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Predicting Polish transport industry equilibrium characteristics as an inverse problem: An Entropy Econometrics Model

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ABSTRACT

The business environment dynamics is governed by a high degree of uncertainty and risk; consequently, in a majority of cases investors face serious difficulties when making business decisions. Additionally, when detailed statistical information relating to industry is missing, any decisions may become a matter of highly risky conjectures.
The present article proposes a simultaneous equation model based on the entropy econometrics estimator for recovering some key industrial subsector long-term equilibrium characteristics in the situation where only sparse, insufficient statistical information is available (e.g. only aggregated data on the whole industry).
The model is applied to the transportation equipment manufacturing industry in Poland, which is composed of eight sub-sectors. As a result of the above procedure, an observation has been made that all firms from different sub-sectors have to increase their steady-state concentration ratios, while the highest concentration corresponds to the lowest increase in profitability. The model outputs conform to the market tendency in this sector and should lead to further applications of the NCEE methodology in business activity on a worldwide scale.

Key words: transport industry, inverse problem, econometrics, non-extensive entropy econometrics.

1. Introduction

One of the most important areas of services is transport, which largely affects the economic development of each country. Not only is it an instrument for the exchange of goods and services but also an important factor in GDP growth and it also influences the development of other sectors of the national economy. It is worth emphasizing that the production of transport equipment is an extremely important determinant of

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transport development[3]. In the European Union, three groups of countries can be distinguished in this respect. The first group includes countries where the average share of the production of transport equipment in the global production in 2012–2018 ranged from 7 to about 12 percent (Slovakia, the Czech Republic, Hungary and Germany). The second group, which includes Romania, Sweden, Poland, Slovenia and Spain, covers countries where this share amounts to around 4 percent, and the remaining countries do not exceed the 3 percent share.

In Poland, in the entities included in the production of transport equipment, after the decline in dynamics in 2012, a successive increase was observed in subsequent years, including the highest in 2015 (by 11.1%). In the last two years of the analysed period, the growth rate of global production slightly slowed down and was lower than the total. Both the pace and the volatility of dynamics in production entities for transport was shaped mainly by the results achieved by entities producing motor vehicles, trailers and semi-trailers, excluding motorcycles. The share of this division's revenues accounts for approximately 90% of production revenues for transport. The remaining production showed significant fluctuations in dynamics. After a period of growth in 2012–2015, in the next two years, global production in this division decreased, while the last year of the analysed period brought a significant increase (by 19.9%).

What is also interesting is the fact that the global production in Poland calculated for entities employing more than 10 people in the years 2012–2016 brought a stable growth (in the range of 1.5% -3.4%). Both 2017 and 2018 saw acceleration in the growth rate of global production.

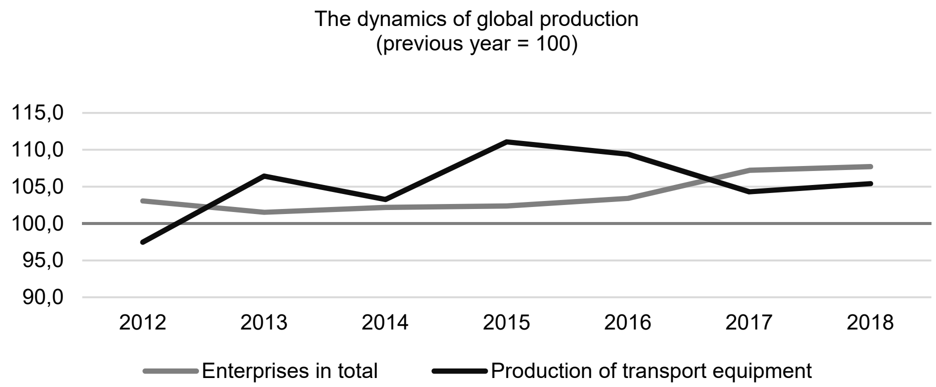

Figure 1. Dynamics of global production

[3] Production of transport equipment consists of two divisions: production of motor vehicles, trailers and semi-trailers, excluding motorcycles (29) and production of other transport equipment (30).

