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The Complex-Number Mortality Model (CNMM) based on orthonormal expansion of membership functions

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ABSTRACT

The paper deals with a new fuzzy version of the Lee-Carter (LC) mortality model, in which mortality rates as well as parameters of the LC model are treated as triangular fuzzy numbers. As a starting point, the fuzzy Koissi-Shapiro (KS) approach is recalled. Based on this approach, a new fuzzy mortality model – CNMM – is formulated using orthonormal expansions of the inverse exponential membership functions of the model components. The paper includes numerical findings based on a case study with the use of the new mortality model compared to the results obtained with the standard LC model.

Key words: exponential membership functions, Legendre's polynomials, mortality modelling, orthonormal system.

1. Introduction

In the last four decades several approaches were proposed to model human mortality and to project future mortality evolution. Among the extrapolative methods, a model proposed by Lee and Carter (1992) is one of the most popular approaches, although other mortality models have been also developed, e.g. Heligman and Pollard (1980), Horiuchi and Coale (1990), Milevsky and Promislow (2001), Currie et al. (2004), Bongaarts (2005), Cairns et al. (2006).

The Lee-Carter model (LC) has been extensively used for many real populations and extended in various directions (see, e.g. Renshaw et al. (1996), Tuljapurkar et al. (2000), Booth et al. (2002), (2006), Brouhns et al. (2002), Renshaw and Haberman (2003), De Jong and Tickle (2006), Koissi and Shapiro (2006), Pitacco et al. (2009), Haberman and Renshaw (2012), Danesi et al. (2015)).

The Lee-Carter method (1992) can be treated as a special case of the principal component analysis with a single component (Bozik and Bell (1987)). The focus of this

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approach is on central age-specific death rates m_{xt} for a range of ages $x = 0,1,2, \dots, X$ and calendar years $t = 1,2, \dots, T$, organized in a two way table with rows referring to one-year age groups and columns referring to one-year period intervals.

The LC method consists of a model of age-specific log-central death rates $y_{xt} = \ln m_{xt}$ with time and age components

$$y_{xt} = a_x + b_x k_t + \varepsilon_{xt}, \quad x = 0,1,2, \dots X, \ \ t = 1,2, \dots, T, \tag{1}$$

and a model of random walk with a drift to forecast time components k_t for $t > T$

$$k_t = d + \ k_{t-1} + \zeta_t, \tag{2}$$

where $\{a_x\}$ in (1) is a set of age-related effects describing the age profile of mortality, $\{k_t\}$ is a set of the time-related effects representing the general trend of mortality, $\{b_x\}$ is a set of age-related effects describing patterns of deviations from the age profile in response to change of the general trend, d in (2) is a constant (a drift), whereas ε_{xt}, ζ_t in (1) and (2), respectively, are random residuals.

Parameters $\{b_x\}$ show which death rates decline rapidly and which slowly over time in response to change of k_t. For some values of x, b_x may be positive while negative for other values, indicating that log-central death rates $y_{xt} = \ln m_{xt}$ are increasing at some ages while decreasing at other ages.

For the full identification of (1), the following two constraints are imposed

$$\textstyle\sum_{x=0}^{X} b_x = 1, \ \sum_{t=1}^{T} k_t = 0. \tag{3}$$

Lee and Carter used the SVD method (Singular Value Decomposition) to estimate a_x, b_x, k_t and assumed that error terms ε_{xt} are normally distributed with a small constant variance. This is rather a strong assumption, which is often violated especially in the case of the imprecise input data. Moreover, prediction errors do not account for the estimation errors of the age-specific parameters a_x, b_x, except of incorporating uncertainty from the forecast of the time component k_t.

It is well-known that various kinds of errors can occur in reporting death statistics. This could be e.g. incorrect year, area or age. Moreover, the midyear population data used to calculate period age-specific mortality rates m_{xt} are also the subject of errors. The midyear population size is the population at July 1 and is assumed to be the point at which half of the deaths during the year have occurred. Such estimates can be actually underestimated or overestimated and this affects the resulting death rates. Therefore, exact age-specific mortality rates are seldom known, hence incorporating the data uncertainty into the model structure seems to be a realistic and expected idea.

The new trends in fuzzy analysis are based on the algebraic approach to fuzzy numbers (e.g. Ishikawa (1997), Kosiński et al. (2003), Rossa et al. (2017), Szymański and Rossa (2014), (2017)). The essential idea in such an approach is representing the membership function of a fuzzy number as an element of the square-integrable function space. We will use this idea to propose a new fuzzy mortality model in the spirit of the Koissi-Shapiro approach.

The log-central mortality rates as well as parameters of the underlying Koissi-Shapiro model are symmetric triangular fuzzy numbers, i.e. numbers with symmetric triangular membership functions. We believe that exponential functions could fit the data better. Therefore, our model is based on exponential membership functions of the model components instead of triangular ones.

The paper is organized as follows. Section 2 recalls the data fuzzification method (Subsection 2.1) and the fuzzy mortality model (Subsection 2.2) as proposed by Koissi and Shapiro. The new complex-number fuzzy mortality model is formulated in Section 3. The concept is presented in six subsections: theoretical backgrounds (Subsection 3.1), formulation of the new mortality model CNMM (Subsection 3.2), estimation of the model parameters (Subsection 3.3), description of the modified fuzzification method (Subsection 3.4), description of the forecasting method (Subsection 3.5) and a case study (Subsection 3.6). Concluding remarks are contained in Section 4. Formal details about orthonormal expansions by means of the Legendre polynomials are included in the Appendix.

2. The Koissi-Shapiro model

2.1. Fuzzification of the input data

In the Koissi-Shapiro model (2006), log-central death rates $y_{xt} = \ln m_{xt}$ are transformed into symmetric triangular fuzzy numbers

$$Y_{xt} = (y_{xt}, e_{xt}), \tag{4}$$

where y_{xt}, e_{xt} are centres and spreads of fuzzy numbers Y_{xt}, respectively.

The addition $\oplus$ and multiplication $\otimes$ of symmetric triangular numbers $A = (a, s_A)$ and $B = (b, s_B)$ defined in the norm T_w are expressed as

$$A \oplus B = (a + b, \max(s_A, s_B)), \tag{5}$$

$$A \otimes B = (ab, \max(s_A|b|, s_B|a|)), \tag{6}$$

and the multiplication of $A = (a, s_A)$ by a scalar $b \in \mathbb{R}$ reduces to

$$A \otimes b = (ab, s_A|b|). \tag{7}$$

Parameters e_{xt} in (4) are also called fuzziness parameters. To determine their values, Koissi and Shapiro postulated using a fuzzy regression model. They assumed existing symmetric triangular fuzzy numbers (c_{0x}, s_{0x}) and (c_{1x}, s_{1x}) satisfying for each age group x the following equalities

$$(y_{xt}, e_{xt}) = (c_{0x}, s_{0x}) \oplus (c_{1x}, s_{1x}) \otimes t, \qquad t = 1,2, \dots, T. \tag{8}$$

This postulate leads to the equalities (9)–(10) of the form

$$y_{xt} = c_{0x} + c_{1x} \cdot t, \qquad t = 1,2, \dots, T. \tag{9}$$

$$e_{xt} = \max(s_{0x}, s_{1x} \cdot t\,), \qquad t = 1,2, \dots, T. \tag{10}$$

To find coefficients in (9) the ordinary least-squares regression is used, i.e. c_{1x} and c_{0x} are found from formulas

$$c_{1x} = \frac{\overline{y_{xt} \cdot t} - \bar{t} \cdot \overline{y_{xt}}}{\overline{t^2} - \bar{t}^2}, \tag{11}$$

$$c_{0x} = \overline{y}_{xt} - c_{1x} \cdot \bar{t}, \tag{12}$$

where $\bar{z}$ means averaging over z_t's.

To find parameters s_{0x}, s_{1x},"the minimum fuzziness criterion" is proposed by minimizing spreads of $Y_{xt} = (y_{xt}, e_{xt})$ and requiring each log-central death rate y_{xt} to fall within the estimated death rates $\hat{y}_{xt}$ at a level $h \in [0,1]$. Since e_{xt} are, by assumption, non-negative numbers and the smallest value they can take is 0, it is necessary to determine such values of s_{0x}, s_{1x}, that at a given x they minimize the sum

$$T \cdot s_{0x} + s_{1x} \cdot \sum_{t=1}^{T} t, \tag{13}$$

subject to the constraints

$$c_{0x} + c_{1x} \cdot t + (1-h)(s_{0x} + s_{1x} t) \geq \ln m_{xt}, \;\; t = 1,2, \dots, T, \tag{14}$$

$$c_{0x} + c_{1x} \cdot t - (1-h)(s_{0x} + s_{1x} t) \leq \ln m_{xt}, \;\; t = 1,2, \dots, T, \tag{15}$$

where $s_{0x}, s_{1x} \geq 0,\; u \in [0,1)$ and $h \in [0,1]$ is a predetermined value representing the degree of fit of the estimated model to the data. As lower h provides a better fit, we can use $h = 0$. After finding the parameters s_{0x}, s_{1x} for each x, the fuzziness parameters e_{xt} can be determined using formula (10).

2.2. The Koissi-Shapiro model

Let us recall the fuzzy mortality model as proposed by Koissi and Shapiro (2006). The structure of their model is analogous to the Lee-Carter one (1992) and takes the form

$$Y_{xt} = A_x \oplus (B_x \otimes K_t), \tag{16}$$

with the difference that $Y_{xt} = (y_{xt}, e_{xt})$ for $x = 0,1, \dots, X,\; t = 1,2, \dots, T$ are fuzzified log-central death rates expressed as triangular numbers with centres y_{xt} and spreads e_{xt}.

Model parameters are assumed to be symmetric triangular numbers $A_x = (a_x, s_{A_x}),\; B_x = (b_x, s_{B_x}), K_t = (k_t, s_{K_t})$ with unknown centres $a_x, b_x, k_t \in \mathbb{R}$ and spreads $s_{A_x}, s_{B_x}, s_{K_t} \geq 0$, respectively.

To find parameters $a_x, b_x, k_t, s_{A_x}, s_{B_x}, s_{K_t}$, Koissi and Shapiro postulated minimizing the Diamond distance D^2 (Diamond (1988)) between the left and right-hand sides of (16). This leads to the criterion function defined for each separate x and t as

$$D^2(Y_{xt}, A_x \oplus (B_x \otimes K_t)) = (a_x + b_x k_t - y_{xt})^2 + \left[a_x + b_x k_t - \max\{s_{A_x}, |b_x| s_{K_t}, |k_t| s_{B_x}\} - (y_{xt} - e_{xt})\right]^2 + \left[a_x + b_x k_t + \max\{s_{A_x}, |b_x| s_{K_t}, |k_t| s_{B_x}\} - (y_{xt} + e_{xt})\right]^2. \tag{17}$$

Unfortunately, the criterion function contains a max-type operator $\max\{s_{A_x}, |b_x| s_{K_t}, |k_t| s_{B_x}\}$, which does not allow using standard derivative based solution algorithms for minimization of (17).

