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From the Editor

This year's latest issue contains twelve articles. Their 34 authors come from 10 different countries such as the United Kingdom, USA, India, Poland, Germany, Nigeria, Iran, Turkey, Pakistan, Australia, Ukraine, Jordan, Saudi Arabia and Egypt, which proves that our journal is a very good place for scientists from different parts of the world to exchange ideas and share research results.

The publication of the final issue of the year is traditionally also a moment to express our gratitude and thanks to our authors, reviewers and all the participants of the editorial process. Our success is to cooperate with an international team of experts and great articles' founders that are the basis to publish high quality scientific papers. A list of the names of these people of merit for our journal is included in the Acknowledgements. On behalf of the Editorial Board, Associate Editors and the journal's readers I sincerely thank all our partners and patrons.

Invited papers

This issue starts with the invited paper ***Unreported standard errors in meta-analysis*** by **Nicholas T. Longford**. The paper discusses how to assess the loss of information caused by the practice excluding from meta-analysis these studies when the standard error of its treatment-effect estimator, or the estimate of the variance of the outcomes, is not reported and cannot be recovered from the available information. The author presents two sets of examples of the methods used, explaining in each section assumptions, difficulties, and conclusions from the analysis, noting also the context of the conducted studies (for example countries with different levels of development or organisation of health care).The examples shown in sections 4 and 5 confirm that even simple methods, using some shortcuts on proper imputation, exploit nearly to the full the information about an incompletely reported study and they estimate the standard error of the overall treatment effect with negligible bias.

Research articles

The article entitled ***Approximately optimum strata boundaries for two concomitant stratification variables under proportional allocation*** by **Faizan Danish** and **S. E. H. Rizvi** deals with a problem of proper choice of the strata boundaries as an important factor as regards the efficiency of the estimator of population characteristic under consideration. For obtaining approximately optimum strata boundaries a Cum

Rule (i=3,4) has been provided based on a single study variable along with two concomitant variables used as the basis of stratification variables. The simulation study proved the superiority of the proposed methods with regard to the existing methods. The research also showed that the use of two stratification variables gains efficiency over a single auxiliary variable and the proposed methods are more precise than the existing methods.

Henryk Gurgul, Jessica Hastenteufel, and **Tomasz Wójtowicz** in their article ***Changes in the impact of US macroeconomic news on financial markets the example of the Warsaw Stock Exchange*** analyse the behaviour of 5-minute returns of the WIG20 in a short period after the announcements of 13 macroeconomic indicators describing the US economy. Authors examine how US macroeconomic news affected the WIG20 in years 2004-2019. The WIG20 reacts significantly to announcements of most of the indicators considered. This reaction is immediate and it is usually limited to the first 5-minute returns. The strongest impact is observed after NFP announcements. The analysis in sub-periods leads to the conclusion that, in general, US macroeconomic news announcements induced the highest averages of abnormal returns during the global financial crisis (2007-2009) and in the first few years after the crisis. In later years, the impact of information from the United States was notably weaker.

The next paper ***Agu-Eghwerido distribution, regression model and applications*** by **Agu Friday Ikechukwu** and **Joseph Thomas Eghwerido** presents a one-parameter distribution called the Agu-Eghwerido (AGUE) distribution with its simple mathematical representation and the regression model of the AGUE distribution. The AGUE parameter was estimated using the method of maximum likelihood estimation. The lifetime applications of the AGUE distribution was illustrated using two lifetime data sets. The characteristic of the introduced model for a larger sample size was examined via simulation study and the simulation results showed that the increase in parameter values decreases the mean squared error value. Similarly, the mean estimate tends towards the true parameter value as the sample sizes increase. Thus, it provides the best fit and more flexible than Pranav, exponential and Lindley distributions for the data sets. Ultimately, the AGUE distribution can serve as an alternative model to Pranav, exponential and Lindley distributions in the literature.

The article entitled ***A new extension of Odd Half-Cauchy Family of Distributions: properties and applications with regression modeling*** prepared by **Subrata Chakraburty, Morad Alizadeh, Laba Handique, Emrah Altun** and **G. G. Hamedani** proposes a new family of continuous distributions called the extended odd half Cauchy-G. The distribution bases on the T −X construction of Alzaatreh et al. (2013) by considering half Cauchy distribution for T and the exponentiated $G(x;\xi)$ as the distribution of X. Authors have outlined several particular cases and a number of important statistical characteristics of this family were investigated. On this basis a new regression model was proposed and its application in modelling data in the presence of covariates was presented.

Muhammad Aslam, Mehreen Afzaal and **Muḥammad Ishaq Bhatti** discuss ***A study on exponentiated Gompertz distribution under Bayesian discipline using informative priors.*** This distribution has been recently used in almost all areas of human endeavours, starting from modelling lifetime data to cancer treatment. This paper explores the important properties of the EGZ distribution under Bayesian discipline using two informative priors: the Gamma Prior (GP) and the Inverse Levy Prior (ILP). The usefulness of the model is illustrated with the use of real-life data in relation to simulated data. The simulated study and real-life data were used for various sample sizes with 10,000 replications. The results for real life data and simulation are identical.

In the next paper entitled ***The problem of statistical assessment of the potential for the development of regional integration processes*** **Oleksandr Osaulenko, Olena Bulatova, Olha Zakharova** and **Natallia Reznikova** show the use of integrated indices to evaluate the potential for the development of regional integration processes. A new research and methodological approach were proposed with regard to the intensity of the influence of internal and external factors on integrative relations development. Countries of the world choose their own strategy for participation in the processes of regional integration depending on challenges determined by their level of socio-economic development, the existing potential, the nature of the development of external relations, etc. Thus, it is advisable to apply integral indicators as they allow providing a comprehensive and quantitative description of processes of economic integration that take place in the world economy at a certain moment of time.

Rama Shanker's and **Umme Habibah Rahman's** article presents ***The Type II Topp-Leone Frechet distribution: properties and applications.*** Authors discuss the properties of the distribution including hazard rate function, reverse hazard rate function, Mills ratio, quantile function and order statistics as well as the maximum likelihood estimation used for estimating the parameters of the proposed distribution. The paper deals also with the problem of applications of the distribution for modelling several data sets relating to temperature and the goodness of fit of the proposed distribution compared with that of the Frechet distribution.

The next paper ***Record data from Kies distribution and related statistical inferences*** by **Nesreen M. Al-Olaimat, Husam A. Bayoud** and **Mohammad Z. Raqab** describes the Kies probability as an alternative to the extended Weibull models due to the fact it provides a more efficient fit to some real-life data sets. The classical and Bayesian inferences for the Kies distribution based on records were proposed and the maximum likelihood estimates were studied jointly with asymptotic and bootstrap confidence intervals. The Bayes estimates, along with credible intervals were discussed assuming squared and LINEX loss functions. The performance of the different estimation methods was assessed via Monte Carlo simulations. From the simulation study it was concluded that the proposed informative Bayes estimates outperform the classical estimates in all considered cases. However, non-informative Bayesian and the classical estimation methods perform almost the same under SE and LINEX under small ν, while better results of the Bayesian methods are obtained under LINEX assuming other positive values of ν. The Bayes credible intervals compete the classical confidence intervals in terms of the coverage probability in all cases.

Amal S. Hassan, Salwa M. Assar, Kareem A. Ali and **Heba F. Nagy** in their paper ***Estimation of the density and cumulative distribution functions of the exponentiated Burr XII distribution*** consider seven different estimators of the PDF and CDF of the EBXII distribution when the shape parameters k and c are assumed to be known. Maximum likelihood estimator, uniformly minimum variance unbiased estimator, least squares estimator, weighted least squares estimator, maximum product spacing estimator, Cramér-von-Mises estimator and Anderson-Darling estimator are obtained. A simulation study was performed to compare the behaviours of the proposed estimates. The results show that the maximum likelihood and uniformly minimum variance unbiased estimates perform better than the other estimators.

The last paper prepared by **Jagdish Saran**, **Narinder Pushkarna** and **Shikha Sehgal** presents ***Relationships for moments of the progressively Type-II right censored order statistics from the power Lomax distribution and the associated inference***. Several recurrence relations between single and product moments of progressively Type-II right censored order statistics from the power Lomax distribution were established. The relations enable the computation of all the single and product moments of progressively Type-II right censored order statistics for all sample sizes n and all censoring schemes (R_1,R_2,…,R_m), m ≤n, in a simple recursive manner. The maximum likelihood approach was used for the estimation of the parameters and the reliability characteristic. A Monte Carlo simulation study was conducted to compare the performance of the estimates for different censoring schemes.

Research Communicates and Letters

The Research Communicates & Letters section presents a paper ***Towards a target employment rate within age and gender groups*** by **Stanisław Jaworski** and **Zofia Zielińska-Kolasińska.** The aim of the article was to state the prognosis about employment rates in European countries. It seems that governments of many countries should revise their economic strategies affecting labour markets if they want to achieve satisfactory employment rates. The research presents a pessimistic prognosis of employment rates in European countries with respect to young and partly to older workers. The German employment rate served as a benchmark for this research. The likelihood was estimated by a Monte-Carlo simulation based on the class of exponential smoothing models.

Włodzimierz Okrasa
Editor

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Submission information for Authors

Statistics in Transition new series (SiT) is an international journal published jointly by the Polish Statistical Association (PTS) and Statistics Poland, on a quarterly basis (during 1993–2006 it was issued twice and since 2006 three times a year). Also, it has extended its scope of interest beyond its originally primary focus on statistical issues pertinent to transition from centrally planned to a market-oriented economy through embracing questions related to systemic transformations of and within the national statistical systems, world-wide.

The SiT-*ns* seeks contributors that address the full range of problems involved in data production, data dissemination and utilization, providing international community of statisticians and users – including researchers, teachers, policy makers and the general public – with a platform for exchange of ideas and for sharing best practices in all areas of the development of statistics.

Accordingly, articles dealing with any topics of statistics and its advancement – as either a scientific domain (new research and data analysis methods) or as a domain of informational infrastructure of the economy, society and the state – are appropriate for *Statistics in Transition new series.*

Demonstration of the role played by statistical research and data in economic growth and social progress (both locally and globally), including better-informed decisions and greater participation of citizens, are of particular interest.

Each paper submitted by prospective authors are peer reviewed by internationally recognized experts, who are guided in their decisions about the publication by criteria of originality and overall quality, including its content and form, and of potential interest to readers (esp. professionals).

Manuscript should be submitted electronically to the Editor:
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It is assumed, that the submitted manuscript has not been published previously and that it is not under review elsewhere. It should include an abstract (of not more than 1600 characters, including spaces). Inquiries concerning the submitted manuscript, its current status etc., should be directed to the Editor by email, address above, or w.okrasa@stat.gov.pl.

For other aspects of editorial policies and procedures see the SiT Guidelines on its Web site: http://stat.gov.pl/en/sit-en/guidelines-for-authors/

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Editorial Policy

The broad objective of *Statistics in Transition new series* is to advance the statistical and associated methods used primarily by statistical agencies and other research institutions. To meet that objective, the journal encompasses a wide range of topics in statistical design and analysis, including survey methodology and survey sampling, census methodology, statistical uses of administrative data sources, estimation methods, economic and demographic studies, and novel methods of analysis of socio-economic and population data. With its focus on innovative methods that address practical problems, the journal favours papers that report new methods accompanied by real-life applications. Authoritative review papers on important problems faced by statisticians in agencies and academia also fall within the journal's scope.

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Abstracting and Indexing Databases

Statistics in Transition new series is currently covered in:

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- CEEOL – Central and Eastern European Online Library
- CEJSH (The Central European Journal of Social Sciences and Humanities)
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Unreported standard errors in meta-analysis

Nicholas T. Longford[1]

ABSTRACT

A study that would otherwise be eligible is commonly excluded from a meta-analysis when the standard error of its treatment-effect estimator, or the estimate of the variance of the outcomes, is not reported and cannot be recovered from the available information. This is wasteful when the estimate of the treatment effect is reported. We assess the loss of information caused by this practice and explore methods of imputation for the missing variance. The methods are illustrated on two sets of examples, one constructed specifically for illustration and another based on a published systematic review.

Key words: empirical Bayes, imputation, meta-analysis, missing value, sensitivity analysis.

1. Introduction

In a typical meta-analysis for comparing two treatments, A and B, there are H studies and for each study i we have an estimate $\hat{\theta}_i$ of the treatment effect θ_i, an estimate $\hat{\sigma}_i^2$ of the variance σ_i^2 of the outcomes and the within-treatment sample sizes $n_{i\mathrm{A}}$ and $n_{i\mathrm{B}}$, from which the standard error of $\hat{\theta}_i$, denoted by τ_i, can be easily estimated. For example, when the subjects in study i are assigned to the treatments completely at random subject to fixed sample sizes $n_{i\mathrm{A}}$ and $n_{i\mathrm{B}}$, we have $\tau_i^2 = \sigma_i^2(1/n_{i\mathrm{A}} + 1/n_{i\mathrm{B}})$, and τ_i^2 is estimated by $\hat{\tau}_i^2 = \hat{\sigma}_i^2(1/n_{i\mathrm{A}} + 1/n_{i\mathrm{B}})$. We assume that the estimators $\hat{\theta}_i$ and $\hat{\sigma}_i^2$ are unbiased for the respective targets θ_i and σ_i^2, and that the variances $\sigma_{i\mathrm{A}}^2$ and $\sigma_{i\mathrm{B}}^2$ within the two treatment groups coincide with σ_i^2. The development presented here can easily be adapted for heteroscedasticity because the key parameter we work with is the standard error τ_i and its estimate. Note that $\hat{\tau}_i$ is not unbiased for τ_i, and neither is $1/\hat{\tau}_i^2$ for $1/\tau_i^2$, even when $\hat{\tau}_i^2$ is unbiased for τ_i^2; see Longford (2010 and 2015) for a discussion of this issue in a wider context.

For background to meta-analysis we refer to Rice, Higgins and Lumley (2018) and references therein. Of historical importance is Glass (1976), credited with coining the term, and Hedges and Olkin (1985), the first comprehensive account of statistical methods for meta-analysis. Nowadays, meta-analysis is applied widely, in social and medical sciences in particular, to pool information across studies in which identical or closely related parameters are estimated.

Systematic reviews are a formalised approach to identifying studies suitable for meta-analysis and related purposes; see Haidich (2010) for an introduction. The CONSORT statement (Begg *et al.*, 1996) and the STROBE initiative (von Elm *et al.*, 2008) formulate guidelines and standards for the conduct and presentation of such reviews and for reporting

[1] School of Public Health, Imperial College London, UK. E-mail: sntlnick@sntl.co.uk.
ORCID: https://orcid.org/0000-0003-4129-9726.

single case studies in a manner conducive to their use in future systematic reviews. They are widely adopted today.