The present paper applies the power-law (PL)-related cross-entropy econometrics (PL-CEE) methodology for recovering the main optimal equilibrium characteristics of the Polish transport industry sub-sectors. In this paper, we will reply to the business question concerning, among others, the optimal number of subsector firms of the transport manufacturing industry consistent with steady-state industry configuration given industry initial characteristic conditions.

In this kind of the problem, we are dealing with ill-behaved inverse problems, suggesting that we want to recover a larger number of model parameters than there are associated data point observations known with uncertainty in this study. As documented in recent publications, the Tsallis power-law (PL)-based non-extensive cross-entropy econometrics (NCEE) approach better deals, conceptually, with such complex non-linear inverse problems. NCEE is based upon the q-generalized Kullback-Leibler (K-L) information divergence criterion function under constraining information characterized by the Bayesian information processing optimal rule. Thus, we consider PL-related non-extensive entropy will remain valuable even in the case of low-frequency series since the outputs provided by Gaussian law correspond to the limiting case of the Tsallis entropy when the Tsallis q-parameter equals unity.

2. Modelling the Polish Industry of Transport

For decades, statistical and mathematical tools to handle ill-posed inverse problem systems have been sought in diverse fields–model parameter estimation, medical imaging, modelling in the life sciences, oil and mineral deposit exploration, shape optimization, etc. For more about inverse problems, see, e.g. Tikhonov regularization theory, Gibbs-Shannon-Jaynes. Interestingly enough, a non-particular hypothesis is required while applying the PL-CEE model in contrast with the traditional econometrics techniques, which generally impose a large number of not always realistic hypotheses on the model.

Contrary to many other fields, the management or economics science, in general, have neglected the link between phenomena and power-law (Gabaix X., September 2008) characterizing complex systems within the class of Levy's processes. In light of recent literature, the amplitude and frequency of socio-economic fluctuations are not considered to substantially diverge from many other extreme events, natural or human-related, once they are explained at the same time (or space) scale. Y. Ikeda and W. Souma (2008) have worked on an international comparison of labour productivity distribution for manufacturing and non-manufacturing firms. A power-law distribution in terms of firms and sector productivity has been found in the US and Japan data. Testing Gibrat's law of proportionate effect, Fujiwara et al. (2004) have found, among other things, that the upper-tail of the distribution of the firm size can be fitted with a power-law (Pareto–Zipf law). According to many studies (e.g. Bottazzi G. et al., 2007; Champernowne D. G., June 1953), a large array of economic laws take the form of PL, in particular, macroeconomic scaling laws, distribution of income, wealth, the size of cities and firms, and the distribution of financial variables such as returns and trading volume. In a recent monograph publication (2019), the author has proposed a theorem linking low-frequency time series socio-economic phenomena–and thus input-output one period systems–with PL distribution. The above citations are not exhaustive.

The PL-CEE is a precious device for econometric modelling even in the case of low-frequency series since outputs provided by the Gibbs-Shannon entropy approach correspond to the Tsallis entropy limiting case of Tsallis q-parameter equal unity. What is more, many complex phenomena involve the long-range correlations which can continuously be seen when data are time (space) scale-aggregated. This could be because of the interaction between the functional relationships describing the phenomena involved and the inheritance properties of power-law (PL). Thus, delimiting the threshold values for a PL (Levy's stable process) transition plausibly towards the Gaussian structure as a function of data frequency level is difficult since each phenomenon may display its rate of convergence towards the central theorem limit attractor. Consequently, a systematic application of the Shannon-Gibbs entropy approach, even based on annual data, could lead to unstable and misleading results. Inversely, since non-extensive Tsallis entropy generalizes the exponential family of laws, it should fit well with high or low-frequency series. In particular, Mantegna R. N. and Stanley H. E. (1999) have studied the dynamics of a general system composed of interacting units each with a complex internal structure comprising many subunits, where they grow in a multiplicative way over 20 years. They found a system following a PL distribution. This is similar to the present case study, where we deal with an industrial sector composed of sub-sectors within which a large number of economic agents provide complex business activities for a given period.