3. The Complex-Number Mortality Model CNMM

3.1. Theoretical backgrounds

The new trends in fuzzy analysis are based on the algebraic approach to fuzzy numbers (see, e.g. Ishikawa (1997), Kosiński et al. (2003), Rossa et al. (2017), Szymański and Rossa (2014), (2017)). The essential idea in such an approach is representing the membership function of a fuzzy number as an element of the square-integrable function space.

Let us consider the membership function of the exponential form

$$\mu(z) = \begin{cases} \exp\left\{-\left(\frac{c-z}{\tau}\right)^2\right\}, & z \le c, \\ \exp\left\{-\left(\frac{z-c}{\nu}\right)^2\right\}, & z > c, \end{cases} \tag{18}$$

where $c \in \mathbb{R}$, $\tau, \nu > 0$ are some scalar parameters.

Note that (18) can be decomposed into two parts – strictly increasing and strictly decreasing functions $\Psi(z)$ and $\Phi(z)$, say, of the form

$$\Psi(z) = \exp\left\{-\left(\frac{c-z}{\tau}\right)^2\right\}, \quad z \le c, \tag{19}$$

$$\Phi(z) = \exp\left\{-\left(\frac{z-c}{\nu}\right)^2\right\}, \quad z > c. \tag{20}$$

Then there exist inverse functions

$$\Psi^{-1}(u) = c + \psi(u), \quad u \in [0,1], \tag{21}$$

$$\Phi^{-1}(u) = c + \varphi(u), \quad u \in [0,1], \tag{22}$$

where $\psi(u)$ and $\varphi(u)$ are expressed as

$$\psi(u) = -\tau(-\ln u)^{\frac{1}{2}}, \quad \varphi(u) = \nu(-\ln u)^{\frac{1}{2}}, \quad u \in [0,1]. \tag{23}$$

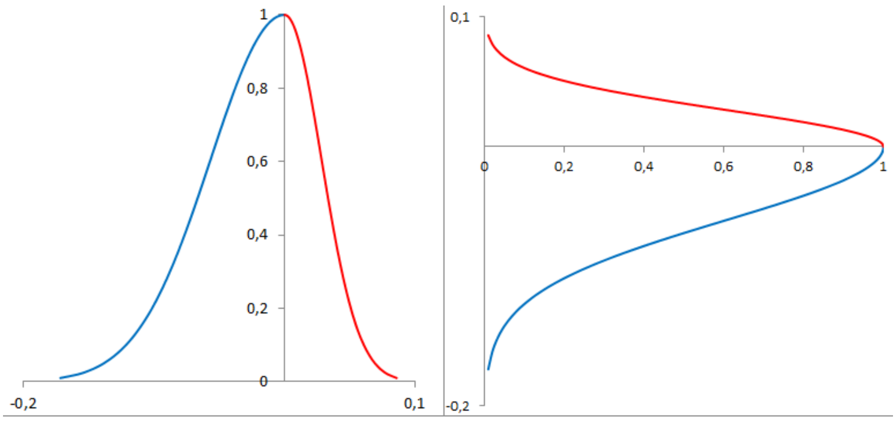
Denoting $f(u) = \Psi^{-1}(u)$ and $g(u) = \Phi^{-1}(u)$ for $u \in [0,1]$, we can write

$$f(u) = c + \psi(u) = c - \tau(-\ln u)^{\frac{1}{2}}, \quad g(u) = c + \varphi(u) = c + \nu(-\ln u)^{\frac{1}{2}}, \tag{24}$$

Functions f, g are square-integrable, so the ordered pair (f, g) belongs to the Cartesian product $L^2(0,1) \times L^2(0,1)$. The scalar product in the space $L^2(0,1)$ is given by the formula

$$\langle f, g \rangle = \int_0^1 f(u)\, g(u) du. \tag{25}$$

Example 1. Figure 1(a) depicts functions $\Psi(z)$ and $\Phi(z)$ as defined in (19) and (20), while Figure 1(b) shows their inverse counterparts (21) and (22), respectively.



(a) (b)

Figure 1. Exponential functions $\Psi(z)$, $\Phi(z)$ and the inverse functions $\Psi^{-1}(u)$, $\Phi^{-1}(u)$ for $c = 0.0$, $\tau = 0.08, \nu = 0.09$.

Source: Developed by the authors.

It is commonly known that a set of vectors $\{P_j\}$ in $L^2(0,1)$ is called an orthonormal set if equalities $\langle P_j, P_k \rangle = 0$ for $j \neq k$ and $\langle P_j, P_j \rangle = 1$ are true.

For any orthonormal set $\{P_j\}$ and $f, g \in L^2(0,1)$ the following relations hold

$$f = \sum_{j=0}^{\infty} \langle P_j, f \rangle P_j, \quad g = \sum_{j=1}^{\infty} \langle P_j, g \rangle P_j. \tag{26}$$

Denoting $\alpha_j = \langle P_j, f \rangle$ and $\beta_j = \langle P_j, g \rangle$, expressions (26) can also be written as

$$f(u) = \sum_{j=0}^{\infty} \alpha_j P_j(u), \quad g(u) = \sum_{j=0}^{\infty} \beta_j P_j(u). \tag{27}$$

Let $A^{(N)}$ be a pair of functions $\left(f^{(N)}, g^{(N)}\right)$, where $f^{(N)}, g^{(N)}$ for $N \in \mathbb{N}$ are some orthonormal expansions of inverse exponential functions (24), i.e.

$$f^{(N)}(u) = \sum_{j=0}^{N} \alpha_j P_j(u), \qquad g^{(N)}(u) = \sum_{j=0}^{N} \beta_j P_j(u), \tag{28}$$

where $\{P_j\}$ is a set of the Legendre polynomials and α_j, β_j are some coefficients of the orthonormal expansion (see Appendix for more details).

Example 2. Let us consider functions $f(u), g(u)$ as depicted in Figure 1(b). Their approximations for $N = 3$ are plotted in Figure 2.

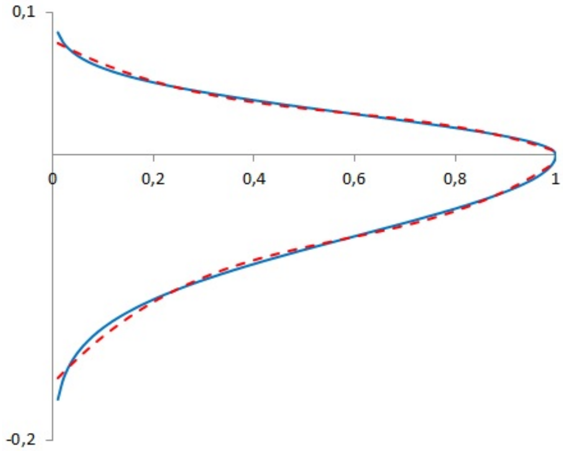


Figure 2. Functions $f(u) = c - \tau(-\ln u)^{\frac{1}{2}}$, $g(u) = c + \nu(-\ln u)^{\frac{1}{2}}$ (solid lines) and their approximations $f^{(3)}(u) = \sum_{j=0}^{3} \alpha_j P_j$, $g^{(3)}(u) = \sum_{j=0}^{3} \beta_j P_j$ (dashed lines).

Source: developed by the authors

Further, we will treat the pairs of functions (f, g) or $\left(f^{(N)}, g^{(N)}\right)$ given in (24), (28), respectively, in terms of the complex analysis. They will be called *complex-valued fuzzy numbers*.

Let the addition, the subtraction and the multiplication of two complex-valued fuzzy numbers $A = (f_A, g_A), B = (f_B, g_B)$ be defined as

$$A \oplus B = (f_A + f_B,\ g_A + g_B), \tag{29}$$

$$A \ominus B = (f_A - f_B, g_A - g_B), \tag{30}$$

$$A \odot B = (f_A f_B - g_A g_B,\ f_A g_B + g_A f_B), \tag{31}$$

while the multiplication of $A = (f_A, g_A)$ by a scalar $d \in \mathbb{R}$ as

$$d \odot A = (d \cdot f_A,\ d \cdot g_A). \tag{32}$$

3.2. The CNMM model formulation

We propose the Complex-Number Mortality Model (CNMM) of the form

$$Y_{xt}^{(N)} = A_x^{(N)} \oplus K_{xt}^{(N)}, \tag{33}$$

where $x = 0,1, \dots, X, t = 1,2, \dots, T$ are age and time indices, respectively, $Y_{xt}^{(N)} = \left(f_{Y_{xt}}^{(N)}, g_{Y_{xt}}^{(N)}\right)$ are complex-valued fuzzy numbers representing fuzzified log-central mortality rates, and $A_x^{(N)} = \left(f_{A_x}^{(N)}, g_{A_x}^{(N)}\right)$, $K_{xt}^{(N)} = \left(f_{K_{xt}}^{(N)}, g_{K_{xt}}^{(N)}\right)$ are some complex-valued fuzzy numbers with functions $f_{A_x}^{(N)}$, $g_{A_x}^{(N)}$, $f_{K_{xt}}^{(N)}$, $g_{K_{xt}}^{(N)}$ and $f_{Y_{xt}}^{(N)}$, $g_{Y_{xt}}^{(N)}$ being orthonormal expansions (28) of the following functions

$$f_{A_x}(u) = a_x - \tau_{A_x}(-\ln u)^{\frac{1}{2}}, \quad g_{A_x}(u) = a_x + \nu_{A_x}(-\ln u)^{\frac{1}{2}}, \tag{34}$$

$$f_{K_{xt}}(u) = b_x k_t - \tau_{B_x}\omega_t(-\ln u)^{\frac{1}{2}}, \quad g_{K_{xt}}(u) = b_x k_t + \nu_{B_x}\varpi_t(-\ln u)^{\frac{1}{2}}, \tag{35}$$

$$f_{Y_{xt}}(u) = y_{xt} - e_{xt}(-\ln u)^{\frac{1}{2}}, \quad g_{Y_{xt}}(u) = y_{xt} + \upsilon_{xt}(-\ln u)^{\frac{1}{2}}, \tag{36}$$

Coefficients $a_x, b_x, k_t, \tau_{A_x}, \nu_{A_x}\ \tau_{B_x}, \nu_{B_x}, \omega_t, \varpi_t$ in (34)–(36) constitute a set of unknown parameters, $y_{xt} = \ln m_{xt}$ are log-central death rates, and e_{xt}, υ_{xt} represent fuzziness of log-central mortality rates evaluated at the fuzzification stage (see Subsection 3.4).