We are concerned with the setting in which an estimate of the sampling variance τ_i^2 is not available for one or a few studies. We deal with the case of a single study for which $\hat{\tau}_i^2$ is not available, but the proposed methods and conclusions carry over to meta-analysis in which several studies have this deficiency. Our focus is on meta-analysis with only a few studies, to which even a single study may contribute with a relatively large amount of information, so we can ill-afford to discard it. We assume that the estimates $\hat{\theta}_i$ and the sample sizes $n_{i\mathrm{A}}$ and $n_{i\mathrm{B}}$ are available for all studies.

There are two generic methods for dealing with missing values in an analysis. By listwise deletion, we apply the planned analysis to the units (studies in a meta-analysis) for which we have complete information. This is wasteful because we discard some studies even though we have their estimates $\hat{\theta}_i$, and sometimes also the sample sizes n_i and other details. By imputation, we substitute a value for each missing data item. However well we may estimate the missing values $\hat{\sigma}_i^2$ (or $\hat{\tau}_i^2$), we overstate the precision of the estimator $\hat{\theta}$ of the overall treatment effect θ because by treating the imputed values $\tilde{\sigma}_i^2$ (or $\tilde{\tau}_i^2$) on par with the corresponding estimates we pretend to have more information than was in fact collected. Multiple imputation (Rubin, 2004) addresses this deficiency in a principled way, although it entails some complexities in our context.

Various forms of sensitivity analysis can hone in on the range of plausible values of the complete-data estimator of the average treatment effect. For outcomes with values in a finite range, imputation of extreme values is an obvious starting point. For an improvement of this method, see Gamble and Hollis (2005). Publication bias is another issue related to missing values. It concerns studies that were conducted but their results were not published. For a landmark contribution to this topic, see Duval and Tweedie (2000). Rothstein, Sutton and Borenstein (2005) is an authoritative edited volume dedicated to this subject. See Lin and Chu (2018) for a recent contribution.

Our problem relates to a study published with incomplete information. On the one hand, we want to rescue such a study for the meta-analysis by using all the available data; on the other hand, we want to reflect in the statements we make the loss due to the incompleteness. In brief, we want to be 'honest' in our inferential statements.

We explore two general approaches, modelling and sensitivity analysis. In Section 3, we specify an empirical Bayes model for the variances σ_i^2 and impute a random draw from the approximated conditional distribution of the missing variance σ_{H+1}^2. This imputation is replicated (independently repeated) several times, to generate a set of plausible completions of the dataset. We assume that the study-specific treatment effects coincide; $\theta_i = \theta$ for all studies i. In Section 3.2 we discuss random-effects meta-analysis (DerSimonian and Laird, 1986), in which this assumption is relaxed and the treatment effects θ_i are a random sample from an unknown distribution.

In Section 4, we apply a method motivated by sensitivity analysis, in which we consider a plausible range of values of σ_i^2, or τ_i^2, and evaluate the corresponding estimates of the overall effect θ and standard errors of $\hat{\theta}$. Section 5 applies the methods to a meta-analysis with complete information, in which the standard error of one study is masked. Section 6 discusses some peripheral issues; they include elicitation of the information about the

Table 1: Examples of sets of five studies included in a meta-analysis, with the sampling variance estimate $\hat{\tau}_i^2$ not available for one study.

	Case A			Case B			Case C		
Study (i)	$\hat{\theta}_i$	$\hat{\tau}_i^2$	n_i	$\hat{\theta}_i$	$\hat{\tau}_i^2$	n_i	$\hat{\theta}_i$	$\hat{\tau}_i^2$	n_i
1	0.467	0.260	80	0.617	0.260	80	0.567	0.260	80
2	0.082	0.365	46	0.232	0.365	46	0.182	0.365	46
3	0.384	0.229	102	0.534	0.229	102	0.484	0.229	102
4	0.163	0.282	66	0.313	0.282	66	0.263	0.282	66
5	0.691	?	92	0.691	?	92	0.621	?	92

missing value(s) and exploiting the information about the mean-variance relationship of the outcomes.

Table 1 presents three examples, A, B and C, of study results for meta-analysis, each with $H+1 = 5$ studies, on which we illustrate the methods we develop. In each example, all five studies have two treatment arms, with equal variances and equal sample sizes within the arms of each study; $\sigma_{i\mathrm{A}}^2 = \sigma_{i\mathrm{B}}^2 = \sigma_i^2$ and $n_{i\mathrm{A}} = n_{i\mathrm{B}} = \frac{1}{2}n_i$, $i = 1,\ldots,5$. The quintets of sample sizes n_i and the quartets of estimates of τ_i^2 are the same across the three cases, only the sets of estimates differ.

By back-calculating the within-treatment variance estimates we can check that the variances are very likely to differ; the estimates are in the range 8.4 – 11.7. Study 5, with $\hat{\tau}_5^2$ not available, has an unexceptional sample size. In each case A – C, we consider the plausible range $(0.17, 0.28)$ for $\hat{\tau}_5^2$. That is, we rule out the possibility that τ_5^2 may be smaller than 0.17 or larger than 0.28. This choice is informed by the sample size and the variances in the other studies. Some leeway at either limit of their range is allowed since the (unknown) variance may be larger or smaller than the four recorded variances. In practice, expert opinion may provide some additional input.

2. Information gained by using imputation

Suppose we have H studies with complete information and another study, $H+1$, with the value of $\hat{\tau}_{H+1}^2$ (or $\hat{\sigma}_{H+1}^2$) missing. The treatment effect common to the H studies is estimated by

$$\hat{\theta}_{-} = \frac{w_1\,\hat{\theta}_1 + \cdots + w_H\,\hat{\theta}_H}{W_H},$$

where $w_i = 1/\hat{\tau}_i^2$ and $W_H = w_1 + \cdots + w_H$. Ignoring the uncertainty about the weights w_i, that is, about the variances τ_i^2, leads to the expression $\mathrm{var}(\hat{\theta}_{-}) = 1/W_H$. This confirms that *information*, defined as the reciprocal of the sampling variance, is additive. Specifically, the information about θ contained in study i is w_i, in the collection of H studies it is W_H and, if τ_{H+1}^2 were available, it would be $W_H + w_{H+1}$ in the $H+1$ studies.

If we had complete information about study $H+1$, we would evaluate the version of the estimator $\hat{\theta}_-$ for $H+1$ studies, that is,

$$\hat{\theta}_+ = \frac{W_H\,\hat{\theta}_- + w_{H+1}\,\hat{\theta}_{H+1}}{W_H + w_{H+1}}\,.$$

If w_{H+1} were known the variance of the estimator of θ would be reduced by

$$\frac{1}{W_H} - \frac{1}{W_H + w_{H+1}} = \frac{w_{H+1}}{W_H\,(W_H + w_{H+1})}\,,$$

or by $100 w_{H+1}/(W_H + w_{H+1})\%$. As w_{H+1} is not known, the potential for reduction is smaller. When w_{H+1} is not known, but a plausible range for it is defined, then we can find the plausible range of this percentage. In the cases in Table 1, this range is $(19.8, 28.9)\%$. The plausible reduction of the standard error is in the range $(9.5, 13.5)\%$. Thus, a lot is at stake; the sampling variance could be reduced by as much as 29%, but the uncertainty about the magnitude of this stake, about 9%, is not trivial either.

3. Empirical Bayes model for σ_i^2

Imputation for a variance estimate is based on an estimate of the distribution underlying the variances of the studies. We assume that this distribution is inverse gamma, and estimate its parameters. First we derive the marginal distribution of the estimator $\hat{\sigma}_i^2$ of the within-treatment group variance σ_i^2 in study $i = 1, \ldots, H$.

We assume that, conditionally on the variance σ_i^2, $k_i\,\hat{\sigma}_i^2/\sigma_i^2$ has χ^2 distribution with k_i degrees of freedom. Thus, the conditional density of $\hat{\sigma}_i^2$, given its estimand σ_i^2, is

$$f(x) = \frac{1}{\Gamma\left(\frac{1}{2}\,k_i\right)} \left(\frac{k_i}{2\sigma_i^2}\right)^{\frac{1}{2}k_i} x^{\frac{1}{2}k_i - 1} \exp\left(-\frac{k_i x}{2\sigma_i^2}\right)\,,$$

where Γ is the gamma function. Further, we assume that the variances σ_i^2 are a random sample from the inverse gamma distribution with parameters α and γ:

$$g(y) = \frac{1}{\Gamma(\gamma)}\,\alpha^{\gamma} \left(\frac{1}{y}\right)^{\gamma+1} \exp\left(-\frac{\alpha}{y}\right)\,.$$

The marginal density of $\hat{\sigma}_i^2$ is obtained by integration of the joint density of $\hat{\sigma}_i^2$ and σ_i^2:

$$\begin{aligned} & C x^{\frac{1}{2}k_i - 1} \int_0^{+\infty} \left(\frac{1}{y}\right)^{\frac{1}{2}k_i + \gamma + 1} \exp\left\{-\frac{1}{y}\left(\alpha + \frac{1}{2}\,k_i x\right)\right\} \mathrm{d}y \\ = \;& C\Gamma\left(\frac{k_i}{2} + \gamma\right) \frac{x^{\frac{1}{2}k_i - 1}}{\left(\alpha + \frac{1}{2}\,k_i x\right)^{\frac{1}{2}k_i + \gamma}}\,, \end{aligned}$$

where C is the standardising constant, for which the expression is a density. We approximate the concluding expression by an inverse gamma density using the relation $(1 + c/k)^k \doteq \mathrm{e}^c$, precise for sufficiently large k. For the denominator, we have

$$\begin{aligned}\left(\alpha+\frac{k_i x}{2}\right)^{\frac{1}{2}k_i+\gamma} &= \left(\frac{k_i x}{2}\right)^{\frac{1}{2}k_i+\gamma}\left(1+\frac{2\alpha}{k_i x}\right)^{\frac{1}{2}k_i+\gamma} \\ &\doteq \left(\frac{k_i x}{2}\right)^{\frac{1}{2}k_i+\gamma}\exp\left\{\frac{2\alpha}{k_i x}\left(\frac{1}{2}k_i+\gamma\right)\right\}.\end{aligned}$$

Hence the approximation to the marginal density of $\hat{\sigma}_i^2$ by an inverse gamma density,

$$\frac{1}{\Gamma(\gamma)}\left\{\frac{\alpha}{k_i}\left(k_i+2\gamma\right)\right\}^{\gamma}\left(\frac{1}{x}\right)^{\gamma+1}\exp\left\{-\frac{\alpha}{k_i x}\left(k_i+2\gamma\right)\right\},$$

where the first two factors standardise the expression to be a density. The expectation of this distribution is $\mu=\alpha(1+2\gamma/k_i)/(\gamma-1)$, assuming that $\gamma>1$, and its variance is $\mu^2/(\gamma-2)$, assuming that $\gamma>2$. The parameters α and γ of this density are estimated by maximising the loglikelihood

$$l=-H\log\{\Gamma(\gamma)\}+H\gamma\log(\alpha)+\gamma\sum_{i=1}^{H}\log\left(\frac{k_i+2\gamma}{k_i}\right)-(\gamma+1)\sum_{i=1}^{H}\log\left(\hat{\sigma}_i^2\right)-\alpha\sum_{i=1}^{H}\frac{k_i+2\gamma}{k_i\,\hat{\sigma}_i^2}.$$

We apply the Newton-Raphson algorithm. The score functions for l are

$$\begin{aligned}\frac{\partial l}{\partial\alpha} &= \frac{H\gamma}{\alpha}-\sum_{i=1}^{H}\frac{k_i+2\gamma}{k_i\,\hat{\sigma}_i^2} \\ \frac{\partial l}{\partial\gamma} &= -H\Gamma'(\gamma)+H\log(\alpha)+\sum_{i=1}^{H}\log\left(\frac{k_i+2\gamma}{k_i}\right)+2\gamma\sum_{i=1}^{H}\frac{1}{k_i+2\gamma}-\sum_{i=1}^{H}\log\left(\hat{\sigma}_i^2\right) \\ &\quad -2\alpha\sum_{i=1}^{H}\frac{1}{k_i\,\hat{\sigma}_i^2},\end{aligned}$$

where Γ' is the digamma function, the derivative of $\log(\Gamma)$. The elements of the Hessian matrix are

$$\begin{aligned}-\frac{\partial^2 l}{\partial\alpha^2} &= \frac{H\gamma}{\alpha^2} \\ -\frac{\partial^2 l}{\partial\alpha\,\partial\gamma} &= -\frac{H}{\alpha}+2\sum_{i=1}^{H}\frac{1}{k_i\,\hat{\sigma}_i^2} \\ -\frac{\partial^2 l}{\partial\gamma^2} &= H\Gamma''(\gamma)-4\sum_{i=1}^{H}\frac{1}{k_i+2\gamma}+4\gamma\sum_{i=1}^{H}\frac{1}{\left(k_i+2\gamma\right)^2},\end{aligned}$$

where Γ'' denotes the trigamma function, the derivative of the digamma function. The Newton-Raphson algorithm converges very fast, as judged by any reasonable criterion for convergence. An initial solution has to be provided; this is difficult to automate because the loglikelihood is not concave throughout the parameter space.

The expression for l implies that the sufficient statistics for α and γ are the average (or total) of $\log(\hat{\sigma}_i^2)$ and, assuming that $\gamma \ll \frac{1}{2}k_i$ for all i, the average (or total) of $1/\hat{\sigma}_i^2$. The 'weights' k_i are therefore not as important for summarising the variances σ_i^2 as they are for the treatment effect θ. We confirm this on an example in Section 5.

We derive a non-iterative estimator of (α, γ) that can be used as an alternative, or as an initial solution for the Newton-Raphson algorithm. Given $\hat{\gamma}$, $\partial l/\partial \alpha$ implies a simple expression for $\hat{\alpha}$;

$$\hat{\alpha} = \frac{H\hat{\gamma}}{\sum_{i=1}^{H} \frac{k_i + 2\hat{\gamma}}{k_i \hat{\sigma}_i^2}}, \tag{1}$$

which is well approximated by $H\hat{\gamma}/(1/\hat{\sigma}_1^2 + \cdots + 1/\hat{\sigma}_H^2)$ when $\hat{\gamma} \ll k_i$ for all i.