Following the above reasoning, the present study based on non-extensive entropy econometrics extends Shannon-Gibbs maximum entropy econometrics to non-ergodic systems. As in statistical physics, socio-economic random events should display two types of stochastic behaviour: ergodic and non-ergodic. Whenever isolated in a closed space, ergodic systems dynamically visit with equal probability all the allowed micro-states (Tsallis, 2009). This is the case for Gibbs-Shannon entropy. Next, since all events are independent or quasi-independent (locally dependent) and equally probable, this means that the above entropy is a linear, positive function of the number of possible states – thus of new data, and then is extensive. In reverse, as a consequence of possible multi-level correlation between system microstates, non-ergodic systems are characterized by entropy which is no longer a linear, positive function of the number of possible states, and then non-extensive. An important fact to be noticed here is the connection between information theory and a Gaussian variable. This connection results from the fact that a Gaussian variable displays the largest entropy among all random variables of equal variance.

Q-generalized Cross-entropy for Inverse Problem Solution

As already said above, the model to be estimated displays more unknown parameters than observed data point observations. In this section, we recall the definition of an ill-posed inverse problem and present a PL-related cross-entropy model in the context of the proposed model. In essence, the canonical ill-posed inverse problem as the one we deal with in this paper can be formally presented as follows:

$$X(\zeta) = \int_D g(Y)h(Y,\zeta)dY + b(\zeta) \tag{1}$$

X : means the observed matrix of updated priors, e.g. the prior data matrix in Table 1,

Y : designates the unknown matrix of the Polish optimal, long-run subsector transport industry configuration to be later estimated,

D: defines the Hilbert support space of the model,

g : is the transformation kernel linking measures X and Y,

b : explains random errors.

This is a basic model which consists in solving an integral equation of the first kind. As said in (Bwanakare, 2014), inverse problem recovery finds application in various fields of science, particularly in the context of Optimal Control Theory. Among different techniques proposed for solving this type of problems, the Tikhonov related regularization theory remains the most applied besides the Gibbs-Shannon-Jaynes maximum (cross) entropy principle and the ill-posed stationary first-order Markov process, in which the operator g can be seen as a generalized transition matrix while X and Y as the Markov states. The contribution of this paper consists in extending the application of the non-extensive cross entropy formalism to search for global regularity—consistent with the maximum (non-extensive) entropy principle—while yielding the smoothest reconstructions of the Polish optimal subsector transport industry configuration, given initial conditions to be presented in the next paragraphs, according to the Jaynes approach.

Next, we follow recent works applying the non-extensive entropy econometrics and define a q-Tsallis-Kullback-Leibler dual entropy criterion function for forecasting the Polish optimal subsector transport industry configuration, as follows:

$$Min\, H_q\left(p//p^0\right) \equiv \lambda \sum p_{kjm} \frac{\left[\frac{p_{kjm}}{p^o{}_{kjm}}\right]^{q-1} - 1}{q-1} + (1-\lambda) \sum \mu_{k\bullet s} \frac{\left[\mu_{j\bullet s}/\mu^o{}_{j\bullet s}\right]^{q-1} - 1}{q-1} \tag{2}$$

Subject to

$$\sum_j P_j = 1 \text{ with} \tag{3}$$

$$\sum_{s>2\ldots S}^{S} \mu_{j\bullet s} = 1 \tag{4}$$

$$\Omega(\text{Nj,Xj,Yj}) = \text{Cij} \tag{5}$$

where:

X_j : transport industry subsector average costs,

Y_j : transport industry subsector average production,

N_j: number of entreprises in a given subsector,

p_{ikm} : probability distribution on the support space point m defining the parameter k in the equation i

$\mu_{j\bullet s}$ is the random error probability on subsector accounts defined on a support space s,

λ: weight on parameters in the criterion function.

The system of equations explained in the equation 5 will be explained later in the next section. Nevertheless, the main fact to underscore is that the system stands for an inverse problem, suggesting that the model presents more parameters to estimate than the observation points.

There exist a few types of constraining forms defining expectations in Tsallis statistics. In the above model we apply the normalized Tsallis-Mendes-Plastino (TMP) constraints (also known as q-averages or an escort distribution) to the reparametrized parameters; see, e.g. Golan (1996) of the system of equations (equ. 5).

