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Analysing the impact of dependency on conditional survival functions using copulas

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ABSTRACT

Nowadays, insurance contract reserves for coupled lives are considered jointly, which has a significant influence on the process of determining actuarial reserves. In this paper, conditional survival distributions of life insurance reserves are computed using copulas. Subsequently, the results are compared with an independence case. These calculations are based on selected Archimedean copulas and apply when the 'death of one individual' condition exists. The estimation outcome indicates that the insurer reserves calculated by means of Archimedean copulas are far more effective than those resulting from an independence assumption. The study demonstrates that copula-based dependency modelling improves the calculations of reserves made for actuarial purposes.

Key words: conditional survival distribution, copula, Kendall's tau, reserves, life table.

1. Introduction

There exist extensive literature on modelling insurance contracts of two lives using the independence assumption between two lives with focusing on the starting time of the contracts. The main problem for calculating reserves (provisions) between two lifetimes is finding a way to reduce costs of insurance companies. In this paper, we relax the independence assumption and show that the dependency between two lives has a great effect on the solvency of financial businesses such as insurance companies.

In the past few years, many works have been done on application of copula for modelling the dependency between two lifetimes. See, for example Hougaard (2000), Spreeuw (2006), Spreeuw and Owadally (2013), and Zaroudi et al. (2018a,b). Lee et al. (2014) studied the dependence between the policyholders in multiple-life of insurance contracts. Also, many works considered the relationship between two lifetimes of coupled lives, such as Carriere (2000) and Ji et al. (2011). This dependency has a significant impact on the determination of pricing of the insurance contracts. If one considers remaining lifetimes of coupled lives dependently at the beginning period of issuing the insurance policy, then the death chance of two lives might depend on the life status of each of them. A copula is a useful tool for building multivariate distributions when the marginal distributions are available. The copula captures the linear and nonlinear dependencies available in dataset simultaneously and is an applicable tool for modelling the dependency structures between different random

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variables (r.v.'s). Moreover, it opens a new vision for considering the properties of joint distributions of marginal r.v.'s such that it separates the study of dependency effects from the effects of the margins (Mari and Kotz, 2004). Following the studies by Spreeuw and Owadally (2013), and Spreeuw (2006), in this paper, we discuss methods of calculating the insurer's reserves based on copula models using the life table of France. Moreover, we apply Kendall's tau for measuring the dependency of variables using copula models. Our method is useful by selecting an optimal copula and bringing an approach for reserves calculation of contracts. Statistical analyses were performed with R version 3.6.2: The R Project for Statistical Computing.

The framework of this paper goes as follows: In Section 2, we review copula models, family of Archimedean copulas, Kendall's tau correlation coefficient, survival functions, and the relationship between copula and survival functions based on 'death of one individual'. In Section 3, we calculate reserves for life insurance policy and three Archimedean copulas for 'death of one individual' using life table of France. In Section 4, we apply a numerical example for reserves of two lifetimes in insurance contracts. Section 5 provides conclusion remarks.

2. Copula

The term copula is a Latin word 'copuler' which means 'tie and link' and was introduced at first by Sklar Theorem (Sklar, 1959) describing the relationship between r.v.'s to build multivariate distribution functions (Nelsen, 2006). In this paper, only bivariate copulas are used for considering the dependency of two lives when one of them dies during the time period of the insurance contract. In the next subsection, the bivariate copulas are introduced and their properties are investigated. For more information on properties of copula in insurance and financial concept, see Frees and Valdez (1998), Katesari and Vajargah (2015), Katesari and Zarodi (2016), and Safari-Katesari and Zaroudi (2020).

2.1. Random Variables and Copula

Copula is a statistical tool for modelling the dependence between r.v.'s. Let us start with the definition of copula from Denuit et al. (2005) as follows:

Definition 2.1 *A bivariate copula* $C : [0,1]^2 \to [0,1]$ *is a non-decreasing and right-continuous function which preserves the following:*

1. $\lim_{u_i \to 0} C(u_1,u_2) = 0$, *for* $i = 1,2$,
2. $\lim_{u_2 \to 1} C(u_1,u_2) = u_1$ *and* $\lim_{u_1 \to 1} C(u_1,u_2) = u_2$,
3. $C(v_1,v_2) - C(u_1,v_2) - C(v_1,u_2) + C(u_1,u_2) \geq 0$,

for any $u_1,u_2,v_1,v_2 \in [0,1]$, $u_1 \leq v_1$ *and* $u_2 \leq v_2$.