The posterior distribution of σ_i^2 is inverse gamma with expectation $\mathrm{E}(\sigma_i^2 \,|\, \hat{\sigma}_i^2) = c$ and variance $\mathrm{var}(\sigma_i^2 \,|\, \hat{\sigma}_i^2) = c^2/(\gamma - 2)$, where $c = \alpha(k + 2\gamma)/\{k(\gamma - 1)\}$. Denote these moments by E and V, respectively. The ratio E^2/V is equal to $\gamma - 2$ for all k_i. This motivates the moment-matching estimator $\hat{\gamma} = 2 + \hat{E}^2/\hat{V}$, based on the naïve estimators of E and V. For α we do not have a moment-matching estimator, but we can use the estimator given by equation (1), without the assumption that $\gamma \ll k_i$. Problems with maximum likelihood are sometimes encountered with small-scale data or large values of $\log\{\Gamma(\hat{\gamma})\}$. We have not come across any, but this non-iterative method can be regarded as a back-up for such an eventuality.

3.1. Imputation

With maximum likelihood estimators $\hat{\alpha}$ and $\hat{\gamma}$, we have several options for imputation for an unknown variance. The simplest is to impute the naïve estimator of the expectation of the fitted distribution, $\hat{c} = \hat{\alpha}(k + 2\hat{\gamma})/\{k(\hat{\gamma} - 1)\}$. This quantity depends on the degrees of freedom k, although only weakly when $k \gg 2\gamma$, when $(k + 2\hat{\gamma})/k \doteq 1$. Next, we could use for imputation values generated by a draw from the fitted (inverse gamma) distribution. And finally, the uncertainty about α and γ could be reflected by drawing first a plausible pair $(\tilde{\alpha}, \tilde{\gamma})$ from the fitted distribution for (α, γ) and then drawing a value $\tilde{\sigma}_i^2$ from the plausible distribution given by $(\tilde{\alpha}, \tilde{\gamma})$. Some approximation cannot be avoided in this process because the joint distribution of $(\hat{\alpha}, \hat{\gamma})$ is known only asymptotically and is estimated by using estimates for the unknown parameters. Bayesian counterparts of these procedures can be implemented; $(\tilde{\alpha}, \tilde{\gamma})$ is drawn from the joint posterior distribution of (α, γ). They also entail some approximation; the paucity of information about α and γ is unavoidable, especially if we have no means of faithfully representing the prior information about them and, indeed, when our prior information is scant. Care has to be exercised also in the choice of a flat prior to represent the absence of any such information.

The maximum likelihood estimators of α and γ, based on studies $1-4$, have very large sampling variances and the two estimators are highly correlated. When maximum likelihood (or any other method) is fitted to a small number of studies the process of using plausible values entails a lot variation.

3.2. Meta-analysis with random effects

We have assumed that the studies have a common expectation θ. It may be more appropriate to assume that the study-specific treatment effects θ_i are a random sample from a distribution with expectation θ and variance $\omega \geq 0$. If this variance were known, the optimal estimator of θ based on the first H studies would be

$$\hat{\theta} = \frac{1}{W_H(\omega)} \sum_{i=1}^{H} w_H(\omega)\, \hat{\theta}_i \,,$$

where $w_i(\omega) = 1/(\omega + \hat{\tau}_i^2)$ and $W_H(\omega) = w_1(\omega) + \cdots + w_H(\omega)$; see DerSimonian and Laird (1986). A profound difficulty in using or adapting this estimator is that ω is not known and, when H is small, is estimated with very low precision. Even if all H studies were very large, so that there would be very little uncertainty about each θ_i, $i = 1, \ldots, H$, ω could be estimated with only $H-1$ degrees of freedom. When the studies have moderate sample sizes and there is appreciable uncertainty about each $\hat{\theta}_i$, the uncertainty about ω is even greater. This is difficult to reflect in the estimation of $\mathrm{var}(\hat{\theta})$, but it is obvious that the conventional estimator $\widehat{\mathrm{var}}(\hat{\theta}) = 1/W_H(\hat{\omega})$ has a negative bias. In fact, even with the assumption of a common treatment effect, $\omega = 0$, the estimator $\widehat{\mathrm{var}}(\hat{\theta}) = 1/W_H$ has a (small) negative bias because the uncertainty about the study weights w_i is ignored. However, this bias is in practice negligible.

The effect of the study-level variance ω on the weights $w_i(\omega)$ is to reduce their dispersion and shrink their relative weights w_i/W_H toward the common value $1/H$. Therefore the effect of uncertainty about the missing value of a sampling variance diminishes with increasing ω. So, the case of $\omega = 0$, explored in the rest of the article represents an extreme case, albeit without taking the uncertainty about ω into account.

3.3. Examples

The fit of the model for the variances σ_i^2, $i = 1, \ldots, 4$, yields the estimates $\hat{\alpha} = 141.23$ and $\hat{\gamma} = 18.60$, with estimated sampling variance matrix

$$\left(\begin{array}{rr} 3870.33 & 611.86 \\ 611.86 & 103.93 \end{array} \right) .$$

The estimated correlation of the two estimators is 0.965. The empirical Bayes estimate of the expected value of $\hat{\sigma}_5^2$ is $141.23/17.60 \times (1 + 2 \times 18.60/92) = 11.27$. The corresponding estimate of $\hat{\tau}_5^2$ is $11.27 \times 2/92 = 0.245$. By substituting this value for $\hat{\tau}_5^2$ we obtain the estimates $\hat{\theta}_+ = 0.382$, 0.499 and 0.445 in the respective cases A, B and C, each with estimated standard error 0.232. The latter is an underestimate in all three cases because we have pretended $\hat{\tau}_5^2$ to be known.

The uncertainty about $\hat{\tau}_5^2$ is partly reflected by averaging the plausible estimates $\tilde{\theta}_+$ obtained by substituting for $\hat{\tau}_5^2$ random draws from its fitted sampling (or posterior) distribution. The estimate of θ is obtained as the average of the plausible estimates. The sampling variance has two components: average of the plausible sampling variances and variance of the plausible estimates of θ. The latter component should be multiplied by $(1+1/m)$; when we choose a large m, this factor makes next to no difference.

We applied $m = 1000$ replications; we can be profligate with the choice of m because the calculations that follow are simple. The averages in the three cases are 0.383, 0.500 and 0.445, and the standard errors are estimated by 0.232 in all three cases. Thus, the results are altered only slightly by using plausible values $\tilde{\tau}_5^2$. In fact, the estimates of the standard errors are greater by less than 0.0002 compared to when $\hat{\tau}_5^2$ is used.

The two kinds of imputation we applied are *improper* in the terminology of Rubin (2004) because they fail to reflect the uncertainty about the missing value(s) in its entirety. Specifically, we have pretended that the parameters α and γ were known and were equal to their estimates. We make amends for this by drawing a random sample of plausible pairs $(\tilde{\alpha}, \tilde{\gamma})$, and then drawing a plausible value $\tilde{\sigma}_{H+1}^2$ from each (plausible) distribution based on the realised pair $(\tilde{\alpha}, \tilde{\gamma})$. In this procedure, we assume that the sampling distribution of $(\hat{\alpha}, \hat{\gamma})$ is bivariate normal, with the sampling variance derived from the fitted information matrix. Relying on asymptotic normality with such a small sample size $H = 4$ is clearly problematic, and some error is committed. However, this is bound to be not as large as if we pretended this variance matrix to vanish.

With this method of multiple imputation, we obtain the estimates 0.378, 0.497 and 0.443 for the respective cases A, B and C, with standard error 0.234 in each case. They do not differ materially from the results obtained by simpler improper methods of imputation. The estimate of the standard error is inflated by only 0.0025.

In generating replicates of $(\tilde{\alpha}, \tilde{\gamma})$, we rejected 32 pairs because they contained at least one negative value. The values of the plausible weight $\tilde{w}_5$ ranged from 0.01 to 15.6; their mean was 4.07 and standard deviation 1.31. The substantial uncertainty about the parameters α and γ translates to substantial uncertainty about τ_5^2 or the weight w_5, but this does not contribute substantially to the uncertainty about θ.

4. Plausible values of $\hat{\theta}$

An approach that involves relatively weak assumptions about the incompletely reported study $H+1$ is based on a plausible range of values of τ_{H+1}^2. A plausible range, an interval $(\tau_{H+1,\mathrm{L}}^2, \tau_{H+1,\mathrm{U}}^2)$, is defined by the condition that all values of τ_{H+1}^2 outside this interval can be ruled out. An interval that subsumes a plausible range is also a plausible range, but a subinterval of a plausible range may not be. In ideal circumstances, the plausible range would be elicited from subject matter experts, such as clinical personnel involved in the meta-analysis or one of its studies. We assume that a plausible range for τ_{H+1}^2 has been specified. A value is said to be plausible if it is contained in the plausible range.

We evaluate the estimator $\hat{\theta}$ conditionally on τ_{H+1}^2 being equal to each value on a fine grid that covers the plausible range. These values can be regarded as plausible for $\hat{\theta}$, and the range they cover, $(\hat{\theta}_\mathrm{L}, \hat{\theta}_\mathrm{U})$, as the plausible range for $\hat{\theta}_+$. The plausible values of $\mathrm{var}(\hat{\theta}_+) = 1/(W_H + w_{H+1})$ can be established similarly. If these two plausible ranges are narrow, then we can conclude the analysis with these two intervals, admitting the uncertainty about $\hat{\theta}_+$ additional to its sampling variation, as well as the uncertainty about the standard error.

The results of the meta-analyses for the three sets of studies introduced in Table 1 are presented in the panels in one row of Figure 1. The first panel at the top (Ae, for case A) presents the plausible value of $\hat{\theta}_+$ as a function of the plausible value of $\hat{\tau}_5^2$ (solid line). The estimate $\hat{\theta}_-$ is indicated by horizontal dashes. By including study 5 in the meta-analysis, the estimate of $\hat{\theta}$ is increased by about 0.10, from $\hat{\theta} = 0.295$ to between $0.373 - 0.409$. The plausible standard errors, plotted in panel As are in the range $0.222 - 0.235$, reduced from the standard error of $\hat{\theta}_-$ equal to 0.263. The t ratio, plotted in panel At, is increased from 1.12 (based on $\hat{\theta}_-$) to between 1.59 and 1.85. So, we conclude with no evidence of a treatment effect, despite an appreciable increase in the estimate of θ and reduction in its standard error.

Panels in the middle row, based on case B, present an example in which study 5 alters the verdict of significance unequivocally, for any plausible value of $\hat{\tau}_5^2$. Panels at the bottom (case C) display an example of *impasse*. As a function of the estimate $\hat{\tau}_5^2$, the t ratio (panel Ct) intersects the horizontal line drawn at 1.96. There would be sufficient evidence of a treatment effect for some plausible values of $\hat{\tau}_5^2$, namely in the range (0.170, 0.217), but 'not significant' would be the verdict for $0.217 < \hat{\tau}_5^2 < 0.240$.

4.1. Plausible verdicts of hypothesis testing

If establishing significance is the sole objective of the analysis, then we can conclude the analysis with an unequivocal statement when the test of the relevant hypothesis yields the same verdict for every plausible value of τ_{H+1}^2. This approach can be reduced to evaluating the t ratio at the limits $\tau_{H+1,\mathrm{L}}^2$ and $\tau_{H+1,\mathrm{U}}^2$ and at most one other point. We assume that the t test is used throughout, and that its assumptions are satisfied.

Let $\tilde{\tau}^2$ be a plausible value of τ_{H+1}^2 and $\tilde{\theta}$ and $\tilde{w}_{H+1}$ the corresponding values of $\hat{\theta}_+$ and w_{H+1}. In Appendix A we show that, except when $\hat{\theta}_{H+1} = 0$, the t ratio, $\tilde{\theta}/\sqrt{W_H + \tilde{w}_{H+1}}$, is either a unimodal or a monotone function of $\tilde{\tau}^2$; its extension to the entire real axis has a single extreme, $w_{H+1}^* = W_H\,(\hat{\theta}_-/\hat{\theta}_{H+1} - 2)$, and is monotone in the two intervals separated by w_{H+1}^*. If w_{H+1}^* lies outside $(w_{H+1,\mathrm{L}}, w_{H+1,\mathrm{U}})$, then the t ratio is a monotone function of $\tilde{w}_{H+1} = 1/\tilde{\tau}^2$ in this range, and so it suffices to evaluate it at the limits $w_{H+1,\mathrm{L}}$ and $w_{H+1,\mathrm{U}}$; these values of t delimit the plausible range of the t ratio. When w_{H+1}^* is contained in $(w_{H+1,\mathrm{L}}, w_{H+1,\mathrm{U}})$, the plausible values of the t ratio have the same sign throughout, and so their range is delimited by the t ratio evaluated at w_{H+1}^* and at either $w_{H+1,\mathrm{L}}$ or $w_{H+1,\mathrm{U}}$. The plausible range of p values is obtained straightforwardly, as the p value is a decreasing function of $|t|$.

An interesting case arises when $\hat{\theta}_-$ and $\hat{\theta}_{H+1}$ have opposite signs and $W_H\,\hat{\theta}_-/\hat{\theta}_{H+1}$ is a plausible value of w_{H+1}. Zero is now a plausible value of the t ratio, so the ratio may be both positive and negative. But significance of the t ratio would be plausible only in some esoteric settings with extremely wide plausible ranges of τ_{H+1}^2.