The form of the TMP is as follows:

$$\left\langle y_q \right\rangle = \sum_i \frac{p_i{}^q}{\sum_i p_i{}^q} y_i \tag{6}$$

The real q stands for the Tsallis parameter, whose value varies between 1 and 5/3, suggesting the case of phenomena evolving within the Gaussian basin of attraction. If the q-Tsallis parameter recovers the value 1, we get the PL limiting Gaussian case already alluded to in the Introduction section.

Above, $H_q\left(p // p^0\right)$ is nonlinear and measures the relative (cross-entropy) entropy in the model. The symbol $//$ is a "distance metric[4]" of divergence information. We need to find the minimum divergence between the prior p^0 and the posterior p (equ. 2) while the imposed restrictions (equ. 3–5) must be fulfilled. For more information about cross-entropy interpretation; see, e.g. Golan (1996), Bwanakare (2014). As far as the parameter confidence area is concerned, we send interested readers, e.g. to the work (Bwanakare, 2019). Finally, it would be worthwhile to summarize below the main steps to be followed while applying the proposed cross-entropy approach:

a) fixing the phenomenon to be modelled, its explicative variables, plus its mathematical form,
b) collecting sample data,
c) setting up parameter support space points for each parameter and for the random component. The support space points are defined over the potential existence area of the parameters,
d) setting up the initial values for each parameter. These values should reflect the highest knowledge about each parameter,
e) building a program code linking all the information provided in Steps *a* through *d*. The main part of this program is of the optimization formulated as follows.

Minimizing the weighted divergence between unknown posterior and prior probabilistic of a non-extensive Tsallis entropy functional subject to the next Bayesian[5] restrictions:

- Moment formulation in the form of econometric model equations according to Steps *c* and *d* above.
- The random component is formulated according to Step *c*.
- The regular conditions must sum the probability space points of each parameter up to unity.

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[4] However, note that K-L divergence is not a true metric since it is not symmetric and does not satisfy the triangle inequality.

[5] For the relationship between the Bayesian and maximum entropy parameter parameterization; see, e.g. (Golan A., 1996).

3. The proposed business model and input data

The system of equations Ω(Nj,Xj,Yj) in equ. (5) relates to the below econometric model (equ. 7–11), which stands for the main constraining component of the cross-entropy system (equ. 2–5).

The econometric model is as follows:

$$N_{jt} = \alpha_{0j} + \lambda_{1j}N_{jt-1} - \alpha_{2j}V_{jt} + \varepsilon_{1j} \quad (7)$$

$$X_{jt} = \beta_{0j} + \lambda_{2j}X_{jt-1} + \beta_{2j}Y_{jt} + \varepsilon_{2j} \quad (8)$$

$$Y_{jt} = \delta_{0j} + \lambda_{3j}Y_{jt-1} - \delta_{2j}X_{jt} + \varepsilon_{3j} \quad (9)$$

$$V_{jt} = X_{jt}/Y_{jt} \quad (10)$$

$$X_{jt}/Y_{jt} \leq 1 \quad (11)$$

N_{jt}: number of firms of the j subsector of the Polish transport industry for the period t,

X_{jt}: inputs of firms in the subsector j of the Polish transport industry for the period t,

Y_{jt}: outputs of firms in the subsector j of the Polish transport industry for the period t,

V_{jt}: level of technology of firms in the subsector j of the Polish transport industry for the period t,

ε_j: common random term in the j subsectors of the transport Polish industry.