By Sklar's Theorem (Sklar, 1959), one can easily consider the dependence between multivariate distribution functions and their corresponding margins with copulas. Now, we consider Sklar's Theorem, which is as follows:

Theorem 2.1 *Consider the joint distribution $H_X(x_1,x_2)$ with continuous marginal cumulative distribution functions (CDFs) F_1 and F_2. Then, a copula C is available for all $x_1,x_2 \in [-\infty,+\infty]$ as follows:*

$$H_X(x_1,x_2) = C(F_1(x_1),F_2(x_2)). \tag{1}$$

Conversely, the function H_X denoted in Eq. (1) *is a bivariate distribution function when C is a copula function with marginal CDFs F_1 and F_2.*

Moreover, we use the following corollary and definition from Nelsen (2006) and Denuit et al. (2005), respectively.

Corollary 2.1 *Following Theorem 2.1, for any $(u_1,u_2) \in [0,1]^2$, we have:*

$$C(u_1,u_2) = H_X(F_1^{-1}(u_1),F_2^{-1}(u_2)).$$

Definition 2.2 *For any $(u_1,u_2) \in [0,1]^2$, $C(u_1,u_2) = u_1u_2$ defined as the independent copula.*

In what follows, we will introduce one of the most famous family of copulas named Archimedean copulas.

2.2. Archimedean Copulas

The copula described by Genest and MacKay (1986a) and Genest and MacKay (1986b) called the Archimedean copulas and there exist rich applications of this copula family in the mathematical and statistical literature. Many actuarial and financial dataset have been analyzed using Archimedean copulas; for more information, see Frees and Valdez (1998), Denuit et al. (2005) and Klugman and Parsa (1999). One of the important features of this family is that they are parametric. A generating function is the general structure of Archimedean copula, which is introduced in the following definition from Denuit et al. (2005).

Definition 2.3 *Assume $\phi : [0,1] \to [0,+\infty]$ is a continuous, strictly decreasing function and has following characteristics:*

$$\phi(1) = 0, \quad \phi^{'}(\tau) < 0, \quad \phi^{''}(\tau) > 0, \quad \forall \tau \in (0,1). \tag{2}$$

Then, the bivariate copula

$$C_\phi(u_1,u_2) = \begin{cases} \phi^{-1}(\phi(u_1)+\phi(u_2)), & if \quad \phi(u_1)+\phi(u_2) \leq \phi(0), \\ \\ 0, & otherwise. \end{cases} \tag{3}$$

is generated by $\phi(\cdot)$ defined in Eq. (2) *where the copula $C_\phi(u_1,u_2)$ is the Archimedean copula for $0 \leq u_1,u_2 \leq 1$. The generator function of the copula is ϕ.*

One of the most important criteria for explaining the relationship between the dependency and the Archimedean copula is expressed in the following Theorem from Nelsen (2006).

Theorem 2.2 *Consider X and Y as r.v.'s with Archimedean copula C generated by ϕ. Then, Kendall's tau (τ) correlation for X and Y is as follows:*

$$\tau = 1 + 4\int_0^1 \frac{\phi(t)}{\phi'(t)}\,dt. \tag{4}$$

In this paper, we apply three copula functions in the Archimedean family, which are defined in Table 1. In this table, the second column displays the values of dependence parameter (θ) indicating the case of independence, the third column shows the relationships between θ and τ, and the fourth and fifth columns display generation functions and domains of θ for these copulas, respectively (Nelsen, 2006).