Apart from adopting the t statistic for the hypothesis $\theta = 0$, the only assumption we have made is about the plausible range for w_{H+1}. For its specification we have to rely on expert opinion formed by information from other studies and the nature of the variation of the outcome variable in the relevant population. Eliciting such opinion is far from trivial. Experts may be ill-at-ease and reticent to cooperate, being concerned that the integrity and veracity

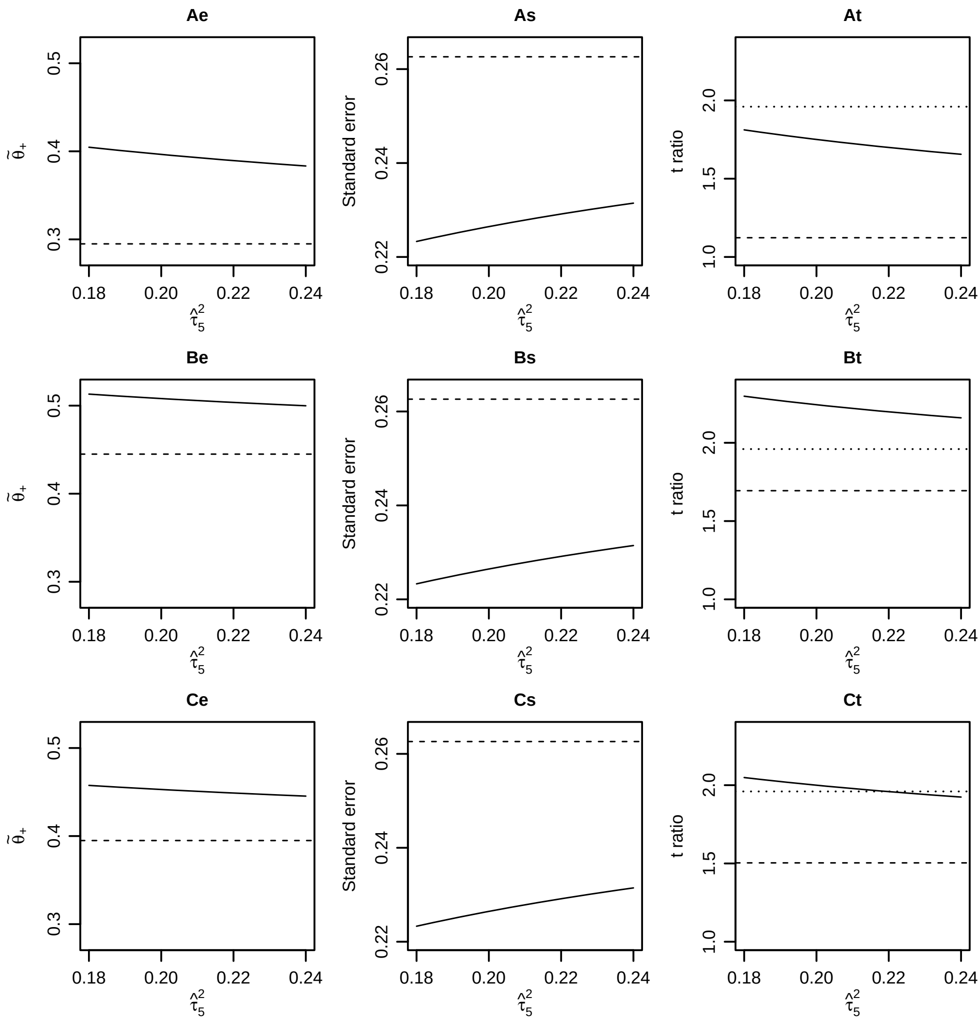

Figure 1: Plausible estimates (e), standard errors (s) and t ratios (t) in the meta-analyses A, B and C (Table 1). Horizontal dashes mark the relevant statistics for the four studies with complete information and the horizontal dots are drawn at the critical value of the t ratio, 1.96. By construction (identical sets of variance estimates $\hat{\tau}_i^2$ in Table 1), the panels As, Bs and Cs are identical.

of their statements may be undermined in the future when new information emerges.

An alternative to this approach involves finding the values of τ_{H+1}^2 for which the p value of 0.05, or another a priori selected value, is attained. The behaviour of the t ratio as a function of $\tau_{H+1}^2 = 1/w_{H+1}$ implies that there are at most two such values. These borderline points are easy to find by the Newton method or another line search algorithm. Within each interval delimited by a pair of these borderline values, the p value is either entirely greater or smaller than the reference value, say 0.05. We then ask the experts whether every interval in which the p values are greater than 0.05 is entirely implausible. If the answer is affirmative, then we conclude with the verdict of significance because it would have been attained for *any* plausible value of τ^2_{H+1}.

A drawback of both approaches is the possibility of an impasse, which arises when significance would have been achieved for some but not all plausible values of τ^2_{H+1}. In such a case, we have to admit that both significance and its negation are plausible outcomes of the analysis. In general, it is preferable to specify as narrow a plausible range for τ^2_{H+1} as possible, to reduce the chances of an impasse. However, it is an imperative that any value of τ^2_{H+1} outside the declared range can be ruled out; otherwise the integrity of the method is breached. In the second variant of this approach, it is important to discourage a hasty or perfunctory dismissal of the plausibility of the intervals in which the p value is greater than the reference (0.05).

4.2. Accounting for the uncertainty about τ^2_{H+1}

The uncertainty that can be attributed to the unknown w_{H+1} is assessed by the conditional distribution of $\hat{\theta}_+$ given $\hat{\theta}_-$ and W_H. The Taylor expansion for $\hat{\theta}_+$ around $\hat{\theta}_-$ yields the approximation

$$\operatorname{var}\left(\hat{\theta}_+ \,|\, \hat{\theta}_-, \hat{\theta}_{H+1}\right) \doteq \left(\hat{\theta}_{H+1} - \hat{\theta}_-\right)^2 \mathrm{E}\left\{(1+r_{H+1})^{-4}\right\} \operatorname{var}(\hat{r}_{H+1}), \tag{2}$$

where $r_i = w_i/W_H$, $i = 1,\ldots,H+1$, is the relative weight. This identity is derived in Appendix B. It implies three factors that have an impact on the uncertainty about $\hat{\theta}_+$: the deviation of $\hat{\theta}_{H+1}$ from $\hat{\theta}_-$, the relative magnitude of w_{H+1} with respect to W_H, and the variance of this ratio r_{H+1}. The first factor does not involve r_{H+1}, and can be evaluated from the available data directly. It vanishes when $\hat{\theta}_{H+1} = \hat{\theta}_-$, and then $\hat{\theta}_+ = \hat{\theta}_-$ for any value of τ^2_{H+1}. The second term has an upper bound of 1.0. If study $H+1$ is large, then w_{H+1} is also large, and then this factor is small. Further, when we have a lot of information about θ, and so W_H is large, then $\operatorname{var}(\hat{r}_{H+1})$ is small. These considerations, however loose and involving approximation, conform with intuition. A study $H+1$ omitted from meta-analysis introduces greater uncertainty about θ when $\hat{\theta}_{H+1}$ is exceptional among the estimates $\hat{\theta}_1, \ldots, \hat{\theta}_H$, when the study contains a lot of information about θ (w_{H+1} is small), and when we are uncertain about w_{H+1}.

The first factor is equal to 0.167, 0.061 and 0.051 for the respective cases A – C presented in Table 1. The other two factors have values common to the three cases. The plausible values of r_{H+1} are in the range 0.265 – 0.406, and for $1/(1+r_{H+1})^4$ they are in the range 0.256 – 0.390. We approximate $\operatorname{var}(\hat{r}_{H+1})$ conservatively by the variance of the uniform distribution on $(0.265, 0.406)$, that is, $0.141^2/12 = 0.00166$, and the expectation of $1/(1+r_{H+1})^4$ by its largest plausible value, $1/1.265^4 = 0.390$. Thus a conservative estimate of the variance in (2) is $0.167 \times 0.00166 \times 0.391 = 1.08 \cdot 10^{-4}$, that is, standard deviation of about 0.0104 in case A, 0.0063 in case B, and 0.0057 in case C. This is a small contribution to the overall uncertainty attributable to the variation of the outcome variable in the studied population(s), as quantified by $1/\sqrt{W_H} = 0.263$.

Table 2: Estimates, standard errors and sample sizes ($n^{(\mathrm{P})}$ — placebo and $n^{(\mathrm{M})}$ — mirtazapine) for the studies in the systematic review of Mavridis *et al.* (2014).

Study	Estimate	St. error	$n^{(\mathrm{P})}$	$n^{(\mathrm{M})}$
S1	–3.1	2.91	18	25
S2	–2.5	2.20	23	26
S3	–1.8	3.02	16	11
S4	–6.8	2.30	23	22
S5	3.6	1.71	20	21
S6	–4.6	2.26	26	29
S7	–2.3	1.97	32	34
S8	–2.9	1.68	47	50

5. Example II

In this section we illustrate the methods on a systematic review conducted by Mavridis *et al.* (2014) for comparing mirtazapine, a drug for treating clinical depression, with placebo. The outcome variable is recorded on the HAMD21 scale constructed originally by Hamilton (1967) using a patient questionnaire. Larger values of HAMD21 correspond to more severe illness.

The systematic review found eight studies. Their results are presented in Table 2, listing the estimate of the treatment effect ($\hat{\theta}_i$), its (estimated) standard error, and the number of observations ($n_i^{(\mathrm{P})}$ for placebo and $n_i^{(\mathrm{M})}$ for mirtazapine) for each study S1 – S8. Study S5, the only one with $\hat{\theta}_i > 0$, is an obvious outlier. The standard errors are in the range 1.68–3.02, and the numbers of observations are in the range 27–97.

We pretend that one of the standard errors is missing and apply the methods that make use of the estimate and sample size for this study. The results are presented in Table 3. Row labelled –Si, $i = 1, \ldots, 8$, represents the setting with the standard error in study Si missing. The first two columns present the estimates of the parameters of the inverse gamma distribution on which imputation for this missing value is based. The next column presents the estimate of the treatment effect based on the seven retained studies ($\hat{\theta}_-$). The next two columns present the imputed standard error $\tilde{\tau}_i$ derived from the (empirical) posterior expectation of the variance σ_i^2 and the estimate of θ based on this standard error ($\hat{\theta}_+$). The right-most column displays the estimates and standard errors based on averaging over 100 random draws from the posterior distribution of σ_i^2.

The results are presented with three decimal places, so that the small differences of the estimates can be discerned. The target of estimation is the treatment effect based on the estimates of all the eight studies, $\hat{\theta} = -2.061$. With no data discarded, the standard error of $\hat{\theta}$ is estimated by 0.751. By imputing the posterior mean, all single-imputation estimates $\hat{\theta}_+$ are close to the target, except for the setting –S5. The estimated standard errors are also close to 0.751, except for –S5. All of them should exceed 0.751 because they are based on less information. The contradiction arises because the uncertainty about the imputed

Table 3: Estimates and estimated standard errors (se) of the treatment effect; metaanalysis of studies summarised in Table 2, with one standard error $\hat{\tau}_i$ deleted; θ_{-} — the study discarded altogether; θ_{+} — the study included, with the posterior expectation of σ_i imputed; $\tilde{\theta}_{+}$ — the study included, with multiple imputation for σ_i.

Row labelled –Sk, $k = 1, \ldots, 8$, indicates that $\hat{\tau}_k$ is regarded as missing.

	$\hat{\alpha}$	$\hat{\gamma}$	$\hat{\theta}_{-}$ (se)	$\tilde{\tau}_i$	$\hat{\theta}_{+}$ (se)	$\tilde{\theta}_{+}$ (se)
–S1	3.03	7.46	–1.987 (0.777)	2.36	–2.096 (0.748)	–2.098 (0.753)
–S2	2.89	6.63	–2.003 (0.799)	2.23	–2.060 (0.752)	–2.061 (0.758)
–S3	3.05	6.87	–2.079 (0.775)	3.26	–2.064 (0.754)	–2.063 (0.764)
–S4	2.89	6.62	–1.496 (0.795)	2.34	–2.043 (0.752)	–2.050 (0.767)
–S5	5.54	13.45	–3.414 (0.836)	2.59	–2.750 (0.795)	–2.753 (0.797)
–S6	2.94	7.00	–1.746 (0.796)	2.05	–2.120 (0.752)	–2.121 (0.753)
–S7	2.93	6.92	–2.021 (0.812)	1.85	–2.066 (0.745)	–2.064 (0.754)
–S8	3.09	7.57	–1.852 (0.840)	1.47	–2.108 (0.739)	–2.113 (0.748)

standard error is ignored. Multiple imputation corrects this bias but the estimates $\tilde{\theta}_{+}$ differ from their single-imputation counterparts $\hat{\theta}_{+}$ only slightly.

The estimates stand out for the setting –S5 because study S5 has a smaller standard error than its sample size suggests. When $\tilde{\tau}_5$ is imputed the contribution of S5 to estimating θ is underrated, and so its influence is reduced. In summary, imputation of the posterior mean of the variance is sufficient for estimating the treatment effect. Multiple imputation yields similar estimates and inflates the standard errors only slightly.

We illustrate sensitivity analysis by pretending that $\hat{\tau}_5$ is not recorded. Since $\hat{\theta}_5$ is an outlier among the estimates, it may be justified to discard the study altogether, especially if a careful review of the literature and of other sources discovers some reason for the exceptional result. Failure to report $\hat{\tau}_5$ might also raise suspicion about both the quality and context of the study. Instead of the dichotomy, to include or exclude the study from the meta-analysis, we define a plausible range of standard errors, $(\tilde{\tau}_{5\mathrm{L}}, \tilde{\tau}_{5\mathrm{U}})$. Exclusion corresponds to $\tilde{\tau}_{5\mathrm{L}} = +\infty$, implying that also $\tilde{\tau}_{5\mathrm{U}} = +\infty$. Exclusion being plausible corresponds to $\tilde{\tau}_{5\mathrm{L}} < +\infty$ and $\tilde{\tau}_{5\mathrm{U}} = +\infty$.

An analyst might impute for $\hat{\tau}_5$ the standard error from a study with a similar sample size, such as S1 or S4, and allowing some larger values to reflect the doubt about $\hat{\theta}_5$. Suppose the plausible range for $\tilde{\tau}_5$ is set to $(2,5)$. Figure 2 displays the plot of the plausible values of $\hat{\theta}_{+}$ as a function of $\tilde{\tau}_5$ (solid line) together with the plausible confidence intervals (shaded area). For $\tilde{\tau}_5 = 2.0$, $\hat{\theta}_{+}$ is close to the target $\hat{\theta}_{+} = -2.06$ (horizontal line of long dashes), so the error caused by the failure to allow for $\hat{\tau}_5 = 1.7 < \tilde{\tau}_{5\mathrm{L}}$ is not harsh.