In the above model, we deal with four interconnected simultaneous dynamic equations of which one equation forms a deterministic relation. Reasoning trough traditional econometrics, we may set the assumption of a component random error reflecting individual behaviour of each of the 8 subsectors and a correlated random error affecting the whole sector. Thus, each of the three first equations accounts for 18 unknown parameters to be estimated, suggesting 54 parameters for the whole model based on data from one period of time (2018). This expectation model describes a partial adjustment of each of the three equations N_{jt}, X_{jt}, Y_{jt} of which the expected values $N^{ex}{}_j$, $X^{ex}{}_j$, $Y^{ex}{}_j$ have to be determined through the estimated parameters of the above equation system. The expected number $N^{ex}{}_j$ (in the steady state) of firms in the different 8 subsectors of the Polish transport industry will depend on the present technological coefficient V_j, which explains the relation between input and output. The $X^{ex}{}_j$ is a response of the present level of output while producers base their future output on the present level of input. Let us recall below basic aspects of a partial adjustment model. Formally, let $y_t *$ be the unknown, targeted level of y_t:

$$y_t * = \alpha + \beta_i x_{it} + \varepsilon_t,\ t=1..T,\ i=1..K \quad (12)$$

And a progressive adjustment equation:

$$y_t - y_{t-1} = (1-\lambda)(y_t * - y_{t-1}, with (0<\lambda<-1) \quad (13)$$

Solving the second equation for y_t and inserting the first expression for y_t, we finally obtain the next equation (Koyck, 1954):

$$y_t = \alpha' + \beta'_i x_{it} + \lambda y_{t-1} + \varepsilon'_t \tag{14}$$

Since this form is linear in parameters and disturbance non-auto correlated (Greene , 2011), the LS estimator will generate consistent and efficient estimates.

Estimated parameters α' and β'_i are the short-run multipliers. To obtain the long-run effect, one transforms $\alpha = \alpha'/(1-\lambda)$ and $\beta_i = \beta'_i/(1-\lambda)$. The long-run disturbance estimates become $\varepsilon_t = \varepsilon'_t/(1-\lambda)$. We then retrieve estimates of equation (12) explaining the targeted value of y_t^*. Next, for λ equal to zero, we may have to deal with a pragmatic agent who prefers to pay the whole attention on the present environment, thereby ignoring information of the past. In the present study, the short-run and long-run effects are estimated and presented below in Tables 2 and 3.

On the epistemological side, the particular advantage of the model is to link the generalized maximum entropy principle with the Bayes optimal information processing rule trough an econometric model embedded in the system as a constraining structure explained as model moments. Finally, Table 1 presents the priors and data used to estimate the model.

Table 1. Some key parameters of the Polish current transport industry Subsectors in 2018

Transport industry Subsectors (Manufacture of)	number of firms(N_j)	Average gross output/(1000*N_j)	Average intermediary Input/1000*N_j	Ratio input-output (V)
Manufacture of motor vehicles	35	2002	1402	0.7
Manufacture of bodies (coachwork) for motor vehicles; manufacture of trailers and semi-trailers	96	77	50	0.649
Manufacture of parts and accessories for motor vehicles	323	301	213	0.708
Building of ships and boats	61	68	48	0.702
Manufacture of railway locomotives and rolling stock	25	322	222	0.,692
Manufacture of air and spacecraft and related machinery	41	241	149	0.619
Manufacture of military fighting vehicles	4	238	101	0.422
Manufacture of transport equipment n.e.c.	38	64	38	0.599

Source: Own based on Statistics Poland data.

4. Outputs and discussion

This section presents the outputs of the proposed model. The computations of the NCEE model were carried out with the GAMS (General Algebraic Modelling System) code. Table 2 presents the new post-entropy outputs of the industry subsector steady state optimal configuration resulting from the NCEE model. Given priors presented in Table 1 and the formal model explained in equ. 2-11, we note a long-run potential increase of about 99% of the total number of firms. The new structure represents the expected steady-state configuration of the Polish transport industry Subsectors. Changes between the present (Table 1) and the expected structure is displayed in Table 2. Precisely, this table provides information on the subsector percent changes of the number of firms, inputs and outputs between initial data (inputs) and model outputs (posteriors) explaining future equilibrium firm activity. One can observe the highest number increase rate in the sub-sector of military fighting vehicles (775%) and a slight decrease in the sub-sector of motor vehicles (-3%).

Next, the same table displays the ratio of input-output change (%). It reveals the long-run equilibrium subsector average profitability change rate (%) (or subsector value added change rate), given the initial conditions and the model formulation presented in equ. 2–11. This ratio is obtained as a difference between the output change (%) and the input change (%) for a given subsector. We notice that this ratio seems to decrease in the long-run and this decrease to be globally proportional to sub-sectorial firm concentration, as shown through column 1 and 2 of Table 2.