Let us express the model in terms of complex analysis using an algebraic representation, i.e.

$$Y_{xt}^{(N)} = f_{Y_{xt}}^{(N)} + i \cdot g_{Y_{xt}}^{(N)}, \quad A_x^{(N)} = f_{A_x}^{(N)} + i \cdot g_{A_x}^{(N)}, \quad K_{xt}^{(N)} = f_{K_{xt}}^{(N)} + i \cdot g_{K_{xt}}^{(N)}, \tag{37}$$

where $i = \sqrt{-1}$ is an imaginary unit.

Then, taking into account (28) we can write

$$A_x^{(N)} = \sum_{j=0}^N \alpha_{xj} P_j + i \sum_{j=0}^N \beta_{xj} P_j = \sum_{j=0}^N (\alpha_{xj} + i\,\beta_{xj}) P_j, \tag{38}$$

$$K_{xt}^{(N)} = \sum_{j=0}^N \eta_{txj} P_j + i \sum_{j=0}^N \lambda_{txj} P_j = \sum_{j=0}^N (\eta_{txj} + i\,\lambda_{txj}) P_j. \tag{39}$$

Thus, the right-hand side of (33) can be expressed as

$$A_x^{(N)} \oplus K_{xt}^{(N)} = \sum_{j=0}^N \left[(\alpha_{xj} + \eta_{xtj}) + i(\beta_{xj} + \lambda_{xtj})\right] P_j. \tag{40}$$

By analogy, the left-hand side of (33) can be written in the form

$$Y_{xt}^{(N)} = \sum_{j=0}^N \epsilon_{xtj} P_j + i \sum_{j=0}^N \theta_{xtj} P_j = \sum_{j=0}^N (\epsilon_{xtj} + i\,\theta_{xtj}) P_j. \tag{41}$$

Coefficients $\alpha_{xj}, \eta_{xtj}, \beta_{xj}, \lambda_{txj}$ and $\epsilon_{xtj}, \theta_{xtj}$ in expansions (40), (41), respectively, correspond to parameters $a_x, b_x, k_t, \tau_{A_x}, \nu_{A_x}\ \tau_{B_x}, \nu_{B_x}, \omega_t, \varpi_t$ via relations (42), (43).

For $j = 0$ we have

$$\begin{aligned} \alpha_{x0} &= a_x - \tau_{A_x} c_0, & \beta_{x0} &= a_x + \nu_{A_x} c_0, \\ \eta_{tx0} &= b_x k_t - \tau_{B_x}\omega_t c_0, & \lambda_{tx0} &= b_x k_t + \nu_{B_x}\varpi_t c_0, \\ \epsilon_{xt0} &= y_{xt} - e_{xt} c_0, & \theta_{xt0} &= y_{xt} + \upsilon_{xt} c_0, \end{aligned} \tag{42}$$

and for $j = 1,2,\dots,N$ there is

$$\begin{aligned} \alpha_{xj} &= -\tau_{A_x} c_j, & \beta_{xj} &= \nu_{A_x} c_j, \\ \eta_{xtj} &= -\tau_{B_x}\omega_t c_j, & \lambda_{xtj} &= \nu_{B_x}\varpi_t c_j, \\ \epsilon_{xtj} &= -e_{xt} c_j, & \theta_{xtj} &= \upsilon_{xt} c_j, \end{aligned} \tag{43}$$

where c_j are some known constants (see Appendix for more details).

For $j = 0,1,2,3$ we get $c_0 = \frac{\sqrt{\pi}}{2}, c_1 = \sqrt{3\pi}\left(\frac{1}{2\sqrt{2}} - \frac{1}{2}\right)$, $c_2 = \sqrt{5\pi}\left(\frac{1}{\sqrt{3}} - \frac{3}{2\sqrt{2}} + \frac{1}{2}\right)$, $c_3 = \sqrt{7\pi}\left(-\frac{5}{\sqrt{3}} + \frac{15}{4\sqrt{2}} - \frac{3}{4\sqrt{2}} + \frac{3}{4}\right)$.

3.3. Estimation of the model parameters

To estimate parameters of the CNMM model we apply the metric in the Hilbert space $L_2(0,1)$ between the left and right-hand sides of (33), i.e. between $Y_{xt}^{(N)}$ and $A_x^{(N)} \oplus K_{xt}^{(N)}$. The estimation problem requires minimizing functional $F^{(N)}$ in the Hilbert space $L_2(0,1)$ of the form

$$F^{(N)} = \sum_{x=0}^{X} \sum_{t=1}^{T} \left\| Y_{xt}^{(N)} \ominus \left(A_x^{(N)} \oplus K_{xt}^{(N)} \right) \right\|^2. \tag{44}$$

Thus, $Y_{xt}^{(N)} \ominus \left(A_x^{(N)} \oplus K_{xt}^{(N)} \right)$ can be expressed as

$$Y_{xt}^{(N)} \ominus \left(A_x^{(N)} \oplus K_{xt}^{(N)} \right) =$$
$$= \sum_{j=0}^{N} \left[\epsilon_{xtj} - (\alpha_{xj} + \eta_{xtj}) + i(\theta_{xtj} - (\beta_{xj} + \lambda_{xtj})) \right] P_j. \tag{45}$$

After some rearrangements, we get

$$F^{(N)} = \sum_{x=0}^{X} \sum_{t=1}^{T} \left\| Y_{xt}^{(N)} \ominus \left(A_x^{(N)} \oplus K_{xt}^{(N)} \right) \right\|^2 = \sum_{x=0}^{X} \sum_{t=1}^{T} \left\| \sum_{j=0}^{N} \left[\epsilon_{xtj} - (\alpha_{xj} + \eta_{xtj}) + i((\theta_{xtj} - (\beta_{xj} + \lambda_{xtj})) \right] P_j \right\|^2. \tag{46}$$

Using Pythagorean theorem for the Hilbert space of complex functions, i.e.

$$\left\| \sum_{j=0}^{N} \alpha_j P_j \right\|^2 = \sum_{j=0}^{N} \left| \alpha_j \right|^2, \tag{47}$$

the criterion function $F^{(N)}$ takes the form

$$F^{(N)} = \sum_{x=0}^{X} \sum_{t=1}^{T} \sum_{j=0}^{N} \left| \epsilon_{xtj} - (\alpha_{xj} + \eta_{xtj}) + i(\theta_{xtj} - (\beta_{xj} + \lambda_{xtj})) \right|^2 =$$
$$\sum_{x=0}^{X} \sum_{t=1}^{T} \sum_{j=0}^{N} \left\{ \left[\epsilon_{xtj} - (\alpha_{xj} + \eta_{xtj}) \right]^2 + \left[\theta_{xtj} - (\beta_{xj} + \lambda_{xtj}) \right]^2 \right\}. \tag{48}$$

On the basis of relations (42) and (43), we have also

$$F^{(N)} = \sum_{x=0}^{X} \sum_{t=1}^{T} \left[y_{xt} - a_x - b_x k_t + c_0 \left(-e_{xt} + \tau_{A_x} + \tau_{B_x} \omega_t \right) \right]^2 +$$
$$\sum_{x=0}^{X} \sum_{t=1}^{T} \left[y_{xt} - a_x - b_x k_t + c_0 (v_{xt} - \nu_{A_x} - \nu_{B_x} \varpi_t) \right]^2 +$$
$$D^{(N)} \sum_{x=0}^{X} \sum_{t=1}^{T} \left\{ \left(-e_{xt} + \tau_{A_x} + \tau_{B_x} \omega_t \right)^2 + \left(v_{xt} - \nu_{A_x} - \nu_{B_x} \varpi_t \right)^2 \right\}, \tag{49}$$

where $D^{(N)} = \sum_{j=1}^{N} c_j^2$.

The criterion function $F^{(N)}$ can also be written as

$$F^{(N)} = \sum_{x=0}^{X}\sum_{t=1}^{T}\Big[2(y_{xt} - a_x - b_x k_t)^2 + C^{(N)}\big(e_{xt} - \tau_{A_x} - \tau_{B_x}\omega_t\big)^2 + C^{(N)}\big(\upsilon_{xt} - \nu_{A_x} - \nu_{B_x}\varpi_t\big)^2 - 2c_0(y_{xt} - a_x - b_x k_t)\big(e_{xt} - \tau_{A_x} - \tau_{B_x}\omega_t\big) + 2c_0(y_{xt} - a_x - b_x k_t)\big(\upsilon_{xt} - \nu_{A_x} - \nu_{B_x}\varpi_t\big)\Big], \tag{50}$$

where $C^{(N)} = c_0^2 + D^{(N)}$.

To satisfy identifiability of the model, we impose constraints analogous to (3) as well as some additional constraints, i.e.

$$\sum_{t=1}^{T} k_t = 0, \quad \sum_{x=0}^{X} b_x = 1,$$
$$\sum_{x=0}^{X} \tau_{B_x} = 1, \ \sum_{x=0}^{X} \nu_{B_x} = 1, \tag{51}$$
$$\sum_{t=1}^{T} \omega_t = C, \ \sum_{t=1}^{T} \varpi_t = D,$$

where $C, D > 0$ are some fixed constants.