For $\tau_5 = 5.0$, study S5 contributes with very small weight; $\hat{\theta}_{+}$ is close to $\hat{\theta}_{-}$, marked by the horizontal dashed line. The standard error increases with $\tilde{\tau}_5$, from 0.771 at $\tilde{\tau}_5 = 2.0$ to 0.824 at $\tilde{\tau}_5 = 5$ — the grey region narrows towards the right. The upper confidence limit for $\hat{\theta}_{+}$ decreases with $\tilde{\tau}_5$. It crosses zero at $\tilde{\tau}_5 = 1.32$. So, there is evidence of a negative effect of mirtazapine so long as $\hat{\tau}_5 > 1.32$. If $\hat{\tau}_5$ were very small, study S5 would dominate

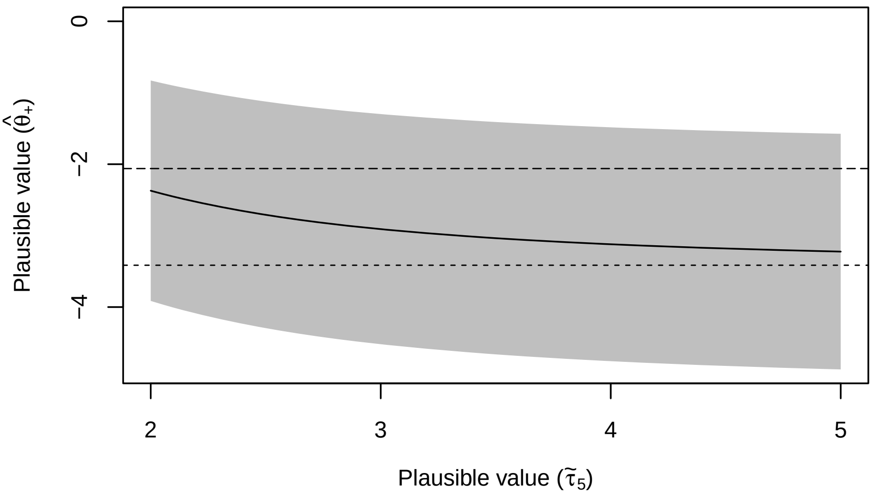

Figure 2: Sensitivity analysis. Plausible estimate of the treatment effect $\hat{\theta}$ as a function of the plausible value of the standard error τ_5 when $\hat{\tau}_5$ for study S5 is masked, regarded as not reported.

the meta-analysis and it would conclude with evidence of a positive (detrimental) effect of mirtazapine. This would happen for $\hat{\tau}_5 < 0.64$. Such an outcome would not be credible given that all but one study yielded a negative estimate.

6. Discussion

The empirical Bayes approach in Section 3 involves some assumptions that are contentious and their plausibility is difficult to assess. The approach motivated by sensitivity analysis in Section 4 carries a lighter burden of assumptions but has a heavier demand on input — it requires the declaration of a plausible range for the missing sampling variance. Also, it may conclude with an impasse, when one conclusion, e.g., of a hypothesis test, is obtained for some plausible values of this variance, and another conclusion for other values that are equally plausible.

The analysis in Section 2 shows that a study with standard error not reported can contribute to the estimation of the overall treatment effect. Accounting for the uncertainty about the imputed standard error makes much less difference. Similar conclusions can be drawn about using plausible values for the missing standard error(s).

The model applied in Section 3 can be expanded to a regression model, and thus strengthen the inference about a missing variance by exploiting the association of the variance and mean implied by the distribution of the outcomes or other auxiliary information. This approach is not always useful. For example, when the outcomes are binary and the events are neither rare nor very frequent the variance is a very flat function of the probability. In any case, the small number of studies precludes any complex modelling and any reliable inference about the mean-variance relationship; prior information may be more useful.

The examples in Sections 4 and 5 confirm that even simple methods, using some short-cuts on proper imputation, exploit nearly to the full the information about an incompletely reported study and they estimate the standard error of the overall treatment effect with negligible bias. Sensitivity analysis using plausible ranges for the missing variance has some potential but this is undermined by the general reluctance to participate in elicitation of these ranges.

The common or average treatment effect θ is ascribed importance, and motivates the attempt to recover information contained in an incompletely reported study $(H+1)$. The problem has some commonality with publication bias, a widely studied issue. The expectation θ can be interpreted as the treatment effect in a set of studies among which the realised studies are a random sample. This interpretation has a flaw in that the treatment effects of these studies, $\theta_1, \ldots, \theta_{H+1}$, would be a random sample from a meaningful distribution only if the populations (constituencies) and contexts of the studies were selected at random from a universe of potential studies, that is, according to a design. In practice, these aspects are selected haphazardly, influenced by the availability of expertise and funding and concern about the specific issue. Also, the contexts of the realised studies, especially those conducted in the more distant past and in countries with different levels of development and organisation of health care, may differ a great deal from the context for which the inferences drawn by a meta-analysis are intended. Such relevance is rarely incorporated in the weights used for estimating the overall (or average) treatment effect.

All the data used in this article are displayed in Tables 1 and 2.

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Appendix A. The t ratio for $\hat{\theta}$ as a function of w_{H+1}

In this appendix we explore the behaviour of the t ratio $\hat{\theta}_+/\sqrt{W_H + w_{H+1}}$ as a function of w_{H+1}. This function is

$$T(w) = \frac{W_H\,\hat{\theta}_- + w\hat{\theta}_{H+1}}{\sqrt{W_H + w}}\,.$$

Its derivative is

$$\begin{aligned}\frac{\partial T}{\partial w} &= \frac{1}{\sqrt{W_H + w}}\left(\hat{\theta}_{H+1} - \frac{1}{2}\,\frac{W_H\,\hat{\theta}_- + w\hat{\theta}_{H+1}}{W_H + w}\right)\\ &= \frac{1}{2\,(W_H + w)^{\frac{3}{2}}}\left\{W_H\left(2\hat{\theta}_{H+1} - \hat{\theta}_-\right) + w\hat{\theta}_{H+1}\right\}.\end{aligned}$$

The sign of this derivative does not depend on w when $\hat{\theta}_{H+1} = 0$. In that case, the derivative has the same sign as $-\hat{\theta}_-$. When $\hat{\theta}_{H+1} \neq 0$, the derivative has a single root at

$$w^*_{H+1} = W_H\left(\frac{\hat{\theta}_-}{\hat{\theta}_{H+1}} - 2\right),$$

where its sign switches from positive to negative or vice versa. Therefore T changes at w^*_{H+1} from decreasing to increasing or vice versa. Its value at w^*_{H+1} is

$$T\left(w^*_{H+1}\right) = \frac{2\,W_H}{\sqrt{W_H + w^*_{H+1}}}\left(\hat{\theta}_- - \hat{\theta}_{H+1}\right).$$

When $\hat{\theta}_{H+1} \neq 0$, the function T has a single root at $w^{(0)}_{H+1} = -W_H\,\hat{\theta}_-/\hat{\theta}_{H+1}$, which is positive when $\hat{\theta}_-$ and $\hat{\theta}_{H+1}$ have opposite signs. In that case, $w^*_{H+1} < 0$, and so T is monotone in the plausible range. In summary, T is either unimodal without changing its sign in the plausible range of w_{H+1}, or is monotone, in which case it may cross zero at one point.

Appendix B. Taylor expansion for $\hat{\theta}_+$

The first-order partial differential of $\hat{\theta}_+$ with respect to w_{H+1} is

$$\begin{aligned}\frac{\partial\hat{\theta}_+}{\partial w_{H+1}} &= \frac{(W_H + w_{H+1})\,\hat{\theta}_{H+1} - W_H\,\hat{\theta}_- - w_{H+1}\,\hat{\theta}_{H+1}}{(W_H + w_{H+1})^2}\\ &= \frac{W_H\left(\hat{\theta}_{H+1} - \hat{\theta}_-\right)}{(W_H + w_{H+1})^2}\\ &= \frac{\hat{\theta}_{H+1} - \hat{\theta}_-}{W_H}\,\frac{1}{\left(1 + \frac{w_{H+1}}{W_H}\right)^2},\end{aligned}$$

from which the expression for the conditional variance in equation (2) follows directly, evaluating $(\partial\hat{\theta}_+/\partial w_{H+1})^2\,\mathrm{var}(\hat{w}_{H+1})$ and substituting $r_{H+1} = w_{H+1}/W_H$.

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Approximately optimum strata boundaries for two concomitant stratification variables under proportional allocation

Faizan Danish[1], **S. E. H. Rizvi**[2]

ABSTRACT

The proper choice of strata boundaries is an important factor determining the efficiency of the estimator of the considered characteristics of a population. In this article, the Cum$\sqrt[3]{D_i(x,z)}$ Rule (i=3,4) for obtaining approximately optimum strata boundaries has been applied, taking into account a single-study variable along with two concomitant variables serving as the basis of the stratification variables. The relative efficiency of the proposed methods has been demonstrated theoretically and empirically by comparing them to a selection of already-existing methods in a simulation study with the use of the proportional allocation method.

Key words: stratification points, proportional allocation, minimal equation.

1. Introduction

Let there be a finite population consisting of N units, for which it is required to estimate the total or mean for the characteristic Y under study, using simple random sampling technique. In order to have this, we partition population L × M strata:

$$\sum_{h=1}^{L}\sum_{k=1}^{M} N_{hk} = N$$

where N_{hk} indicates the number of units in $(h\,,k)^{th}$ stratum.

Let 'n' be the number of units to be drawn from the whole population and suppose that the allocation of sample size n_{hk} such that

$$\sum_{h=1}^{L}\sum_{k=1}^{M} n_{hk} = n$$

[1] Corresponding author. Division of Biostatistics, Department of Population Health, School of Medicine, New York University, USA. E-mail: Faizan.Danish@nyulangone.org. ORCID: https://orcid.org/0000-0002-0476-9744.

[2] Division of Statistics and Computer Sciences, SKUAST-J, J&K, India. E-mail: sehrizvi_stats@yahoo.co.in.

Let y_{hki} (i = 1 , 2 , 3,…, N_{hk}) be the population unit and then the population total is

$$Y = \sum_{h=1}^{L} \sum_{k=1}^{M} \sum_{i=1}^{N_{hk}} y_{hki}$$

For the study variable, the unbiased estimate of $\overline{Y}$, $\overline{y}_{st} = \sum_{h=1}^{L} \sum_{k=1}^{M} W_{hk} \overline{y}_{hk}$,where $\overline{y}_{hk} = \frac{1}{n_{hk}} \sum_{i=1}^{n_{hk}} y_{hki}$ and W_{hk} denotes the weight of the (h,k)th stratum.

For stratified simple random sampling, the sample estimate $\overline{y}_{st}$ is unbiased with sampling variance as below:

$$V\left(\overline{y}_{st}\right) = \sum_{h} \sum_{k} \left(1 - f_{hk}\right) \frac{W_{hk}^2 \sigma_{hky}^2}{n_{hk}}$$

where $f_{hk} = \frac{n_{hk}}{N_{hk}}$ and if f.p.c is ignored, we have

$$V\left(\overline{y}_{st}\right) = \sum_{h} \sum_{k} \frac{W_{hk}^2 \sigma_{hky}^2}{n_{hk}}$$

σ_{hky}^2 represents the population variance for the character Y and is defined as

$$\sigma_{hky}^2 = \frac{1}{N_{hk}} \sum_{i=1}^{N_{hk}} \left(y_{hki} - \overline{y}_{hk}\right)^2$$

$\overline{y}_{hk}$ being the population mean of all the N_{hk} units in the $\left(h, k\right)^{th}$ stratum.

Construction of stratification points was pioneered by Dalenius (1950), while minimizing variance set of equations as the functions of population parameters were obtained and due to their implicit nature, it becomes complicated to obtain solutions. Cochran (1961,1963) has also discussed the cases regarding the optimum boundaries. Yadav and Singh (1984), Rizvi *et al.* (2000), Danish *et al.* (2020), Khan *et al.* (2008), Khan *et al.* (2014), Danish *et al.* (2017), Danish *et al.* (2018), Danish and Rizvi (2018,2019) and Danish, F. (2018). Rizvi and Danish (2018) made an attempt to summarise the proposed contribution towards obtaining stratification points.

The allocation procedure in which a sample size is selected as per proportion of the stratum is known as proportional allocation. In such allocation, the sample size is selected as

$$n_{hk} = \frac{nN_{hk}}{N} = nW_{hk}$$

thus, $n_{hk} \propto N_{hk}$ and $\sum_{h=1}^{L}\sum_{k=1}^{M} n_{hk} = n$

Hence, under such allocation, variance is

$$V\left(\bar{y}_{st}\right)_p = \frac{1}{n}\sum_h \sum_k W_{hk}\sigma_{hky}^2 \tag{1.1}$$

In this paper, for obtaining stratification points using classical approach for two concomitant variables as the basis of stratification variables and a single study variable under the proportional allocation method by assuming different distributions of the concomitant variables and both dependent and independent cases have been discussed as well.

2. Variance expression

Let the regression model of the response variable Y and the two information variables X & Z be given as

$$Y = C\ (X, Z) + e$$

where 'e' is error term such that

$E(e \mid x,z) = 0$ and $V(e|x,z) = \eta(x,z) > 0, \forall\ x\varepsilon(a,b)\ z\varepsilon(c,d)$,

$(\mathrm{b} - \mathrm{a}) < \infty, (\mathrm{c} - \mathrm{d}) < \infty$

If joint marginal of X and Z is f(x, z) and f(x) and f(z) denotes marginal densities of individual variables, respectively, then under above regression model,

we have $W_{hk} = \int_{x_{h-1}}^{x_h}\int_{z_{k-1}}^{z_k} f(x,z)\partial x\partial z$ is weight of a stratum.

$\mu_{hky} = \mu_{hkc} = \dfrac{1}{W_{hk}}\int_{x_{h-1}}^{x_h}\int_{z_{k-1}}^{z_k} c(x,z)\, f(x,z)\partial x\partial z$ and

$\sigma_{hky}^2 = \sigma_{hkc}^2 + \mu_{hk\eta}$ denotes its mean and variation respectively,

where $(x_{h-1}, x_h, z_{k-1}, z_k)$ be the stratification points and $\mu_{hk\eta}$ is the average value of the function $\eta(x,z)$ and σ_{hkc}^2 as

$$\sigma_{hkc}^2 = \frac{1}{W_{hk}}\int_{x_{h-1}}^{x_h}\int_{z_{k-1}}^{z_k} c^2(x,z)\, f(x,z)\partial x\partial z - (\mu_{hkc})^2$$

Using these relations, the variance can be expressed in terms of the population parameters of the function of X and Y and $V(e|x,z)$. The variance expression for the case of proportional allocation is therefore given by

$$V\left(\bar{y}_{st}\right)_{prop}=\frac{\left[\sum_h\sum_k W_{hk}\left(\sigma_{hkc}^2+\mu_{hk\eta}\right)\right]}{n} \tag{2.1}$$

The expression for various terms can be in terms of Singh and Sukhatme (1969) and Danish *et al.* (2018).