Table 1: Three copulas in Archimedean copula family

Family	Independence	τ	$\phi(\tau)$	$\theta \in$
Family 3	$\theta = 0$	$(1/\theta) - e$	$\exp[\tau^{-\theta}] - e$	$(0, \infty)$
G-H	$\theta = 1$	$\theta - 1/\theta$	$(-\ln \tau)^{\theta}$	$[-1, \infty)$
Clayton	$\theta = 0$	$\theta/(\theta+2)$	$\tau^{-\theta} - 1$	$[-1, \infty)\backslash\{0\}$

2.3. Survival Function and Copula

A contract of life insurance defines on couple (two persons) lives with x and y ages at time 0, respectively. The remaining lifetimes of persons x and y are defined by T_x and T_y, where the marginal survival functions and the copula survival function are denoted by $S_{T_x}(s_1)$, $S_{T_y}(s_2)$ and $S_{T_x,T_y}(s_1,s_2)$, respectively. The survival function of the remaining lifetime at time t for the person aged x, given the death of person aged y at time t_y is defined as (Spreeuw, 2006):

$$\begin{aligned} S_{T_x;t}(s|T_y = t_y) &= P(T_x > t+s | T_x > t, T_y = t_y) = \frac{-\frac{\mathrm{d}}{\mathrm{d}t_y} P(T_x > t+s, T_y > t_y)}{-\frac{\mathrm{d}}{\mathrm{d}t_y} P(T_x > t, T_y > t_y)} \\ &= \frac{-\frac{\mathrm{d}}{\mathrm{d}t_y} S_{T_x,T_y}(t+s, t_y)}{-\frac{\mathrm{d}}{\mathrm{d}t_y} S_{T_x,T_y}(t, t_y)} = \frac{\left(C_2\left(S_{T_x}(t+s), \xi\right)\right)_{\xi = S_{T_y}(t_y)}}{\left(C_2\left(S_{T_x}(t), \xi\right)\right)_{\xi = S_{T_y}(t_y)}}. \end{aligned} \tag{5}$$

Notice that the function defined in Eq. (5) is a survival copula function where the partial derivative of copula C for the second argument denoted with C_2 such that $C_2\left(S_{T_x}(t), \xi\right) \neq 0$ is defined for $\xi = S_{T_y}(t_y)$. Then, following Spreeuw (2006) and using Eq. (3), we have:

$$C(S_{T_x}(t+s), S_{T_y}(t_y)) = \phi^{-1}(\phi(S_{T_x}(t+s)) + \phi(S_{T_y}(t_y))) = S_{T_x,T_y}(t+s, t_y),$$

and

$$\begin{aligned}(C_2(S_{T_x}(t+s),\xi))_{\xi=S_{T_y}(t_y)} &= \phi^{'}(S_{T_y}(t_y))(\phi^{-1})^{'}(\phi(S_{T_x,T_y}(t+s,t_y))) \\ &= \phi^{'}(S_{T_y}(t_y))(\phi^{-1})^{'}(\phi(S_{T_x}(t+s))+\phi(S_{T_y}(t_y))).\end{aligned} \tag{6}$$

3. Reserves

For calculating the insurer's reserves, we use the expected *present* value for the payment of insurance contract at the beginning of each year until the death of one individual. For a whole life annuity-due, the net single premium is defined as (Gerber, 1997):

$$\ddot{a}_x = \sum_{k=0}^{\infty} v^k {}_k p_x, \tag{7}$$

where $v = (1+i)^{-1}$ is a discount factor and i is an annual interest rate. Notice that we consider i the same for the entire years.

Definition 3.1 *(Gerber, 1997). The survival function for remaining lifetime T_x is denoted by $S_{T_x}(t)$ and defined as:*

$${}_t p_x = S_{T_x}(t) = P(T_x > t).$$

Using definition 3.1, we can rewrite Eq. (7) for 'death of one individual' status where person aged x is still alive until t and person aged y will die between t_y and $t_y + \mathrm{d}t$ as follows:

$$\ddot{a}_{x;t|t_y} = \sum_{s=0}^{\infty} v^s \, S_{T_x;t}(s|T_y = t_y). \tag{8}$$

By combining Eq. (5) and Eq. (6), we obtain:

$$S_{T_x;t}(s|T_y = t_y) = \frac{(\phi^{-1})^{'}(\phi(S_{T_x}(t+s))+\phi(S_{T_y}(t_y)))}{(\phi^{-1})^{'}(\phi(S_{T_x}(t))+\phi(S_{T_y}(t_y)))}. \tag{9}$$