Moreover, we impose also boundary constraints of the form

$$\sum_{t=1}^{T} y_{xt} = \sum_{t=1}^{T}(a_x + b_x k_t), \quad \sum_{x=0}^{X} y_{xt} = \sum_{x=0}^{X}(a_x + b_x k_t), \tag{52}$$

$$\sum_{t=1}^{T} e_{xt} = \sum_{t=1}^{T}\big(\tau_{A_x} + \tau_{B_x}\omega_t\big), \quad \sum_{x=0}^{X} e_{xt} = \sum_{x=0}^{X}\big(\tau_{A_x} + \tau_{B_x}\omega_t\big), \tag{53}$$

$$\sum_{t=1}^{T} \upsilon_{xt} = \sum_{t=1}^{T}\big(\nu_{A_x} + \nu_{B_x}\varpi_t\big), \quad \sum_{x=0}^{X} \upsilon_{xt} = \sum_{x=0}^{X}\big(\nu_{A_x} + \nu_{B_x}\varpi_t\big). \tag{54}$$

It follows from requirements (51)–(54) that the following equalities hold

$$a_x = \frac{1}{T}\sum_{t=1}^{T} y_{xt}, \tag{55}$$

$$k_t = \sum_{x=0}^{X}(y_{xt} - a_x), \tag{56}$$

$$\tau_{A_x} = \frac{1}{T}\sum_{t=1}^{T} e_{xt} - \frac{C}{T}\tau_{B_x}, \quad \nu_{A_x} = \frac{1}{T}\sum_{t=1}^{T} \upsilon_{xt} - \frac{D}{T}\nu_{B_x}. \tag{57}$$

$$\omega_t = \sum_{x=0}^{X}\big(e_{xt} - \tau_{A_x}\big), \quad \varpi_t = \sum_{x=0}^{X}\big(\upsilon_{xt} - \nu_{A_x}\big). \tag{58}$$

Partial derivatives of $F^{(N)}$ with respect to the remaining parameters b_x and τ_{B_x}, ν_{B_x} are of the form

$$\frac{\partial F^{(N)}}{\partial b_x} = -\sum_{t=1}^{T} k_t\big\{4(y_{xt} - a_x - b_x k_t) - 2c_0\big[\big(e_{xt} - \tau_{A_x} - \tau_{B_x}\omega_t\big) - \big(\upsilon_{xt} - \nu_{A_x} - \nu_{B_x}\varpi_t\big)\big]\big\}, \tag{59}$$

$$\frac{\partial F^{(N)}}{\partial \tau_{B_x}} = -2\sum_{t=1}^{T} \omega_t\big[C^{(N)}\big(e_{xt} - \tau_{A_x} - \tau_{B_x}\omega_t\big) - c_0(y_{xt} - a_x - b_x k_t)\big], \tag{60}$$

$$\frac{\partial F^{(N)}}{\partial \nu_{B_x}} = -2\sum_{t=1}^{T} \varpi_t\big[C^{(N)}\big(\upsilon_{xt} - \nu_{A_x} - \nu_{B_x}\varpi_t\big) + c_0(y_{xt} - a_x - b_x k_t)\big]. \tag{61}$$

Setting (59)–(61) equal to zero we receive

$$b_x = \frac{\sum_{t=1}^{T} k_t[2y_{xt} - c_0(e_{xt} - v_{xt} - \tau_{B_x}\omega_t + \nu_{B_x}\varpi_t)]}{2\sum_{t=1}^{T} k_t^2}, \tag{62}$$

$$\tau_{B_x} = \frac{C^{(N)}\sum_{t=1}^{T}\omega_t(e_{xt} - \tau_{A_x}) - c_0\sum_{t=1}^{T}\omega_t(y_{xt} - a_x - b_x k_t)}{C^{(N)}\sum_{t=1}^{T}\omega_t^2}, \tag{63}$$

$$\nu_{B_x} = \frac{C^{(N)}\sum_{t=1}^{T}\varpi_t(v_{xt} - \nu_{A_x}) + c_0\sum_{t=1}^{T}\varpi_t(y_{xt} - a_x - b_x k_t)}{C^{(N)}\sum_{t=1}^{T}\varpi_t^2}. \tag{64}$$

The exact solution can be found using an iterative procedure. After choosing a set of starting values for unknown parameters, expressions (57), (58) and (62)–(64) can be computed sequentially using the most recent set of estimates.

It is worth noting that coefficients $k_t, b_x, \tau_{B_x}, \nu_{B_x}, \omega_t, \varpi_t$ satisfy conditions (51). Indeed, we have

$$\sum_{t=1}^{T} k_t = \sum_{t=1}^{T}\sum_{x=0}^{X}(y_{xt} - a_x) = \sum_{t=1}^{T}\sum_{x=0}^{X} y_{xt} - \sum_{x=0}^{X}\sum_{t=1}^{T}\left(\frac{1}{T}\sum_{t=1}^{T} y_{xt}\right) = \sum_{t=1}^{T}\sum_{x=0}^{X} y_{xt} - \sum_{x=0}^{X}\sum_{t=1}^{T} y_{xt} = 0, \tag{65}$$

and similarly, there is

$$\sum_{x=0}^{X}\tau_{B_x} = \frac{1}{C^{(N)}\sum_{t=1}^{T}\omega_t^2}\sum_{x=0}^{X}\left[C^{(N)}\sum_{t=1}^{T}\omega_t(e_{xt} - \tau_{A_x}) - c_0\sum_{t=1}^{T}\omega_t(y_{xt} - a_x - b_x k_t)\right] = \frac{1}{C^{(N)}\sum_{t=1}^{T}\omega_t^2}\left[C^{(N)}\sum_{t=1}^{T}\omega_t\sum_{x=0}^{X}(e_{xt} - \tau_{A_x}) - c_0\sum_{t=1}^{T}\omega_t\sum_{x=0}^{X}(y_{xt} - a_x) + c_0\sum_{x=0}^{X} b_x\sum_{t=1}^{T}\omega_t k_t\right]. \tag{66}$$

From (51), (56), (58) we have $\sum_{x=0}^{X} b_x = 1$, $\sum_{x=0}^{X}(y_{xt} - a_x) = k_t$, $\sum_{x=0}^{X}(e_{xt} - \tau_{A_x}) = \omega_t$. Thus, we can write

$$\sum_{x=0}^{X}\tau_{B_x} = \frac{1}{C^{(N)}\sum_{t=1}^{T}\omega_t^2}\left[C^{(N)}\sum_{t=1}^{T}\omega_t^2 - c_0\sum_{t=1}^{T}\omega_t k_t + c_0\sum_{t=1}^{T}\omega_t k_t\right] = 1. \tag{67}$$

We also have

$$\sum_{t=1}^{T}\omega_t = \sum_{t=1}^{T}\sum_{x=0}^{X}(e_{xt} - \tau_{A_x}) = \sum_{x=0}^{X}\sum_{t=1}^{T}(e_{xt} - \tau_{A_x}) = \sum_{x=0}^{X}\sum_{t=1}^{T} e_{xt} - T\sum_{x=0}^{X}\tau_{A_x} = \sum_{x=0}^{X}\sum_{t=1}^{T} e_{xt} - \sum_{x=0}^{X}\sum_{t=1}^{T} e_{xt} + C\sum_{x=0}^{X}\tau_{B_x} = C\cdot 1 = C. \tag{68}$$

Similar derivations refer to $\sum_{x=0}^{X}\nu_{B_x}$ and $\sum_{t=1}^{T}\varpi_t$. Hence, there following equalities hold

$$\sum_{x=0}^{X}\tau_{B_x} = \sum_{x=0}^{X}\nu_{B_x} = 1 \quad \text{and} \quad \sum_{t=1}^{T}\omega_t = C, \quad \sum_{t=1}^{T}\varpi_t = D. \tag{69}$$

There is also

$$\sum_{x=0}^{X} b_x = \sum_{x=0}^{X}\frac{\sum_{t=1}^{T} k_t[2y_{xt} + c_0(e_{xt} - v_{xt} - \tau_{B_x}\omega_t + \nu_{B_x}\varpi_t)]}{2\sum_{t=1}^{T} k_t^2} = \frac{1}{2\sum_{t=1}^{T} k_t^2}\left[2\sum_{t=1}^{T} k_t\sum_{x=0}^{X}(y_{xt} - a_x) + c_0\sum_{t=1}^{T} k_t\sum_{x=0}^{X}(e_{xt} - \tau_{A_x}) - c_0\sum_{t=1}^{T} k_t\sum_{x=0}^{X}(v_{xt} - \nu_{A_x}) - c_0\sum_{t=1}^{T} k_t\left(\omega_t\sum_{x=0}^{X}\tau_{B_x} - \varpi_t\sum_{x=0}^{X}\nu_{B_x}\right)\right]. \tag{70}$$

Using relations (56), (58) and (51) we obtain

$$\sum_{x=0}^{X} b_x = \frac{2\sum_{t=1}^{T} k_t^2 + c_0\left[\sum_{t=1}^{T} k_t(\omega_t - \varpi_t) - \sum_{t=1}^{T} k_t(\omega_t - \varpi_t)\right]}{2\sum_{t=1}^{T} k_t^2} = 1. \tag{71}$$

The special case. Let us assume that $e_{xt} = v_{xt}$ for $x = 0,2, \dots, X,\ t = 1,2 \dots, T$, then the criterion function (50) reduces to

$$F^{(N)} = 2\sum_{x=0}^{X} \sum_{t=1}^{T} \left[(y_{xt} - a_x - b_x k_t)^2 + C^{(N)}\left(e_{xt} - \tau_{A_x} - \tau_{B_x}\omega_t\right)^2\right] \tag{72}$$

and formulas (62) and (63) defining parameters b_x and τ_{B_x} simplify to the following ones

$$b_x = \frac{\sum_{t=1}^{T} k_t y_{xt}}{\sum_{t=1}^{T} k_t^2}. \tag{73}$$

$$\tau_{B_x} = \frac{\sum_{t=1}^{T} \omega_t (e_{xt} - \tau_{A_x})}{\sum_{t=1}^{T} \omega_t^2}, \tag{74}$$

where $\sum_{x=0}^{X} b_x = 1, \sum_{x=0}^{X} \tau_{B_x} = 1$.

It follows from these derivations that the main parameters a_x, b_x, k_t have similar interpretation as in the standard Lee-Carter model (see Section 1). The age-related effects a_x describe the age profile of mortality, time-related effects k_t describe the overall trend of mortality, and b_x represent the mean change of log-central mortality rate y_{xt} in response to change of the time component k_t. However, the CNMM model also has additional parameters $\tau_{A_x}, \tau_{B_x}, \omega_t$ and $\nu_{A_x}, \nu_{B_x}, \varpi_t$ treated as fuzziness of the model parameters. They will be used to determine the fuzziness boundaries of mortality forecasts.

3.4. Data fuzzification

There are several methods proposed to fuzzify the data. One of them is an approach proposed by Koissi and Shapiro (2006) discussed in Subsection 2.1.

What we propose here is to consider a modified version of the Koissi-Shapiro fuzzification method. Let the fuzziness parameters e_{xt} and v_{xt} satisfy the following respective equations for each fixed x

$$e_{xt} = s_{0x} + s_{1x}t, \quad v_{xt} = r_{0x} + r_{1x}t, \qquad t = 1,2, \dots, T, \tag{75}$$

where $s_{0x}, s_{1x}, r_{0x}, r_{1x}$ are found by solving the following optimization problem

$$\text{minimize } \sum_{t=1}^{T}(e_{xt} + v_{xt}) = T \cdot (s_{0x} + r_{0x}) + (s_{1x} + r_{1x})\sum_{t=1}^{T} t, \tag{76}$$

subject to the constraints

$$a_x + b_x \cdot k_t + (s_{0x} + s_{1x}t) \geq \ln m_{xt}, \quad t = 1,2, \dots, T, \tag{77}$$

$$a_x + b_x \cdot k_t - (r_{0x} + r_{1x}t) \leq \ln m_{xt}, \quad t = 1,2, \dots, T, \tag{78}$$

where a_x, k_t, b_x are defined in (55), (56) and (73), and $s_{0x}, s_{1x} \geq 0$ as well as $r_{0x}, r_{1x} \geq 0$ are the smallest values satisfying inequalities (77) and (78), respectively. Once, the coefficients $s_{0x}, s_{1x}, r_{0x}, r_{1x}$ are found, the fuzziness parameters e_{xt} and υ_{xt} can be determined from equations (75).