3. Minimal equations for proportional allocation

Since $\sum_h\sum_k W_{hk}\mu_{hk\eta}=\mu_\eta$, which is the population parameter and therefore is a fixed constant. Hence, minimization of (2.1) is equivalent to minimization

$$V_p=\sum_h\sum_k W_{hk}\sigma_{hkc}^2 \tag{3.1}$$

Thus, to obtain minimal equations, we minimize V_p by with respect of x_h and equate to zero, we get

$$\frac{\partial}{\partial x_h}V_p=\sum_k\left[W_{hk}\frac{\partial}{\partial x_h}\sigma_{hkc}^2+\sigma_{hkc}^2\frac{\partial}{\partial x_h}W_{hk}+W_{ik}\frac{\partial}{\partial x_h}\sigma_{ikc}^2+\sigma_{ikc}^2\frac{\partial}{\partial x_h}W_{ik}\right]=0$$

After further simplification, we get

$$\sum_k\left[W_{hk}\int_{z_{k-1}}^{z_k}\frac{f\left(x_h,z\right)}{W_{hk}}\left\{\left[c\left(x_h,z\right)-\mu_{hkc}\right]^2-\sigma_{hkc}^2\right\}\partial z\right]+\sigma_{hkc}^2\int_{z_{k-1}}^{z_k}f\left(x_h,z\right)\partial z$$
$$=\sum_k\left[W_{ik}\int_{z_{k-1}}^{z_k}\frac{f\left(x_h,z\right)}{W_{ik}}\left\{\left[c\left(x_h,z\right)-\mu_{ikc}\right]^2-\sigma_{ikc}^2\right\}\partial z\right]+\sigma_{ikc}^2\int_{z_{k-1}}^{z_k}f\left(x_h,z\right)\partial z \tag{3.2}$$

For obtaining minimal equations we also differentiate V_p partially w.r.t. z_k in a similar way, we get

$$\sum_h\left[W_{hk}\int_{x_{h-1}}^{x_h}\frac{f\left(x,z_k\right)}{W_{hk}}\left\{\left[c\left(x,z_k\right)-\mu_{hkc}\right]^2-\sigma_{hkc}^2\right\}\partial x\right]+\sigma_{hkc}^2\int_{x_{h-1}}^{x_h}f\left(x,z_k\right)\partial x$$
$$=\sum_h\left[W_{hj}\int_{x_{h-1}}^{x_h}\frac{f\left(x,z_k\right)}{W_{hj}}\left\{\left[c\left(x,z_k\right)-\mu_{hjc}\right]^2-\sigma_{hjc}^2\right\}\partial x\right]+\sigma_{hjc}^2\int_{x_{h-1}}^{x_h}f\left(x,z_k\right)\partial x \tag{3.3}$$

However, for obtaining minimal equations we minimize V_p on equating the partial derivative of this expression with respect of x_h and z_k to zero, we get

$$W_{hk} f(x_h, z_k) \frac{\left\{\left[c(x_h, z_k) - \mu_{hkc}\right]^2 - \sigma_{hkc}^2\right\}}{W_{hk}} + f(x_h, z_k)\sigma_{hkc}^2$$

$$= W_{ij} f(x_h, z_k) \frac{\left\{\left[c(x_h, z_k) - \mu_{ijc}\right]^2 - \sigma_{ijc}^2\right\}}{W_{ij}} + f(x_h, z_k)\sigma_{ijc}^2$$

This gives the equation as

$$c(x_h, z_k) = \frac{\left(\mu_{hkc} + \mu_{ijc}\right)^2}{2}, \qquad \begin{array}{l} i = h+1, h = 1, 2, \ldots, L-1 \\ j = k+1, k = 1, 2, \ldots, M-1 \end{array} \tag{3.4}$$

On the condition that $\lambda(x, z) = c'(x, z) f(x, z)$ belongs to class Ω functions, solutions to the system of equation (3.4) give OSB in the sense of minimization of variance $V\left(\overline{y}_{st}\right)_{prop}$. These equations are also very difficult to solve and, therefore, for these equations also we shall find methods of obtaining approximation to the exact solutions $\left[x_h, z_k\right]$. Further better approximation can be obtained by using some approximate iterative procedures.

4. Some miscellaneous results

In the case of complexities in the equations, let us impose few regularity conditions on $f(x, z), c(x, z)$ *and* $\eta(x, z)$.We state that $\zeta(x, z)$ belongs to class Ω if it satisfies

i) $0 < \zeta(x, z)$

ii) $\zeta(x, z) < \infty$

iii) $\zeta(x, z), \zeta'(x, z)$ *and* $\zeta''(x, z)$ exist and are continuous $\forall\,(x, z)$ in [(a, b), (c, d)] respectively such that $(b-a) < \infty$ *and* $(d-c) < \infty$.

Let us suppose $f(x, z)$ *and* $\eta(x, z)$ belong to class Ω and the function c(x, z) satisfies the conditions (ii) and (iii).

Before we proceed to prove the results, let us define the symbol 'O', which has been used in the present investigation.

For two functions $T_1(x,z)$ *and* $T_2(x,z)$, such that the ratio $T_1(x,z)/T_2(x,z)$ remains bounded as x and z tends to their limits, then we can write $T_1(x,z) = O(T_2(x,z))$.

Lemma 4.1: If the function $I_{ij}(x,z)$ is defined as

$$I_{ij}(x,z) = \int_{z_1}^{z_2}\int_{x_1}^{x_2} (t_1 - x_1)^i (t_2 - z_1)^j f(t_1,t_2)\partial t_1 \partial t_2 \text{ , } x_1 < x_2 \;\&\; z_1 < z_2$$

then

$$I_{ij}(x,z) = \left[\begin{array}{l} \dfrac{k_1^{i+1}k_2^{j+1}}{(i+1)(j+1)} f + \dfrac{k_1^{i+1}k_2^{j+1}}{(i+1)(j+1)} f_x + \dfrac{k_1^{i+1}k_2^{j+2}}{(i+1)(j+2)} f_z \\ + \dfrac{1}{2!}\left[\dfrac{k_1^{i+3}k_2^{j+1}}{(i+3)(j+1)} f_{xx} + 2\dfrac{k_1^{i+2}k_2^{j+2}}{(i+2)(j+2)} f_{xz} + \dfrac{k_1^{i+1}k_2^{j+3}}{(i+1)(j+3)} f_{zz}\right] + O\left(k^{i+j+5}\right) \end{array}\right] \tag{4.1}$$

where $f(t_1,t_2) = f, \dfrac{\partial f}{\partial t_1} = f_x, \dfrac{\partial f}{\partial t_2} = f_z, \dfrac{\partial^2 f}{\partial t_1^2} = f_{xx}, \dfrac{\partial^2 f}{\partial t_2^2} = f_{zz}, \dfrac{\partial^2 f}{\partial t_1 \partial t_2} = f_{xz}$,

$k_1 = x_2 - x_1$ and $k_2 = z_2 - z_1$.

Lemma 4.2: Let $\mu_\eta(x,z)$ denote the conditional expectation of the function $\eta(t_1,t_2)$, so that

$$\mu_\eta(x,z) = \frac{\int_{z_1}^{z_2}\int_{x_1}^{x_2} \eta(t_1,t_2)\; f(t_1,t_2)\; \partial t_1 \partial t_2}{\int_{z_1}^{z_2}\int_{x_1}^{x_2} f(t_1,t_2)\quad \partial t_1 \partial t_2}$$

Then, the series expansion of $\mu_\eta(x,z)$ at point (t_1 , t_2) is given by

$$\mu_\eta(x,z) = \eta\left[\begin{array}{l} 1 + \dfrac{\eta'}{2\eta}(k_1+k_2) + \left(\dfrac{\eta'(f_x+f_z) + 2f\eta''}{12 f\eta}\right)(k_1+k_2)^2 \\ + \left(\dfrac{f(f_{xx}+f_{zz}+f_{xz})\eta' + f(f_x+f_z)\eta'' + f^2\eta''' - \eta'(f_x+f_z)^2}{12 f^2 \eta}\right)(k_1+k_2)^3 \\ + O\left((k_1+k_2)^4\right) \end{array}\right] \tag{4.2}$$

Lemma 4.3: If $\sigma_\eta^2(x,z)$ denotes the conditional variance of the function $\eta(t_1,t_2)$ defined in the interval (x, z) , then

$$\sigma_\eta^2(x,z)=\frac{(k)^2}{12}\left(\eta^{'}\right)^2\left[1+\frac{\eta^{''}}{\eta^{'}}(k)^1+O(k)^2\right] \tag{4.3}$$

where $(k)^1$ *and* $(k)^2$ denote all k_i's with power '1' and '2' respectively.

Lemma 4.4:

$$(k_1k_2)^{\lambda-1}\int_{z_1}^{z_2}\int_{x_1}^{x_2} f(t_1,t_2)\partial t_1\partial t_2 =\left[\int_{z_1}^{z_2}\int_{x_1}^{x_2}\sqrt[\lambda]{f(t_1,t_2)}\partial t_1\partial t_2\right]^{\lambda}\left[1+O(k^2)\right] \tag{4.4}$$

Lemma 4.5: With $I_{00}(x,z)$ and $\sigma_\eta^2(x,z)$ defined as in Lemma 4.1 and Lemma 4.3 respectively, we have

$$I_{00}(x,z)\sigma_\eta^2(x,z)=\frac{k^2}{12}\int_{z_1}^{z_2}\int_{x_1}^{x_2}\eta'^2(t_1,t_2)\quad f(t_1,t_2)\partial t_1\partial t_2 \tag{4.5}$$

where k denotes any k_h or k_k.

5. Minimal equations and their approximate solutions

In this section we will obtain expansion of the series given in (3.4) about the points x_h and z_k the common boundary of $(h,k)^{th}$ *and* $(h+1,k+1)^{th}$ strata and obtain the approximate systems of equation, which will give approximately optimum points of stratification as their solutions. In doing so we shall make use of the Lemma's already 4.1-4.5.

The minimal equations for this method are given by

$$c(x_h,z_k)-\mu_{hkc}=\mu_{ijc}-c(x_h,z_k)$$

i= h +1,h = 1, 2, … L , j= k+1, k= 1,2,…,M

For R.H.S., the corresponding expansion for the L.H.S. may be obtained by changing the signs of coefficients of even powers of $\left(k_i, k_j\right)$ the width of $(i,j)^{th}$ stratum, where $k_i = x_{h+1} - x_h$, $k_j = z_{k+1} - z_k$. From (4.2), after replacing $\left(x_1, x_2\right)$ *by* $\left(x_h, x_{h+1}\right)$ and $\left(z_1, z_2\right)$ *by* $\left(z_k, z_{k+1}\right)$, we have

$$\mu_{ikc} = c\left[1 + \frac{c^{'}}{2c}k_i + \left(\frac{c^{'} f_x + 2 fc^{''}}{12 fc}\right)k_i^2 + \left(\frac{ff_{xx}c^{'} + ff_x c^{''} + f^2 c^{'''}}{24 f^2 c}\right)k_i^3 + O\left(k_i^4\right)\right]$$

where $k_i = x_{h+1} - x_h$ and derivatives are evaluated at x_h.

However, when the same functions are differentiated w.r.t. z_k, we have

$$\mu_{hjc} = c\left[1 + \frac{c^{'}}{2c}k_j + \left(\frac{c^{'} f_z + 2 fc^{''}}{12 fc}\right)k_j^2 + \left(\frac{ff_{zz}c^{'} + ff_z c^{''} + f^2 c^{'''}}{24 f^2 c}\right)k_j^3 + O\left(k_j^4\right)\right]$$

where $k_j = z_{k+1} - z_k$

Here, the derivatives are evaluated at both x_h *and* z_k, we get

$$\mu_{ijc} = c\left[1 + \frac{c^{'}}{2c}\left(k_i k_j\right) + \left(\frac{c^{'}\left(f_x + f_z\right) + 2fc^{''}}{12fc}\right)\left(k_i k_j\right)^2 + \left(f_4\right)\left(k_i k_j\right)^3 + O\left(k_i k_j\right)^4\right]$$

where

$$f_4 = \frac{f\left(f_{xx} + f_{zz} + f_{xz}\right)c^{'} + f\left(f_x + f_z\right)c^{''} + f^2 c^{'''}}{24 f^2 c}$$

Thus, we have

$$\mu_{ijc} - c\left(x_h, z_k\right) = \frac{\left(k_i k_j\right)}{2}\left[c^{'} + \left(\frac{c^{'}\left(f_x + f_z\right) + 2fc^{''}}{12f}\right)\left(k_i k_j\right) + O\left(k_i k_j\right)^2\right]$$

Similarly, we get

$$\mu_{hkc} - c\left(x_h, z_k\right) = \frac{\left(k_h k_k\right)}{2}\left[c^{'} + \left(\frac{c^{'}\left(f_x + f_z\right) + 2fc^{''}}{12f}\right)\left(k_h k_k\right) + O\left(k_h k_k\right)^2\right]$$

Evaluating derivatives at x_h and z_k. Therefore, equation (3.4) can be put as

$$\frac{(k_h k_k)}{2}\left[c' + \left(\frac{c'(f_x+f_z)+2fc''}{12f}\right)(k_h k_k) + O(k_h k_k)^2\right]$$

$$= \frac{(k_i k_j)}{2}\left[c' + \left(\frac{c'(f_x+f_z)+2fc''}{12f}\right)(k_i k_j) + O(k_i k_j)^2\right] \tag{5.1}$$

Now, let us consider an expansion of the function

$$B_{hk} = \int_{z_{k-1}}^{z_k}\int_{x_{h-1}}^{x_h} c'^2(t_1,t_2) f(t_1,t_2)\partial t_1 \partial t_2$$

about the point $[x_h, z_k]$. Expanding the integral about x_h and z_k with the help of the Taylor's expansion for two variables, we have

$$B_{hk} = fc'^2 k_h k_k\left[1 - \frac{(f_x+f_z)c' + 2fc''}{fc'}\frac{(k_h k_k)}{2} + O(k_h k_k)^2\right] \tag{5.2}$$

where in (5.2) also the function of f, η and their derivatives are evaluated at x_h and z_k. Thus, we find that

$$\frac{(k_h k_k)^2}{8}\frac{B_{hk}c'}{f} = \frac{(k_h k_k)^3 c'^3}{8}\left[1 - \left(\frac{(f_x+f_z)c' + 2fc''}{2fc'}\right)(k_h k_k) + O(k_h k_k)^2\right]$$

or

$$\left[\frac{(k_h k_k)^2}{8}\frac{B_{hk}c'}{f}\right]^{\frac{1}{3}} = \frac{(k_h k_k)c'}{2}\left[1 - \left(\frac{(f_x+f_z)c' + 2fc''}{2fc'}\right)(k_h k_k) + O(k_h k_k)^2\right]^{\frac{1}{3}}$$

$$= \frac{(k_h k_k)c'}{2}\left[1 - \left(\frac{(f_x+f_z)c' + 2fc''}{6fc'}\right)(k_h k_k) + O(k_h k_k)^2\right] \tag{5.3}$$

Similarly, we obtain

$$\left[\frac{(k_i k_j)^2}{8}\frac{B_{ij}c'}{f}\right]^{\frac{1}{3}} = \frac{(k_h k_k)c'}{2}\left[1 - \left(\frac{(f_x+f_z)c' + 2fc''}{6fc'}\right)(k_i k_j) + O(k_i k_j)^2\right]$$

Therefore, the minimal equations (5.1) can be put as

$$\left[\frac{(k_h k_k)^2}{8}\frac{B_{hk}c'}{f}\right]^{\frac{1}{3}}\left[1+O(k_h k_k)^2\right]=\left[\frac{\left(k_i k_j\right)^2}{8}\frac{B_{ij}c'}{f}\right]^{\frac{1}{3}}\left[1+O\left(k_i k_j\right)^2\right] \tag{5.4}$$

Hence, if the terms of order $O\left(\underset{((a,b),(c,d))}{Sup}\left(k_h k_j\right)\right)^3$ can be neglected, we can replace the minimal equations approximately by

$$\left[\frac{(k_h k_k)^2}{8}\frac{B_{hk}c'}{f}\right]^{\frac{1}{3}}=\left[\frac{\left(k_i k_j\right)^2}{8}\frac{B_{ij}c'}{f}\right]^{\frac{1}{3}}$$

or $(k_h k_k)^2 B_{hk}$ = Constant (5.5)

In the case when it is possible to find a function $Q_1'(x_{h-1}, x_h, z_{k-1}, z_k)$ such that

$$\begin{aligned}(k_h k_k)^2 B_{hk} &= (k_h k_k)^2 \int_{z_{k-1}}^{z_k}\int_{x_{h-1}}^{x_h} c'^2(t_1,t_2) f(t_1,t_2)\partial t_1 \partial t_2\\ &= Q_1'(x_{h-1}, x_h, z_{k-1}, z_k)\left[1+O(k_h k_k)^2\right]\end{aligned} \tag{5.6}$$

Thus, the system of equations (5.5) to the same degree of accuracy can be put as

$$Q_1'(x_{h-1}, x_h, z_{k-1}, z_k) = \text{Constant} \tag{5.7}$$

The above results can be put in the form of a note as follows.