As far as the model interval confidence is concerned (see Table 2), we observe a global cross-entropy norm I(m) of around 0.368. This index compounds the parameter cross-entropy norm of around 0.391 and the error term index of around 0.125. These two values will depend on the value level of the weight λ in the criterion function (equ. 2). The higher value of this parameter tends to increase the parameter precision while worsening the prediction level of the model through the error component. Finally, as the cross-entropy index varies between zero and one, its higher value suggests weaker discrimination of the model (data) against the prior. Its value closer to zero, in the contrary, reveals a higher significance of the model in terms of discriminating against the prior. Table 3 presents estimate values of the model and model parameter inference index value I(m). For instance, based on the global cross-entropy norm displayed in Table 3, one can say that the model has discriminated in favour of the posterior (the proposed model outputs) for approximately 63.2% (1- Global cross entropy norm I(m)). Readers interested in information theory statistical inference can find details on the subject, e.g. in Golan et al. (1996), or for the non-extensive entropy, e.g. in Bwanakare (2014).

Finally, as presented in Table 2, we notice that all the transport subsectors but one have increased their firm concentration ratio, while the highest increase corresponds to the lowest increase in profitability.

Table 2. The post-entropy expected steady-state configuration of the Polish transport industry Subsectors

Sub-sectors of the Polish transport industry	Change of number of firms in %	Ratio input output change (%)	Change of inputs in %	Change of outputs in %
Motor vehicles	-3.0	2.398	10	13
Bodies (coachwork) for motor vehicles; trailers and semi-trailers	96.0	-20.646	45	20
Parts and accessories for motor vehicles	85.0	-3.123	4	1
Building of ships and boats	115.0	-29.814	6	-18
Railway locomotives and rolling stock	168	1.279	21	23
Air and spacecraft and related machinery	134	-16.139	27	9
Military fighting vehicles	775	-80.679	37	-24
Transport equipment n.e.c.	137	-51.218	26	-17
Total number of enterprises of all subsectors	1239			
Average input	2536			
Average output	3606			
q-Tsallis parameter	1.000			
Global cross-entropy norm	0.368			
Parameter cross-entropy norm	0.391			

Source: Own work.

Table 3. Model parameter estimates and statistical inference

	α_{0j}	λ_{1j}	α_{2j}	β_{0j}	λ_{2j}	β_{2j}	δ_{0j}	λ_{3j}	δ_{2j}
Estimate values	0.879	0.835	-0.067	-0.882	0.449	-0.899	-0.708	0.099	0.696
Normed index I(p)	0.349	0.418	0.900	0.353	0.867	0.295	0.637	0.872	0.626

Source: Own work.

Table 3 presents model system parameter estimates and their normalized statistical precision I(p). As already explained, I(p) close to unity means that prior and posterior parameters are identical, which suggests that the model (new data, in Bayesian interpretation) is no pertinent. In terms of entropic formalism, no entropy reduction is reached through the incorporated econometric model system (equ. 2–11). If we adopt the rule of thumb presented in Golan et al. (1996), all parameters are more or less significant as all precision indices I(p) are lower than 0.99.

5. Concluding remarks

The proposed model aimed at predicting the subsector's most plausible, long-run financial configuration of firms consistent with current information on their inputs, outputs and structure through a generalized maximum entropy principle. It consisted of minimizing information divergence between unknown posteriors related to industry subsector main characteristics and corresponding priors and model initial data. This model proposed NCEE as a recent approach for solving complex inverse problems. As revealed through the model outputs, the long-run change of different profitability ratios is diversified while the highest increase in firm concentration corresponds to the lowest increase in profitability. The model could be developed to take into account recent theoretical developments in management and economics. Based on the above outputs, we can now expect a further dynamic development of the transport industry in Poland evidenced by sub sector firm concentration. This phenomenon significantly leads to the increase in firm competition while negatively impacting on different ratios of profitability as reported in this paper.

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