3.5. Mortality prediction

To forecast log-central mortality rates, time component k_t can be viewed, analogously to the Lee-Carter approach, as a stochastic process. The estimated or forecasted values $\hat{y}_{xt}$ of log-central death rates y_{xt} will be derived for from the following formula

$$\hat{y}_{xt} = a_x + b_x k_t, \tag{79}$$

where a_x, b_x are time invariant, and k_t is a time dependent component. For $t > T$, the time component will be forecasted via a time series model of the form

$$k_t = \delta + k_{t-1} + \zeta_t, \tag{80}$$

with δ and ζ_t's denoting, respectively, a drift and independent and identically distributed Gaussian random terms.

Similar approach applies to parameters e_{xt}, υ_{xt} expressing fuzziness of log-central death rates. The estimated or forecasted values $\hat{e}_{xt}, \hat{\upsilon}_{xt}$ will be derived from the following formulas

$$\hat{e}_{xt} = \tau_{A_x} + \tau_{B_x}\omega_t, \qquad \hat{\upsilon}_{xt} = \nu_{A_x} + \nu_{B_x}\varpi_t, \tag{81}$$

where $\tau_{A_x}, \tau_{B_x}, \nu_{A_x}, \nu_{B_x}$ are time invariant, while ω_t, ϖ_t are time dependent model parameters. Thus, for $t > T$, both ω_t and ϖ_t will be forecasted using the following time series models

$$\omega_t = \mu + \omega_{t-1} + \varsigma_t, \qquad \varpi_t = \gamma + \varpi_{t-1} + \xi_t, \tag{82}$$

with μ, γ denoting some drifts and ς_t, ξ_t denoting independent and identically distributed Gaussian random terms.

The ML estimates $\hat{\delta}, \hat{\mu}, \hat{\gamma}$ of parameters δ, μ, γ are as follows

$$\hat{\delta} = \frac{k_T - k_1}{T-1}, \qquad \hat{\mu} = \frac{\omega_T - \omega_1}{T-1}, \quad \hat{\gamma} = \frac{\varpi_T - \varpi_1}{T-1}. \tag{83}$$

3.6. The case study

To illustrate theoretical discussions presented in this section, the estimates of a_x, b_x, k_t and $\tau_{A_x}, \tau_{B_x}, \omega_t, \nu_{A_x}, \nu_{B_x}, \varpi_t$ were estimated using the real mortality data. Next, the *ex-post* forecasts from the model (33) were derived and the prediction accuracy with results yielded by the Lee-Carter model compared.

The analysis was based on the central death rates in Poland from the years 1965–2019. For computational reasons, age-specific death rates multiplied by 1000 were used. The necessary data were sourced from the Human Mortality Database (www.mortality.org), separately for males and females. The 2014–2019 death rates served the purpose of evaluating predicted rates and were not used in the estimation. Estimates of the parameters were obtained using scaled central death rates for males and females recorded in the years 1965–2013. To ensure the clarity of data presentation, estimates of a_x, b_x, k_t's vs. x or t are plotted in the separate Figures 3–5.

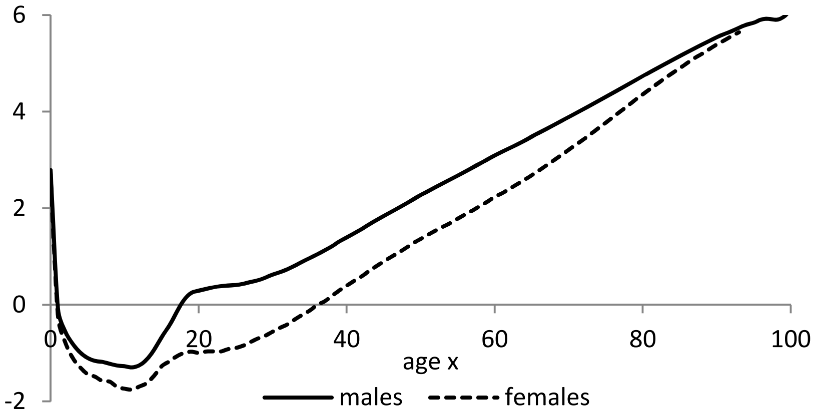


Figure 3. Estimates of parameters a_x, $x = 0,1,2,\ldots,X$ (Poland, males and females)

Source: Developed by the authors.

Curves illustrated in Figure 3 show the average profiles of mortality for males and females over the age range [0,100]. Both curves exhibit a typical "bath tube" shapes with high values around the infant ages, followed by minimal rates at the childhood ages, higher accidental mortality at young adulthood ages and increasing mortality at adulthood and old ages with nearly constant rate of increase. The "accident hump" at adolescence stands for higher mortality rates due to accidental deaths caused by augmented risk-taking behaviour as well as increased suicide rates. Note that the more demonstrable hump refers to the subpopulation of males.

The arrangement of curves in Figure 4 shows that log-central mortality rates for males in young and old age groups are more sensitive to temporal changes in mortality than analogous rates for females. The reverse relationship applies to other age groups.

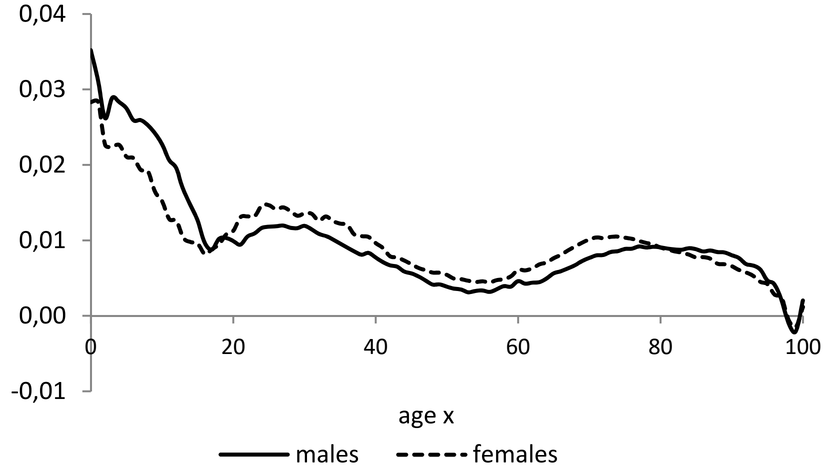


Figure 4. Estimates of parameters b_x, $x = 0,1,2, \dots, X$ (Poland, males and females)

Source: Developed by the authors.

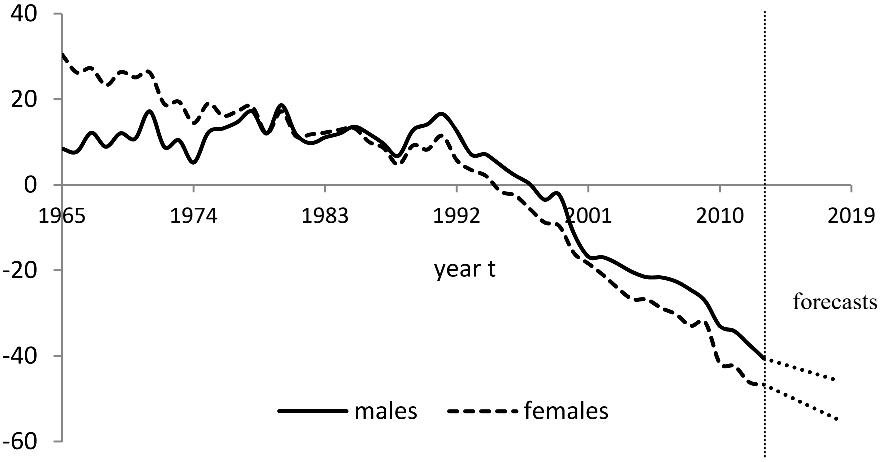


Figure 5. Estimates of parameters $k_t, t = 1,2, \dots, T$ (Poland, males and females) and forecasts up to 2019

Source: Developed by the authors.

Figure 5 illustrates the trend of mortality both for males and females and forecasts up to 2019. It can be seen that curves are generally decreasing, with the decline being faster for women. However, the trend in mortality before 1991 shows a slight flattening, apart from certain fluctuations, which can be explained by the health crisis of the 1970s and 1980s in Poland.

Figures 6—11 exhibit both the real and estimated mortality rates for selected age groups. Estimates of log-central death rates y_{xt} were obtained for males and females by using formula (79). In this case, as before, the estimation period was 1965–2013 and the period of *ex-post* forecasts spanned the years 2014—2019. Forecasts of k_t after 2013 were determined from the model (80). Similar models (81), (82) were used to estimate and forecast fuzziness parameters e_{xt}, v_{xt} necessary to determine fuzziness boundaries for mortality forecasts up to 2019.

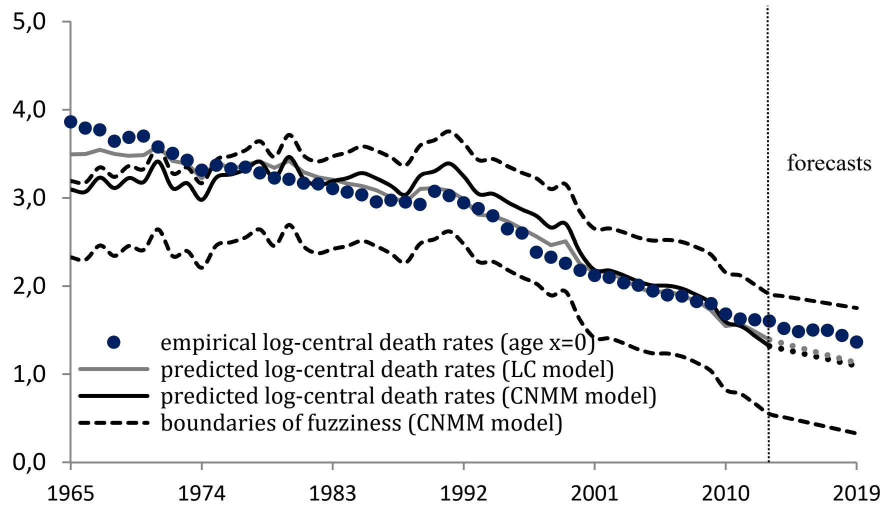


Figure 6. The real and predicted log-central death rates obtained with the LC and CNMM models together with the fuzziness areas (Poland, males aged 0 years)

Source: Developed by the authors.