Remark 1: If the regression of the dependent variable Y and stratification variables X and Z in an unbounded population is given by

$$Y = C(X,Z)+e$$

where 'e' is the error component such that $E(e\,|\,x,z)$ = 0 and $V(e\,|\,x,z)=\eta(x,z)>0$ $,\forall x\in(a,b)$ *and* $z\in(c,d)$ with finite deviation of the intervals, and in addition if $c'^2(x,z)\,f(x,z)$ belong to Ω, then the system of equations (3.4) giving strata boundaries $\left[x_h, z_k\right]$, which correspond to the minimum of $V\left(\overline{y}_{st}\right)_{prop}$, can be put as

$$\left\{\left(k_h k_k\right)^2 \int_{z_{k-1}}^{z_k} \int_{x_{h-1}}^{x_h} c'^2\left(t_1, t_2\right) f\left(t_1, t_2\right) \partial t_1 \partial t_2\left[1+O\left(k_h k_k\right)^2\right]\right\}^{\frac{1}{3}}$$

$$=\left\{\left(k_i k_j\right)^2 \int_{z_{k-1}}^{z_k} \int_{x_{h-1}}^{x_h} c'^2\left(t_1, t_2\right) f\left(t_1, t_2\right) \partial t_1 \partial t_2\left[1+O\left(k_i k_j\right)^2\right]\right\}^{\frac{1}{3}}.$$

If the terms of order $O\left(\underset{((a,b),(c,d))}{Sup}\left(k_h k_j\right)\right)^3$ can be neglected, these equations can be replaced by the approximate system of equations

$$\left(k_h k_k\right)^2 \int_{z_{k-1}}^{z_k} \int_{x_{h-1}}^{x_h} c'^2\left(t_1, t_2\right) f\left(t_1, t_2\right) \partial t_1 \partial t_2\left[1+O\left(k_h k_k\right)^2\right]=\text{Constant}$$

Or equivalently by

$$Q_1'\left(x_{h-1}, x_h, z_{k-1}, z_k\right)=\text{Constant}$$

Therefore,

$$\therefore \; Q_1'\left(x_{h-1}, x_h, z_{k-1}, z_k\right)\left[1+O\left(k_h k_k\right)^2\right] =\left(k_h k_k\right)^2 \int_{z_{k-1}}^{z_k} \int_{x_{h-1}}^{x_h} c'^2\left(t_1, t_2\right) f\left(t_1, t_2\right) \partial t_1 \partial t_2\left[1+O\left(k_h k_k\right)^2\right]$$

The same result can also be obtained by minimizing the function

$$\sum_h \sum_k \int_{z_{k-1}}^{z_k} \int_{x_{h-1}}^{x_h} c'^2\left(t_1, t_2\right) f\left(t_1, t_2\right) \partial t_1 \partial t_2\left[1+O\left(k_h k_k\right)^2\right]$$

as in the light of Lemma 5, $12 \sum_h \sum_k W_{hk} \sigma_{hkc}^2$ equals to this function

Thus, we find that if the function $c'^2(x, z)\, f\left(t_1, t_2\right)$ belongs to the class Ω the minimum value of $\sum_h \sum_k W_{hk} \sigma_{hkc}^2$ and therefore $V\left(\overline{y}_{st}\right)_{prop}$ exists and the set of strata boundaries $\left[x_h, z_k\right]$, corresponding to this minimum, is the solution of the systems of equations (3.4) or equivalently of (5.4). These equations are very difficult to solve exactly and it becomes essential to find some approximation to stratification points.

It may be precisely solved by substituting the exact minimal equations by other systems of equations which are comparatively easy to solve but are only asymptotically equivalent to the exact equations. The error is introduced because we neglect terms of higher powers of the strata widths which can be justified when the total stratum is large. The approximate systems of equations are obtained by neglecting terms of order $O\left(m^3\right)$ where $m = \underset{((a,b),(c,d))}{Sup}(k_h k_k)$, on both sides of (5.4). For large strata, the terms of order $O\left(m^3\right)$ are small and therefore the error involved in the approximate systems of equations is small, although this error is comparatively larger than the one involved in the case of optimum allocation. Here, we shall develop the approximate systems of equations given in (5.5) and (5.7).

6. Approximate systems of equations

I. If in the expansion of the minimal equations (3.4) we neglect all terms except the first on both sides of the equation, the solution is obtained by taking

$$x_h = \text{constant} = \frac{b-a}{L}, \text{ h=1,2,...,L and } z_k = \text{constant} = \frac{d-c}{M}, \quad k = 1,2,\ldots,M \,. \tag{6.1}$$

Therefore

$$x_h = a + \left(\frac{(b-a)}{L}\right)h \quad with \quad x_0 = a \quad and \quad x_L = b$$

and
$$z_k = c + \left(\frac{(d-c)}{M}\right)k \quad with \quad z_0 = c \quad and \quad z_M = d$$

It cannot be suspected that this set of approximations can give good solutions as these are simple to obtain. However, the method is not applicable in the case of infinite range.

II. An approximation to the optimum points of stratification is obtained by solving the systems of equations

$$\left(k_h k_k\right)^2 \int_{x_{h-1}}^{x_h} \int_{z_{k-1}}^{z_k} c'^2\left(t_1,t_2\right) f\left(t_1,t_2\right) \partial t_1 \partial t_2 = C_1 \tag{6.2}$$

as shown in (5.5). The solutions of this system of equations and also of those that will now follow, are expected to be closer to the optimum stratification points as compared to the solutions obtained from (6.1).

III. From Lemma 4 and equation (6.2), we get a general class of approximate systems of equations as

$$\left[(k_h k_k)^{3\lambda-1}\int_{x_{h-1}}^{x_h}\int_{z_{k-1}}^{z_k}\left(c'^2(t_1,t_2)\,f(t_1,t_2)\right)^{\lambda}\partial t_1\partial t_2\right]^{\frac{1}{\lambda}} = \text{Constant}$$

However, for $\lambda = \frac{1}{2}$,we have

$$\left[(k_h k_k)\int_{x_{h-1}}^{x_h}\int_{z_{k-1}}^{z_k} c'^2(t_1,t_2)\sqrt{f(t_1,t_2)}\partial t_1\partial t_2\right]^2 = C_2$$

and for $\lambda = \frac{1}{3}$, we have a system of equations as

$$\left[\int_{x_{h-1}}^{x_h}\int_{z_{k-1}}^{z_k}\sqrt[3]{c'^2(t_1,t_2)\,f(t_1,t_2)}\partial t_1\partial t_2\right]^3 = C_3$$

giving approximations to stratification points $[x_h, z_k]$. As remarked in the case of the optimum allocation method, in some particular cases some of the approximate systems given in the above equations may be meaningless. Therefore, depending upon the situation, one should make the approximate choice of the systems of equations for obtaining the approximations to optimum points $[x_h, z_k]$.

7. Cum $\sqrt[3]{D_3(x,z)}$ Rule

If the function $D_3(x,z) = c'^2(x,z)\,f(x,z)$ is bounded and its first two derivatives exists $\forall\ x \in [a,b]\ \&\ z \in [c,d]$, then taking equal intervals with a given values of L and M on the cumulative cube root of $D_3(x,z)$ will give AOSB $[x_h, z_k]$.

Remarks:

I. If we take either $c(x,z) = \alpha + \beta x$ or $c(x,z) = \alpha + \gamma z$ in $D_3(x,z)$ it reduces to the method proposed by Singh and Sukhatme (1969).

II. If the function $c'^2(x,z)$ is constant, therefore the proposed method reduces to Cum $\sqrt[3]{f(x,z)}$ rule.

Further, for any distribution and given number of strata the set of AOSB will remain unchanged with respect to the form of conditional variance. However, the efficiency of the stratification will differ from stratified simple random sampling estimators as well as other estimators with the choice of various forms of conditional variance.

8. Empirical study

For the purpose of empirical study, the effectiveness of the methods of finding approximation to the optimum points of stratification, we have considered the system of minimal equations obtained for the case of proportional allocation. In this illustration we shall consider equal interval approximation and the system of approximations given in (6) article. The former approximation is specially considered due to its simplicity. From all the later approximations we have only chosen one suitable method. Since the order of approximation involved in all these methods is the same, this one approximation will give the idea about the effectiveness of all other approximations given in article (6). For the sake of simplicity, the linear regression line Y on X and Z have been taken as the form $y = \alpha + \beta x + \gamma z + e$. Here, it is considered that the two auxiliary variables used for stratification are dependent. From all the subsequent approximations we have only chosen one suitable method. Since the order of approximation involved in all these methods is the same, this one approximation will give the idea about the effectiveness of all other approximations. For obtaining the stratification points under proportional allocation let us assume $c(x,z) = \alpha + \beta x + \gamma z$. Further, let us assume that the correlation coefficient between X and Z is denoted by ρ and is equal to 0.65. Let us consider the following examples:

Empirical study 1:
Suppose

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \qquad , x \geq 0$$

and the variable Z has

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \qquad , z \geq 0$$

In order to obtain the OSB when both the variables are standard normally distributed by assuming the value of regression coefficients $\beta = 0.65$ and $\gamma = 0.57$. For obtaining total 16 strata, 4 along X variable and 4 along Z variable using the proposed Cum $\sqrt[3]{D_3(x,z)}$ rule, by solving it in Mathematica Software assuming that the distribution of X and Z are truncated at x = 6 and z = 4, respectively, we get the stratification points as below:

Table 1. OSB and Variance, for standard normally distributed auxiliary variables

OSB (x_h, z_k)	**Variance** Cum $\sqrt[3]{D_3(x,z)}$ Rule	**Variance (Singh 1975)**	**% R.E.**
(0.3347,0.2673) (0.5779,0.2673) (1.9004,0.2673) (6.0000,0.2673) (0.3347,0.5284) (0.5779,0.5284) (1.9004,0.5284) (6.0000,0.5284) ((0.3347,0.9865) (0.5779,0.9865) (1.9004,0.9865) (6.0000,0.9865) (0.3347,4.0000) (0.5779,4.0000) (1.9004,4.0000) (6.0000,4.0000)	0.06798628	0.182346	268.21

Empirical study 2:

Let

$$f(x) = \begin{cases} \frac{1}{\sigma x\sqrt{2\pi}} e^{-\frac{(\log x-\mu)^2}{2\sigma^2}}; x > 0, \sigma > 0 \\ 0 \qquad , elsewhere \end{cases}$$

and

$$f(z) = \begin{cases} \frac{1}{b-a}, a \leq z \leq b \\ 0 \quad , elsewhere \end{cases}$$

To obtain the OSB in proportional allocation using the proposed Cum $\sqrt[3]{D_3(x,z)}$ rule for uncorrelated auxiliary variables having densities as defined above. Standardised log-normal distribution is defined in the interval $x \in [0,10]$ and the other variable $z \in [0,1]$, and β=0.82 & γ=0.437. For 3×2 (L×M) = 6 strata, i.e. 3 along X variable and

2 along Z variable, the results obtained after solving the function using Mathematica Software are presented in the following tables as:

Table 2. OSB and Variance, for standard lognormal and uniform distributions

OSB $\left(x_h, z_k\right)$	**Variance** (Cum $\sqrt[3]{D_3(x,z)}$ Rule)
(3.4216,0.4759)	0.035281796
(6.5319,0.4759)	
(10.0000,0.4759)	
(3.4216,1.0000)	
(6.5319,1.0000)	
(10.0000,1.0000)	

9. For independent auxiliary variables under proportional allocation (ρ = 0)

In order to propose a technique under proportional allocation when the two auxiliary variables are independent to each other we need to proceed in the same way as proceeded in the case when they were dependent only with the difference that here in this case we have to take marginal densities rather than joint densities under consideration. We can write (5.3) as

$$\left[\frac{(k_h)^2}{8}\frac{B_h c'(x)}{f(x)}\right]^{\frac{1}{3}}\left[\frac{(k_h)^2}{8}\frac{B_h c'(z)}{f(z)}\right]^{\frac{1}{3}}$$

$$=\frac{(k_h)c'(x)}{2}\left[1-\left(\frac{(f_x)c'+2f(x)c''(x)}{6f(x)c'(x)}\right)(k_h)+O(k_h)^2\right]$$

$$\frac{(k_k)c'(z)}{2}\left[1-\left(\frac{(f_z)c'+2f(z)c''(z)}{6f(z)c'(z)}\right)(k_k)+O(k_k)^2\right]$$

