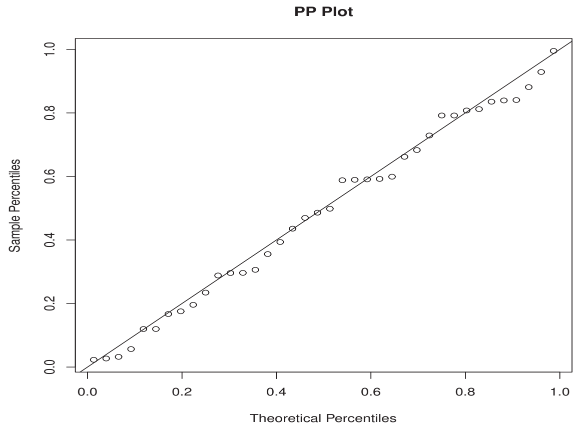

Figure 4: The PP plot.

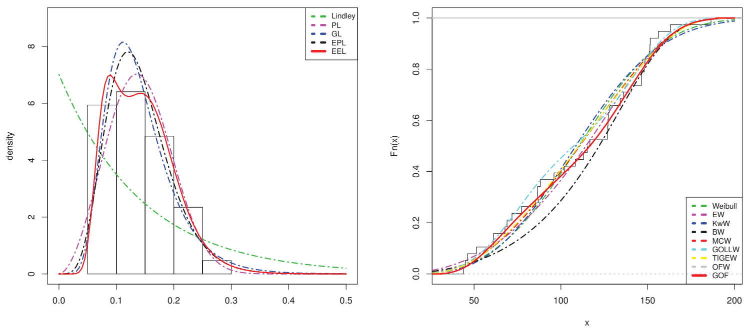

Figure 5: Fitted densities of distributions.

6. Conclusion

In this paper, we present a new class of distributions called the Generalized Odd Frechet (GOF) family of distributions. The statistical properties of the GOF distribution including the hazard and reverse hazard functions, quantile function, moments, incomplete moments, generating functions, mean deviations, Bonferroni and Lorenz curves, order statistics and maximum likelihood estimation for the model parameters are given. Simulation studies were conducted to examine the performance of the new GOF distribution. We also present applications of this new model to a real life data set in order to illustrate the usefulness of the distribution.

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