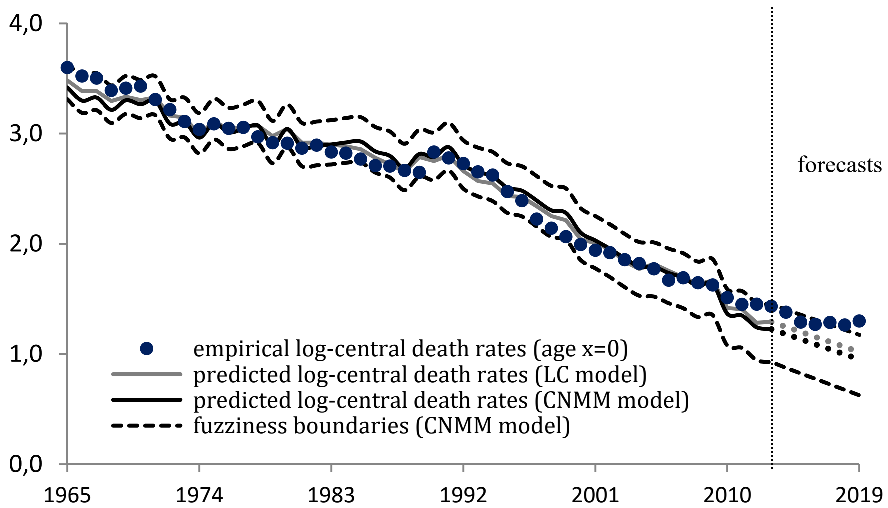


Figure 7. The real and predicted log-central death rates obtained with the LC and CNMM models together with the fuzziness boundaries (Poland, females aged 0 years)

Source: Developed by the authors.

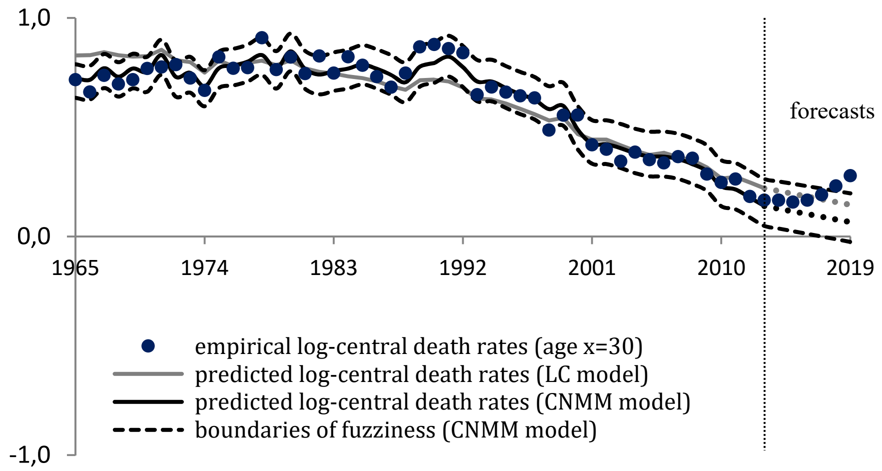


Figure 8. The real and predicted log-central death rates obtained with the LC and CNMM models together with the fuzziness boundaries (Poland, males at the age of 30 years)

Source: Developed by the authors.

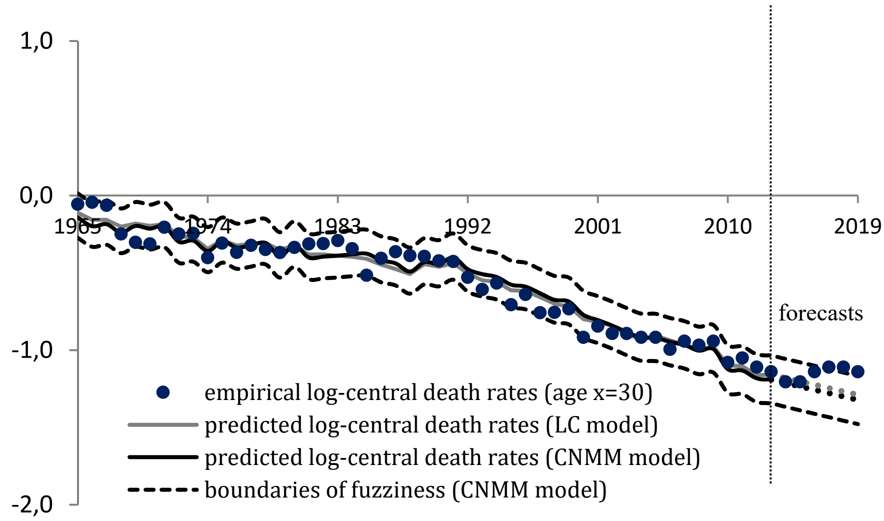


Figure 9. The real and predicted log-central death rates obtained with the LC and CNMM models together with the fuzziness boundaries (Poland, females at the age of 30 years)

Source: Developed by the authors.

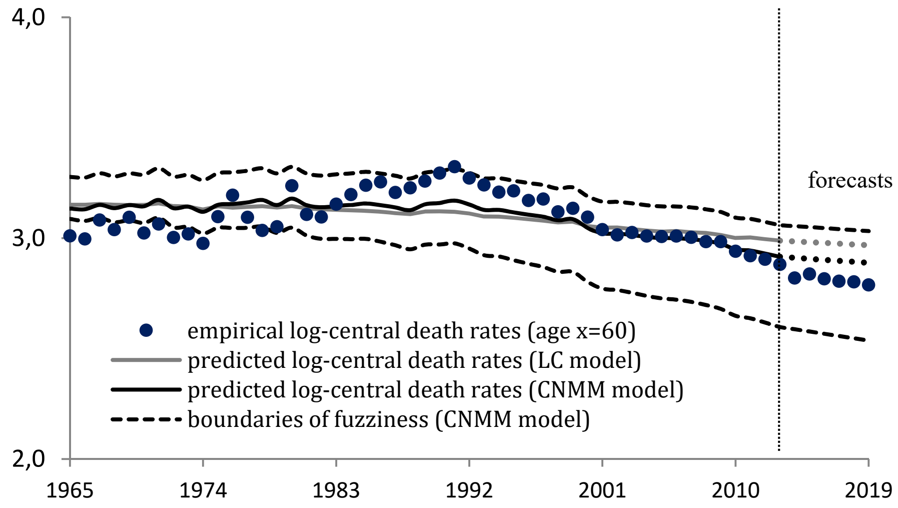


Figure 10. The real and predicted log-central death rates obtained with the LC and CNMM models together with the fuzziness boundaries (Poland, males at the age of 60 years)

Source: Developed by the authors.

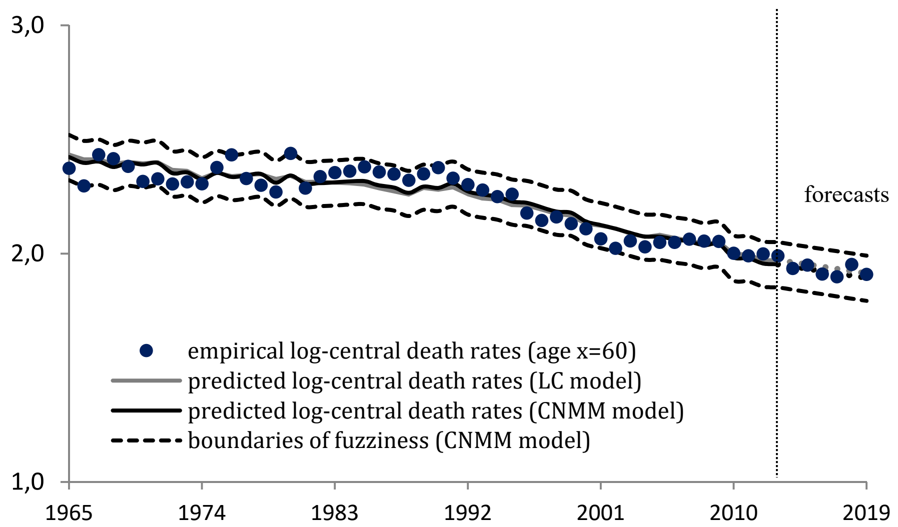


Figure 11. The real and predicted log-central death rates obtained with the LC and CNMM models together with the fuzziness boundaries (Poland, females at the age of 60 years)

Source: Developed by the authors.

The CNMM model as well as the LC model were then compared using *ex-post* mean squared prediction error (MSE) based on the differences between real and estimated log-central mortality rates, i.e.

$$MSE_t = \sqrt{\frac{1}{X+1}\sum_{x=0}^{X}(y_{xt} - \hat{y}_{xt})^2}, \quad t > T, \tag{84}$$

where $\hat{y}_{xt}$ are estimated log-central death rates obtained from the CNMM or LC model.

Table 1. Prediction accuracy of the LC model vs. the CNMM model in terms of the *ex-post* MSE errors

Year	Males		Females	
	LC	**CNMM**	**LC**	**CNMM**
POLAND				
2014	0.166	0.112	0.118	0.116
2015	0.152	0.107	0.111	0.105
2016	0.167	0.116	0.140	0.131
2017	0.174	0.124	0.126	0.125
2018	0.158	0.117	0.150	0.158
2019	0.171	0.129	0.134	0.142
NORWAY				
2014	0.265	0.243	0.305	0.302
2015	0.297	0.273	0.272	0.234
2016	0.294	0.270	0.255	0.248
2017	0.302	0.269	0.340	0.328
2018	0.308	0.283	0.316	0.310
CHECHIA				
2014	0.227	0.220	0.230	0.227
2015	0.281	0.276	0.247	0.238
2016	0.253	0.247	0.235	0.222
2017	0.245	0.242	0.265	0.251

Source: Developed by the authors.

Table 1 summarizes the results of comparisons between the LC and CNMM models in terms of their prediction accuracy for Poland and for two selected European countries. MSE errors were assessed for those years for which the real mortality rates were available.

On the basis of the results obtained, it can be noticed that the CNMM model utilizing complex-valued fuzzy numbers provides comparable or smaller *ex-post* forecast errors, in terms of the MSE measure, than the LC model.

4. Concluding remarks

In the paper an algebraic approach to mortality modelling was introduced. For the formal purposes, the concept of complex-valued fuzzy numbers was also discussed.

The popularity of the widely used Lee-Carter mortality model lies in its simplicity and ease of interpretation. However, due to the uncertainty and imprecision of empirical age-specific mortality rates, it seems justified to use a fuzzy mortality model instead. In our approach, the log-central death rates were viewed as complex-valued fuzzy numbers derived for each age-time cell. The parameters of fuzzified log-central death rates were found in the data fuzzification stage, which was the first step of the model estimation. Next, fuzzy log-central death rates were transformed into complex-valued fuzzy numbers and modelled using the complex analysis.