Notice that Eq. (8) is the annual reserves of the whole-life annuity (defined at Gerber, 1997) for the 'death of one individual' status, which by Eq. (9) can be written as follows:

$$\ddot{a}_{x;t|t_y} = \sum_{s=0}^{\infty} v^s \, \frac{(\phi^{-1})^{'}(\phi(S_{T_x}(t+s))+\phi(S_{T_y}(t_y)))}{(\phi^{-1})^{'}(\phi(S_{T_x}(t))+\phi(S_{T_y}(t_y)))}. \tag{10}$$

Table 2 provided the reserve for some Archimedean copulas at 'death of one individual' status using Eq. (10).

Table 2: Reserve for Archimedean copulas at ‘death of one individual’ status

Copula	Reserves ($\ddot{a}_{x;t\|t}$)
Clayton	$\sum_{s=0}^{\infty} v^s \left(\frac{S_{T_x}(t+s)^{-\theta}+S_{T_y}(t_y)^{-\theta}-1}{S_{T_x}(s)^{-\theta}+S_{T_y}(t_y)^{-\theta}-1}\right)^{-\frac{1}{\theta}-1}$
G-H	$\sum_{s=0}^{\infty} v^s \frac{\exp\{-[(-\ln S_{T_x}(t+s))^{\theta}+(-\ln S_{T_y}(t))^{\theta}]^{\frac{1}{\theta}}\}}{\exp\{-[(-\ln S_{T_x}(t))^{\theta}+(-\ln S_{T_y}(t))^{\theta}]^{\frac{1}{\theta}}\}} \times \left(\frac{(-\ln S_{T_x}(t+s))^{\theta}+(-\ln S_{T_y}(t_y))^{\theta}}{(-\ln S_{T_x}(t))^{\theta}+(-\ln S_{T_y}(t_y))^{\theta}}\right)^{\frac{1}{\theta}-1}$
Family 3	$\sum_{s=0}^{\infty} v^s \left(\frac{\ln(\exp(S_{T_x}(t+s)^{-\theta})+\exp(S_{T_y}(t_y)^{-\theta})-e)}{\ln(\exp(S_{T_x}(t)^{-\theta})+\exp(S_{T_y}(t_y)^{-\theta})-e)}\right)^{-\frac{1}{\theta}-1} \times \frac{\exp(S_{T_x}(t)^{-\theta})+\exp(S_{T_y}(t_y)^{-\theta})-e}{\exp(S_{T_x}(t+s)^{-\theta})+\exp(S_{T_y}(t_y)^{-\theta})-e}$

4. Application

In this section, we calculate reserves using copula models for life insurance companies. The results are compared with the independence case, which demonstrates the advantage of the copula model in decreasing the amount of reserves and impact on the solvency of the insurance companies. Moreover, statistical computations were conducted with R version 3.6.2: The R Project for Statistical Computing. In order to calculate the insurer’s reserves based on the copulas defined in Table 2 for the ‘death of one individual’ status, one needs the marginal distributions of male and female’s survival. For this goal, we use the life table of France in the year 2008. The complete life table can be found at https://www.ined.fr/fr/tout-savoir-population/chiffres/france/mortalite-cause-deces/table-mortalite/.

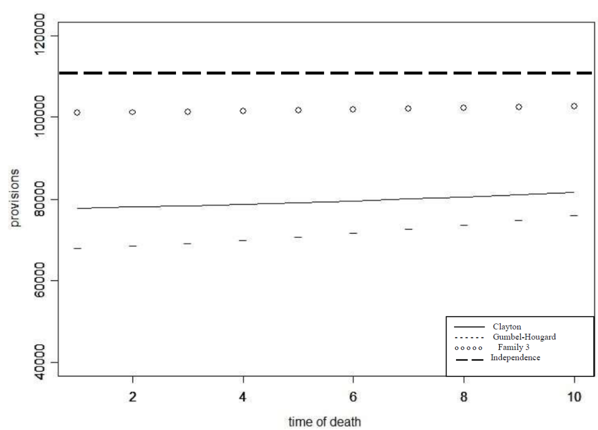

Figure 1: Reserves for time of death of person aged y at period 10.