What makes the CNMM model superior to the standard LC model is that the proposed approach allows for determination of fuzziness boundaries for the mortality trajectories. For the standard LC model, the confidence intervals for log-central mortality rates can also be derived, but they reflect the error term in the random walk model, ignoring the estimation errors of other parameters, so the confidence intervals can only be derived for the prediction window.

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References

Bongaarts, J., (2005). Long-range trends in adult mortality: Models and projection methods, Demography, 42(1), pp. 23–49.

Booth, H., Hyndman, R. J., Tickle, L., De Jong, P., (2006). Lee-Carter mortality forecasting: A multi-country comparison of variants and extensions models, Demographic Research, 15(9), pp. 289–310.

Booth, H., Maindonald, J., Smith, L., (2002). Applying Lee-Carter under conditions of variable mortality decline. Population Studies, 56 (3), 325–333.

Bozik, J. E., Bell, W. R., (1987). Forecasting Age Specific Fertility Using Principal Components, Bureau of the Census Statistical Research Division Washington D.C., CENSUS/SRD/RR-87/19, https://www.census.gov/srd/papers/pdf/rr87-19.pdf.

Brouhns, N., Denuit, M., Vermunt, J. K., (2002). A Poisson log-bilinear regression approach to the construction of projected lifetables, Insurance: Mathematics and Economics, 31(3), pp. 373–393.

Cairns, A. J. G., Blake, D., Dowd, K., (2006). A two-factor model for stochastic mortality with parameter uncertainty: Theory and calibration, Journal of Risk and Insurance, 73(4), pp. 687–718.

Currie, I. D., Durban, M., Eilers, P. H. C., (2004). Smoothing and forecasting mortality rates, Statistical Modelling, 4(4).

Danesi, I. L., Haberman, S., Millossovich, P., (2015). Forecasting mortality in subpopulations using Lee-Carter type models: A comparison, Insurance: Mathematics and Economics, 62(4), pp. 151–161.

De Jong, P., Tickle, L., (2006). Extending Lee-Carter mortality forecasting. Mathematical Population Studies, 13(1), pp. 1–18.

Diamond, P., (1988), Fuzzy least-squares, Information Sciences, 46(3), pp. 141–157.

Haberman, S., Renshaw, A., (2012). Parametric mortality improvement rate modelling and projecting, Insurance: Mathematics and Economics, 50(3), pp. 309–333.

Heligman, L., Pollard, J. H., (1980). The age pattern of mortality, Journal of the Institute of Actuaries, 170, pp. 49–80.

Horiuchi, S., Coale, A. J., (1990). Age patterns of mortality for older women: An analysis using the age-specific rate of mortality change with age, Mathematical Population Studies, 2(4), 245–267.

Human Fertility Database. Max Planck Institute for Demographic Research (Germany) and Vienna Institute of Demography (Austria). Available at www.humanfertility.org.

Ishikawa, S., (1997). Fuzzy inferences by algebraic method, Fuzzy Sets and Systems, 87, pp. 181–200.

Koissi, M.-C., Shapiro, A. F., (2006). Fuzzy formulation of the Lee-Carter model for mortality forecasting, Insurance: Mathematics and Economics, 39, pp. 287–309.

Kosiński, W., Prokopowicz, P., Ślęzak, D., (2003). Ordered Fuzzy Numbers, Bull. Polish Acad. Sci. Math., 51, pp. 327–338.

Lee, R. D., Carter, L., (1992). Modeling and forecasting the time series of U.S. mortality, Journal of the American Statistical Association, 87, pp. 659–671.

Milevsky, M. A., Promislow, S. D., (2001). Mortality Derivatives and the Option to Annuitise,Insurance: Mathematics and Economics, 29, pp. 299–318

Pitacco, E., Denuit, M., Haberman, S., Olivieri, A., (2009). Modelling Longevity Dynamics for Pensions and Annnuity Business, Oxford University Press.

Prokopowicz, P., Czerniak, J., Mikołajewski, D., Apiecionek, Ł., Ślęzak, D. (eds.), (2017). Ordered Fuzzy Numbers: Definitions and Operations, Studies in Fuzziness and Soft Computing, vol. 356, Springer Open.

Renshaw, A. E., Haberman, S., (2003). Lee-Carter mortality forecasting with age specific enhancement, Insurance: Mathematics and Economics, 33(2), pp. 255–272.

Renshaw, A., Haberman, S., Hatzopoulos, P., (1996). The modelling of recent mortality trends in United Kingdom male assured lives. British Actuarial Journal, 2, pp. 449–477.

Rossa, A., Socha, L., Szymański, A., (2011). Analiza i modelowanie umieralności w ujęciu dynamicznym (in Polish), University of Lodz Press, Łódź.

Rossa, A., Socha, L., Szymański, A., (2017). Hybrid Dynamic and Fuzzy Models of Mortality. University of Lodz Press.

Szymański, A., Rossa, A., (2014). Fuzzy mortality model based on Banach algebra, International Journal of Intelligent Technologies and Applied Statistics, 7,pp. 241–265.

Szymański, A., Rossa, A., (2017). Improvement of fuzzy mortality model by means of algebraic methods, Statistics in Transition, 18, pp. 701–724.

Tuljapurkar, S., Li, N., Boe, C., (2000). A universal pattern of mortality decline in the G7 countries, Nature, 405, pp. 789–792.

APPENDIX

A.1. The orthogonal expansion

Two vectors $\varphi, \psi \in L^2(0,1)$ are called orthogonal ($\varphi \perp \psi$) if $\langle \varphi, \psi \rangle = 0$ and parallel if one is multiple of the other. If φ and ψ are orthogonal ($\varphi \perp \psi$), then the Pythagorean theorem is satisfied

$$\| \varphi + \psi \|^2 = \| \varphi \|^2 + \| \psi \|^2.$$

A vector φ is called a unit vector if $\| \varphi \| = 1$.

Suppose φ is a unit vector. Then, the projection of ψ in the direction of φ is given by

$$\psi_{\|} = < \varphi, \psi > \varphi$$

and $\psi_{\perp}$, defined as

$$\psi_{\perp} = \psi - < \varphi, \psi > \varphi,$$

is orthogonal to φ.

It is commonly known that a set of vectors $\{P_j\}$ in $L^2(0,1)$ is called an orthonormal set if $\langle P_j, P_k \rangle = 0$ for $j \neq k$ and $\langle P_j, P_j \rangle = 1$.

A.2. The Legendre polynomials as the basis of the orthonormal expansion

Let us consider the set of orthonormal Legendre polynomials. The first four polynomials take the form

$$P_0(u) = 1,$$

$$P_1(u) = \sqrt{3}(2u - 1),$$

$$P_2 = \frac{\sqrt{5}}{2}[3(2u - 1)^2 - 1],$$

$$P_3 = \frac{\sqrt{7}}{2}[5(2u - 1)^3 - 3(2u - 1)],$$

and recursively

$$P_{n+1}(u) = \frac{\sqrt{(2n + 1)(2n + 3)}}{(n + 1)}(2u - 1)P_n(u) - \frac{n}{n + 1}\sqrt{\frac{2n + 3}{2n - 1}}P_{n-1}(u).$$

By putting $j = n + 1$ we have for $n = 2, 3, \ldots$

$$P_j(u) = \frac{\sqrt{(2j - 1)(2j + 1)}}{j}(2u - 1)P_{j-1}(u) - \frac{j - 1}{j}\sqrt{\frac{2j + 1}{2j - 3}}P_{j-2}(u).$$

We will use the recursive formula to find the Legendre polynomials P_4, P_5 orthonormal on the interval $[0, 1]$. We get

$$P_4 = \frac{\sqrt{9}}{8}[35(2u - 1)^4 - 30(2u - 1)^2 + 3],$$

$$P_5 = \frac{\sqrt{11}}{8}[63(2u-1)^5 - 70(2u-1)^3 + 15(2u-1)].$$

For $j = 3$ there is $P_3 = \frac{\sqrt{7}}{2}[5(2u-1)^3 - 3(2u-1)]$.

Let us calculate the scalar products $\langle P_0, P_3\rangle, \langle P_1, P_3\rangle, \langle P_2, P_3\rangle$.

For $P_0(u) = 1,\ P_3 = \frac{\sqrt{7}}{2}[5(2u-1)^3 - 3(2u-1)]$ there is

$$\langle P_0, P_3\rangle = \frac{\sqrt{7}}{2}\left[5\int_0^1 (2u-1)^3\,du - 3\int_0^1 (2u-1)du\right] = 0.$$

For $P_1(u) = \sqrt{3}(2u-1),\ P_3 = \frac{\sqrt{7}}{2}[5(2u-1)^3 - 3(2u-1)]$ we have

$$\langle P_1, P_3\rangle = \frac{\sqrt{21}}{2}\left[5\int_0^1 (2u-1)^4 - 3\int_0^1 (2u-1)^2 du\right] = 0.$$

For $P_2 = \frac{\sqrt{5}}{2}[3(2u-1)^2 - 1], P_3 = \frac{\sqrt{7}}{2}[5(2u-1)^3 - 3(2u-1)]$ we get

$$\langle P_2, P_3\rangle = \frac{\sqrt{35}}{4}\left[15\int_0^1 (2u-1)^5 du - 14\int_0^1 (2u-1)^3 du + 3\int_0^1 (2u-1)du\right] = 0.$$

Hence, it follows that $P_0 \perp P_3,\ \ P_1 \perp P_3,\ \ P_2 \perp P_3$.

Now, let us verify the normality of the element $P_3 \in L_2(0,1)$, i.e. we will verify the equality $\| P_3 \|^2 = 1$. Note that P_3^2 is equal to

$$P_3^2 = \frac{7}{4}[5(2u-1)^3 - 3(2u-1)]^2$$
$$= \frac{7}{4}[25(2u-1)^6 - 30(2u-1)]^4 + 9(2u-1)^2].$$

The squared norm of the element $P_3 \in L_2(0,1)$ is as follows

$$\| P_3 \|^2 = \frac{7}{4}\left[25\int_0^1 (2u-1)^6\,du - 30\int_0^1 (2u-1)^4 du + 9\int_0^1 (2u-1)^2 du\right]$$
$$= \frac{7}{4}\left(\frac{25}{7} - \frac{30}{5} + \frac{9}{3}\right) = 1.$$

Thus, $P_3 \in L_2(0,1)$ belongs to the orthonormal system $\{P_j\}_{j=0}^{3}$.

A.3. Orthonormal expansions of inverse triangular functions

Let us assume that f, g take the following forms

$$f(u) = a - s(1-u), \quad g(u) = a + s(1-u), \quad u \in [0,1],$$

and let $\{P_j\}$ be an orthonormal set of Legendre polynomials in $L^2(0,1)$.

First, we will find coefficients α_j, β_j for $j = 0, 1$. We have

$$\alpha_0 = \langle P_0, f\rangle = \int_0^1 f(u)du = a - s\int_0^1 (1-u)du = a - \frac{s}{2},$$

$$\beta_0 = \langle P_0, g\rangle = \int_0^1 f(u)du = a + s\int_0^1 (1-u)du = a + \frac{s}{2},$$

$$\alpha_1 = \langle P_1, f\rangle = \int_0^1 \sqrt{3}(2u-1)[a-(1-u)s]du = \frac{s}{2\sqrt{3}},$$

$$\beta_1 = \langle P_1, g\rangle = \int_0^1 \sqrt{3}(2u-1)[a+(1-u)s]du = -\frac{s}{2\sqrt{3}}.$$

Thus, we obtain

$$f(u) = f^{(1)}(u) = \sum_{j=0}^{1} \alpha_j P_j = a - \frac{s}{2} + s\left(u - \frac{1}{2}\right) = a - s(1-u),$$

$$g(u) = g^{(1)}(u) = \sum_{j=0}^{1} \beta_j P_j = a + \frac{s}{2} + s\left(-u + \frac{1}{2}\right) = a + s(1-u).$$

A.4. Orthonormal expansions of inverse exponential functions

Suppose that f, g are expressed as

$$f(u) = c - \tau(-\ln u)^{\frac{1}{2}}, \quad g(u) = c + \nu(-\ln u)^{\frac{1}{2}}, \quad u \in [0,1]. \tag{A.1}$$

First, we will find coefficients α_j, β_j for $j = 0,1,2,3$. We have

$$\alpha_j = \langle P_j, f\rangle = \langle P_j, c+\psi\rangle = \langle P_j, c\rangle + \langle P_j, \psi\rangle,$$

$$\beta_j = \langle P_j, g\rangle = \langle P_j, c+\varphi\rangle = \langle P_j, c\rangle + \langle P_j, \varphi\rangle.$$

For scalar products $\langle P_j, \psi\rangle$ and $\langle P_j, \varphi\rangle$ we need to calculate the integral $\int_0^1 u^j(-\ln u)^{\frac{1}{2}}du$. After some basic calculations we obtain

$$\int_0^1 u^j(-\ln u)^{\frac{1}{2}}du = \frac{\sqrt{\pi}}{2(j+1)^{\frac{3}{2}}}.$$

For $j = 0$, we get $P_0(u) = 1$ and

$$\langle P_0, c\rangle = \langle 1, c\rangle = \int_0^1 c\,du = c.$$

Thus,

$$\alpha_0 = \langle P_0, f\rangle = \langle 1, c\rangle + \langle 1, f\rangle = c + \int_0^1 \psi(u)du,$$

$$\beta_0 = \langle P_0, g\rangle = \langle 1, c\rangle + \langle 1, g\rangle = c + \int_0^1 \varphi(u)du.$$

Hence, there is

$$\alpha_0 = c - \tau \int_0^1 (-\ln u)^{\frac{1}{2}}\, du,$$

$$\beta_0 = c + \nu \int_0^1 (-\ln u)^{\frac{1}{2}} du.$$

For $j = 0$, we have $\int_0^1 (-\ln u)^{\frac{1}{2}} du = \frac{\sqrt{\pi}}{2}$, and α_0, β_0 can be reduced to

$$\alpha_0 = \int_0^1 f(u) du = c - \tau \frac{\sqrt{\pi}}{2}, \tag{A.2}$$

$$\beta_0 = \int_0^1 g(u) du = c + \nu \frac{\sqrt{\pi}}{2}. \tag{A.3}$$

Using the recursive formula, we can obtain next orthonormal expansion for $j = 1,2,3, \ldots$.

Let us take $j = 1$, then $P_1(u) = \sqrt{3}(2u - 1)$ and

$$\langle P_1, c\rangle = \sqrt{3} c \int_0^1 (2u - 1)\, du = \sqrt{3}(c - c) = 0.$$

We have also

$$\langle P_1, \psi\rangle = -\tau \int_0^1 P_1 (-\ln u)^{\frac{1}{2}}\, du = -\tau \sqrt{3} \int_0^1 (2u - 1)(-\ln u)^{\frac{1}{2}}\, du = -\tau \frac{\sqrt{3\pi}}{2}\left(\frac{1}{\sqrt{2}} - 1\right),$$

$$\langle P_1, \varphi\rangle = \nu \int_0^1 P_1 (-\ln u)^{\frac{1}{2}}\, du = \nu \sqrt{3} \int_0^1 (2u - 1)(-\ln u)^{\frac{1}{2}}\, du = \nu \frac{\sqrt{3\pi}}{2}\left(\frac{1}{\sqrt{2}} - 1\right).$$

Thus, we receive

$$\alpha_1 = \langle P_1, f\rangle = -\tau \frac{\sqrt{3\pi}}{2}\left(\frac{1}{\sqrt{2}} - 1\right), \tag{A.4}$$

$$\beta_1 = \langle P_1, g\rangle = \nu \frac{\sqrt{3\pi}}{2}\left(\frac{1}{\sqrt{2}} - 1\right). \tag{A.5}$$

For $j = 2$, there is $P_2 = \frac{\sqrt{5}}{2}[3(2u - 1)^2 - 1]$ and

$$\alpha_2 = \langle P_2, f\rangle = \langle P_2, c\rangle + \langle P_2, \psi\rangle, \quad \beta_2 = \langle P_2, g\rangle = \langle P_2, c\rangle + \langle P_2, \varphi\rangle,$$

where

$$\langle P_2, c\rangle = c\frac{\sqrt{5}}{2} \int_0^1 (3(2u - 1)^2 - 1) du = 2c\sqrt{5} - 3c\sqrt{5} + \frac{3c\sqrt{5}}{2} - \frac{c\sqrt{5}}{2} = 0$$

$$\langle P_2, \psi\rangle = -\tau \int_0^1 P_2 (-\ln u)^{\frac{1}{2}}\, du = -\tau\sqrt{5\pi}\left(\frac{1}{\sqrt{3}} - \frac{3}{2\sqrt{2}} + \frac{1}{2}\right),$$

$$\langle P_2, \varphi\rangle = \upsilon \int_0^1 P_2 (-\ln u)^{\frac{1}{2}}\, du = \upsilon\sqrt{5\pi}\left(\frac{1}{\sqrt{3}} - \frac{3}{2\sqrt{2}} + \frac{1}{2}\right).$$

Hence,

$$\alpha_2 = \langle P_2, f\rangle = -\tau\sqrt{5\pi}\left(\frac{1}{\sqrt{3}} - \frac{3}{2\sqrt{2}} + \frac{1}{2}\right),$$

$$\beta_2 = \langle P_2, g\rangle = \upsilon\sqrt{5\pi}\left(\frac{1}{\sqrt{3}} - \frac{3}{2\sqrt{2}} + \frac{1}{2}\right).$$

Let us find coefficients α_3 and β_3, i.e.

$$\alpha_3 = \langle P_3, f\rangle = \langle P_3, c\rangle + \langle P_3, \psi\rangle,$$

$$\beta_3 = \langle P_3, g\rangle = \langle P_3, c\rangle + \langle P_3, \varphi\rangle.$$

We have $P_3 = \frac{\sqrt{7}}{2}[5(2u-1)^3 - 3(2u-1)]$ and

$$\langle P_3, c\rangle = c\frac{\sqrt{7}}{2}\int_0^1 (5(2u-1)^3 - 3(2u-1))du = \frac{c\sqrt{7}}{2}(10 - 20 + 12 - 2) = 0,$$

$$\langle P_3, \psi\rangle = -\tau\int_0^1 P_3(-\ln u)^{\frac{1}{2}}du = -\tau\sqrt{7\pi}\left(-\frac{5}{\sqrt{3}} + \frac{15}{4\sqrt{2}} - \frac{3}{4\sqrt{2}} + \frac{3}{4}\right).$$

$$\langle P_3, \varphi\rangle = \upsilon\int_0^1 P_3(-\ln u)^{\frac{1}{2}}du = \upsilon\sqrt{7\pi}\left(-\frac{5}{\sqrt{3}} + \frac{15}{4\sqrt{2}} - \frac{3}{4\sqrt{2}} + \frac{3}{4}\right)$$

Hence,

$$\alpha_3 = \langle P_3, f\rangle = -\tau\sqrt{7\pi}\left(-\frac{5}{\sqrt{3}} + \frac{15}{4\sqrt{2}} - \frac{3}{4\sqrt{2}} + \frac{3}{4}\right), \quad (A.6)$$

$$\beta_3 = \langle P_3, g\rangle = \upsilon\sqrt{7\pi}\left(-\frac{5}{\sqrt{3}} + \frac{15}{4\sqrt{2}} - \frac{3}{4\sqrt{2}} + \frac{3}{4}\right). \quad (A.7)$$

Thus, orthonormal expansions of $f(u)$ and $g(u)$ defined in (A.1) are as follows

$$f(u) \cong f^{(3)}(u) = \sum_{j=0}^{3}\alpha_j P_j, \quad g(u) \cong g^{(3)}(u) = \sum_{j=0}^{3}\beta_j P_j,$$

where

$$\alpha_0 P_0(u) = c - \tau\frac{\sqrt{\pi}}{2}, \qquad \beta_0 P_0(u) = c + \nu\frac{\sqrt{\pi}}{2},$$

$$\alpha_1 P_1(u) = -\tau\,\frac{3\sqrt{\pi}}{2}\left(\frac{1}{\sqrt{2}} - 1\right)(2u-1), \quad \beta_1 P_1(u) = \nu\,\frac{3\sqrt{\pi}}{2}\left(\frac{1}{\sqrt{2}} - 1\right)(2u-1),$$

$$\alpha_2 P_2(u) = -\tau\frac{5\sqrt{\pi}}{2}\left(\frac{1}{\sqrt{3}} - \frac{3}{2\sqrt{2}} + \frac{1}{2}\right)[3(2u-1)^2 - 1],$$

$$\beta_2 P_2(u) = \nu\frac{5\sqrt{\pi}}{2}\left(\frac{1}{\sqrt{3}} - \frac{3}{2\sqrt{2}} + \frac{1}{2}\right)[3(2u-1)^2 - 1],$$

$$\alpha_3 P_3(u) = -\tau\frac{7\sqrt{\pi}}{2}\left(-\frac{5}{\sqrt{3}} + \frac{15}{4\sqrt{2}} - \frac{3}{4\sqrt{2}} + \frac{3}{4}\right)[5(2u-1)^3 - 3(2u-1)],$$

$$\beta_3 P_3(u) = \nu\frac{7\sqrt{\pi}}{2}\left(-\frac{5}{\sqrt{3}} + \frac{15}{4\sqrt{2}} - \frac{3}{4\sqrt{2}} + \frac{3}{4}\right)[5(2u-1)^3 - 3(2u-1)].$$