We assume an insurance contract for a couple of 60 years old. As long as a person (male regardless of the state of the woman) is alive, the insurer liabilities are 10,000 units at the beginning of each year. Table 3 provided the percentage of difference in calculated reserves average using some copulas in Archimedean family (Clayton, Gumbel-Hougaard and Family 3) relative to the independence copula for the time period of 10, 20, and 30

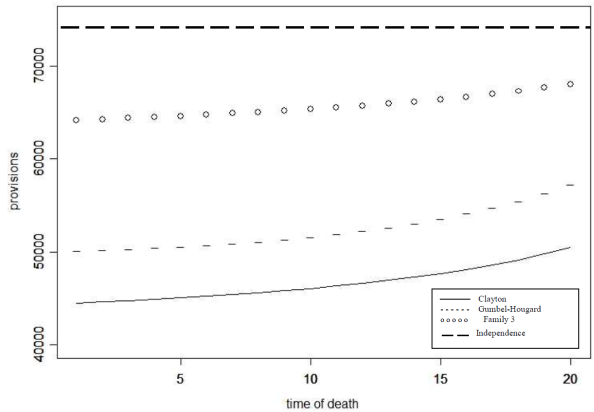

Figure 2: Reserves for time of death of person aged y at period 20.

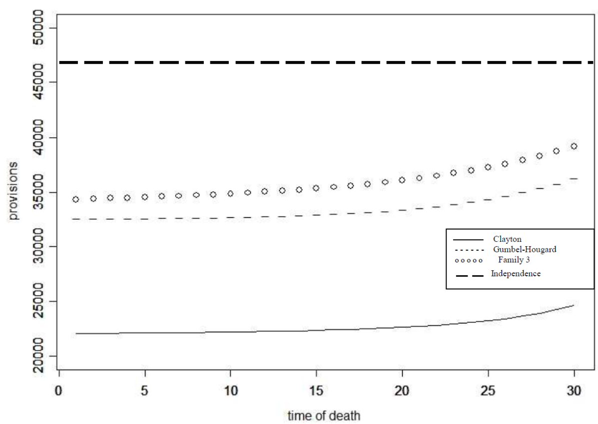

Figure 3: Reserves for time of death of person aged y at period 30.

years. Moreover, the Kendall's tau coefficient defined in Eq. (4) is 0.5 at the start point of the contract with an annual interest rate of 4%. Table 3 demonstrated that applying copula in

Table 3: The percentage of difference in calculated reserves average using Archimedean copula relative to the independence copula

Copula	period 10	period 20	period 30
Clayton	-29.2	-36	-44
Gumbel-Hougaard	-38.1	-28.7	-15.5
Family 3	-11.5	-11.3	-14

Family 3 reduced the reserve from 11% through 14% compared to the independence copula in the case that if one of the persons died at the age of y for 10, 20, and 30 years. As can be seen from Figures 1, 2, and 3, the reserves for all three copulas are less than independence case during the time. Figure 1 shows the reserve at the time period of 10 years which the copula of Family type 3 has the highest and the Gumbel-Hougaard copula has the lowest amount of reserve. Figure 2 and Figure 3 displayed the reserves at the time period of 20 years and 30 years, respectively, which the Family 3 has the highest, and the Clayton has the lowest amount of reserves. To this end, our estimation results indicate that the insurer's reserves with using Archimedean copula families are less than the independence case, which increases the solvency of insurance companies.

5. Conclusion

Since life insurance reserves are calculated by the death of one individual, dependency of the two lifetimes plays an important role for actuarial computations of insurance companies. In this paper, the insurer's reserves are calculated using some Archimedean copulas for different time periods. The results showed that fitting the appropriate copula optimizes the amount of insurers' funds, which can be spent for reserves of future liabilities. Thus, considering dependency between two lifetimes for calculating the optimal reserves by the insurer is highly recommended.

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