In a similar way, we have

$$\left[\frac{(k_i)^2}{8}\frac{B_i c'(x)}{f(x)}\right]^{\frac{1}{3}}\left[\frac{\left(k_j\right)^2}{8}\frac{B_j c'(z)}{f(z)}\right]^{\frac{1}{3}}$$

$$=\frac{(k_i)c'(x)}{2}\left[1-\left(\frac{(f_x)c'+2f(x)c''(x)}{6f(x)c'(x)}\right)(k_i)+O(k_i)^2\right]$$

$$\frac{\left(k_j\right)c'(z)}{2}\left[1-\left(\frac{(f_z)c'+2f(z)c''(z)}{6f(z)c'(z)}\right)\left(k_j\right)+O\left(k_j\right)^2\right]$$

where $B_h = \int_{x_{h-1}}^{x_h} c'^2(t_1) f(t_1)\, dt_1$ *and* $B_k = \int_{z_{k-1}}^{z_k} c'^2(t_2) f(t_2)\, dt_2$

However, if the terms of order $O\left(\underset{a,b}{Sup}(k_h)\right)^3$ *and* $O\left(\underset{c,d}{Sup}(k_k)\right)^3$ can be neglected, we replace the minimal equations approximately by

$$\left[\frac{(k_h)^2}{8}\frac{B_h c'(x)}{f(x)}\right]^{\frac{1}{3}}\left[\frac{(k_h)^2}{8}\frac{B_h c'(z)}{f(z)}\right]^{\frac{1}{3}} = \left[\frac{(k_i)^2}{8}\frac{B_i c'(x)}{f(x)}\right]^{\frac{1}{3}}\left[\frac{(k_j)^2}{8}\frac{B_j c'(z)}{f(z)}\right]^{\frac{1}{3}}$$

or in other words we have that $k_h^2 B_h$ *and* $k_k^2 B_k$ are constants. In the case when it is possible to find a function $Q_1'(x_{h-1}, x_h)$ *and* $Q_1'(z_{k-1}, z_k)$ such that

$$\begin{aligned}(k_h)^2 B_h &= (k_h)^2 \int_{x_{h-1}}^{x_h} c'^2(t_1) f(t_1)\, dt_1 \\ &= Q_1'(x_{h-1}, x_h)\left[1 + O(k_h)^2\right]\end{aligned}$$

and

$$\begin{aligned}(k_k)^2 B_k &= (k_k)^2 \int_{z_{k-1}}^{z_k} c'^2(t_2) f(t_2)\, dt_2 \\ &= Q_1'(z_{k-1}, z_k)\left[1 + O(k_k)^2\right]\end{aligned}.$$

The Remark 1 can be proceeded in the case of independent variables too. Similarly, an approximate system of equations can be proposed in the same way as proposed in the case when auxiliary variables are dependent, and can be written as

$$(k_h k_k)^2 \int_{x_{h-1}}^{x_h} \int_{z_{k-1}}^{z_k} c'^2(t_1) c'^2(t_2) f(t_1) f(t_2)\, \partial t_1 \partial t_2 = C_1.$$

The solution of this system of equations and also those that will now follow are expected to be closer to the optimum points of stratification as compared to the strata obtained composed from $k_h = \frac{b-a}{L}$ = Constant and $k_k = \frac{d-c}{M}$ = Constant. Therefore, depending on the situation one should make the approximate choice of the system of equations for obtaining the approximation of optimum points of stratification.

10. Cum $\sqrt[3]{D_4(x,z)}$ Rule

For equal intervals with given values of L and M on the cumulative cube root of $D_4(x,z)$ will give AOSB if the function $D_4(x,z) = c'^2(x)c'^2(z) f(x) f(z)$ is bounded and its first derivative exists in all $x \in [a,b]$ and $z \in [c,d]$.

Remarks:

1. If the functions $c'^2(x)$ *and* $c'^2(z)$ are constants, then the proposed method is reduced to cum $\sqrt[3]{f(x) f(z)}$ rule.
2. If we take $c(x) = c(z)$ and $f(x) = f(z)$, then the proposed method is reduced to the Yadava and Singh (1984) method of Cum $\sqrt[3]{B_2(x)}$

   where $B_2 = \dfrac{f(x)x^2c'^2(x) + x\phi'(x) - \phi(x)}{x^3}$

11. Empirical study

We shall demonstrate empirically the efficiency of the given method obtaining approximately optimum strata boundaries (AOSB). For this purpose, we have considered the system of minimal equations obtained for the case of proportional allocation when the two auxiliary variables used for stratification are independent. From all the subsequent approximations we have only chosen one suitable method. Since the order of approximation involved in all these methods is the same, this one approximation will give the idea about the effectiveness of all other approximations. For obtaining the stratification points under proportional allocation let us assume $c(x,z) = \alpha + \beta x + \gamma z$. Let us consider the following examples:

Empirical study 3:

Let $f(x) = 2(2-x), 1 \le x \le 2$ and $f(z) = e^{-z+1}, 1 \le z \le 6$

In order to obtain stratification points when the auxiliary variable X follows right-triangular distribution defined in [1,2] and auxiliary variable Z follows exponential distribution defined in [1,6] we assume the values of $\beta = 0.567$ and $\gamma = 0.257$. While execution for obtaining OSB using Cum $\sqrt[3]{D_4(x,z)}$ Rule by solving the function using Mathematica Software for 6 strata, 2 along X variable and 3 along Z variable. The results obtained are presented in the following table.

Table 3. OSB and Variance, with right-triangular and exponential distribution

OSB (x_h, z_k)	Variance		% R.E.
	Cum $\sqrt[3]{D_4(x,z)}$ Rule	Yadava and Singh, 1984	
(1.5000,1.9474) (1.0000,1.9474) (1.5000,3.3368) (1.0000,3.3368) (1.5000,6.0000) (1.0000,6.0000)	0.089542	0.152122	169.89

Empirical study 4: Let us consider the distribution of X as right-triangular having

$$f(x) = 2(2-x), 1 \le x \le 4$$

and Z variable is having a uniformly distributed having

$$f(z) = \frac{1}{b-a}, 1 \le z \le 2$$

In order to find the OSB when one of the auxiliary variable is following right-triangular distribution and the other uniform distribution, we assume the value of $\beta = 0.56$ and $\gamma = 0.762$. The stratification points obtained for total 6 strata among that 3 along X variable and 2 along Z variable for the Cum $\sqrt[3]{D_4(x,z)}$ Rule using Mathematica Software for solving the function are presented in the following table.

Table 4. Uncorrelated variables having right-triangular and exponential distribution, OSB and Variance

OSB (x_h, z_k)	Variance		% R.E.
	Cum $\sqrt[3]{D_4(x,z)}$ Rule	Khan *et al.* (2008)	
(1.7880,1.5000) (2.6870,1.5000) (4.0000,1.5000) (1.7880,2.0000) (2.6870,2.0000) (4.0000,2.0000)	0.0354952	0.08293	233.64

12. Simulation Study

In this section, we conduct a simulation study to investigate the effectiveness of the proposed dynamic programming with the following methods (1-3) in stratification package in the R statistical software and 4 & 5 in LINGO:

1. Dalenius and Hodges [1959] cum f method, which is the most frequently used and better known method.
2. Gunning and Horgan [2004] method.
3. Lavallée -Hidiroglou [1988] method with Kozak's [2004] algorithm.
4. Khan et al. [2015] method
5. Proposed method.

In this study, a data set (with N = 5000) following the uniform and exponential distribution with a = 0, b=1, c= 0 and d = 2 was randomly generated by the R software. Then, the OSBs using the proposed method as discussed earlier are obtained for the three different number of strata, that is, (L,M)= (3 ,4). Then, the OSBs using the proposed method are obtained for (L,M)= (3 ,4). The OSBs are determined using cum f method, geometric method and the Lavallée-Hidiroglou (Kozak's) method using the stratification package with CV = 0.75 and Khan et al. [10] and the proposed method using LINGO.

Table 5. The variance of variables for different stratification methods

Stratification Method	Variance (in e-09)
Dalenius and Hodges [1959] cum $\sqrt{f}$ method	312.8371
Gunning and Horgan [2004] method	2891.916
Lavallee-Hidiroglou [1988] method using Kozak's [2004] method	728.3791
Khan et al. [2015]	589 .7021
Proposed method	203.107

From the table above, it is noted that the OSBs obtained by the cum f method and the proposed dynamic programming method are very close to each other, whereas the OBSs in the other methods, geometric, Lavallée-Hidiroglou method with Kozak's algorithm and Khan et al. (2015) differ widely from that of the proposed method. However, the table reveals that the proposed method yields the smallest variances of the estimate for all (L, M) = (3,4) as compared to all the other methods. Although the variances for the dynamic programming method are closed to the cum f method, the other two methods produce a greater variance than the dynamic programming technique. Thus, the study reveals that the proposed dynamic programming technique is more efficient than the other methods while stratifying a population with a uniform and exponential distributions.

13. Conclusion

The optimum stratification is defined as subdividing heterogeneous population into the best possible manner that makes the homogeneity within subpopulation and heterogeneity between them. Demarcation of strata boundaries is one of the main factors for efficient results in stratified random sampling. In this regard, we have proposed Cum $\sqrt[3]{D_i(x,z)}$ Rule (i = 3,4) for obtaining approximately OSB for two stratification variables having single study variables for both the dependent as well as independent cases for concomitant variables. Thus, comparing the proposed method Cum $\sqrt[3]{D_3(x,z)}$ Rule for standard normally distributed auxiliary variables with the Singh (1975), the %RE obtained is 268.21, which indicates the efficiency of the proposed method. Further, the %RE obtained while making comparisons between the proposed method (Cum $\sqrt[3]{D_4(x,z)}$ Rule) and the method given by Yadava and Singh (1984) results in 169.89 for right-triangular and exponential auxiliary variables. In the same case for right-triangular and uniform auxiliary variables the %RE comes out to be 233.64 as compared with Khan *et al.* (2008) under proportional allocation. Further, the simulation study also proved the superiority of the proposed methods with regard to the existing methods. Thus, it can be concluded that the use of two stratification variables gains efficiency over a single auxiliary variable and the proposed methods are more precise than the existing methods. The proposed strategy can be entirely applied to different distributions that describe the concomitant variables.

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Changes in the impact of US macroeconomic news on financial markets the example of the Warsaw Stock Exchange

Henryk Gurgul[1], Jessica Hastenteufel[2], Tomasz Wójtowicz[3]

ABSTRACT

Due to the high importance of the American economy, in the past, announcements of US macroeconomic data were shown to have a significant impact on financial markets in general, and on European stock markets in particular. However, as this effect may vary in time, this paper examines the changes in the impact of US macroeconomic news on the WIG20, the main index of the Warsaw Stock Exchange. Based on intraday data from 2004-2019 we study the changes in significance and in the strength of the reaction of WIG20 to announcements of unexpected values of 13 indicators describing the American economy. On the basis of the event study analysis, we describe the reaction of the WIG20 index in the first few minutes after these kinds of announcements.

Key words: event study, macroeconomic announcements, intraday data, Warsaw Stock Exchange.

1. Introduction

Information on the state of an economy is important for investors in financial markets, and it affects both the foreign exchange markets as well as the stock markets. Due to globalisation and the dominant role of the US economy, American macroeconomic news is very important in this context. This has been shown in various scientific papers dealing with this issue. In the beginning, these studies mainly examined the impact of US macroeconomic data on the USA, and on markets in other developed countries (e.g. Schwert, 1981; Pearce and Roley, 1985; Li and Hu, 1998; Nikkinen and Sahlström, 2004; Boyd et al., 2005; Andersen et al., 2007; Harju and

[1] Department of Applications of Mathematics in Economics, Faculty of Management, AGH University of Science and Technology, Al. Mickiewicza 30, 30-059 Cracow, Poland. E-mail: henryk.gurgul@gmail.com. ORCID: https://orcid.org/0000-0002-6192-2995.

[2] Department of Business and Management, IU International University of Applied sciences, Bad Honnef, Germany. E-mail: j.hastenteufel@iubh-fernstudium.de. ORCID: https://orcid.org/0000-0002-4386-5771.

[3] Department of Applications of Mathematics in Economics, Faculty of Management, AGH University of Science and Technology, Al. Mickiewicza 30, 30-059 Cracow, Poland. E-mail: twojtow@agh.edu.pl. ORCID: https://orcid.org/0000-0002-9742-3493.

Hussain, 2011). Over time, this analysis was extended to investigate the impact of US macroeconomic data on emerging markets, including markets in Central and Eastern Europe (e.g. Hanousek et al., 2009; Gurgul and Wójtowicz, 2013, 2014, 2015). However, the number of papers concerning the reaction of European emerging markets to macroeconomic news from the USA is still very limited.

Previously mentioned studies of stock market reactions to information about the US economy were carried out by various authors who applied various methods and used different data covering the periods of different economic conditions. Taking that into account, it is difficult to compare the results of these studies, and to draw conclusions from the changes in the impact of news from the United States. For this reason, in this paper we examine the reactions of the WIG20 (the main index of the Warsaw Stock Exchange) to announcements of 13 macroeconomic indicators containing current information about the state of the US economy over time. The WSE is the largest stock market among post communist countries in Central and Eastern Europe. Hence, the results of this paper can be seen as a reflection of changes observed on other emerging markets in the CEE region. Additionally, previous research from Gurgul and Wójtowicz (2014; 2015) suggests that the reaction of investors on the WSE is similar to that on the Vienna Stock Exchange. Therefore, the results of the study can be at least partially transferred to the VSE.

To describe the reaction of the WSE to US data in detail, we consider 5-minute WIG20 returns from January 2004 until the end of July 2019. Application of this data allows us to comprehensively investigate the changes in strength and significance of the impact of US macroeconomic news over time. It is also important to mention that the period considered includes both the time of bull market before the global financial crisis, the time of the crisis itself, as well as the period of changes following the crisis. Additionally, it also covers periods of other crises in the world including the government-debt crisis in Greece or the financial crisis in Spain (2008–2014).

By applying intraday data this paper is an extension of previous research about the event study methodology. To verify the significance and the strength of the impact of announcements of US macroeconomic data on the WIG20, we apply a nonparametric rank test proposed by Kolari and Pynnönen (2011). It is a generalization of the widely applied test of Corrado and Zivney (1992). The application of this methodology, instead of the commonly used GARCH models or regressions with dummy variables, allows us to analyse the significance of the reaction of index returns to American macroeconomic news more precisely.

The remainder of the paper is organised as follows. In the next section we provide an overview of the existing literature focussing on the impact of macroeconomic announcements on financial markets. In Section 3 we present the US macroeconomic indicators and the returns used in this study. We also briefly describe the methodology applied. Afterwards the empirical analysis and the discussion of its results are presented. The final section critically concludes the paper.

2. Literature review

Early studies on the impact of US macroeconomic data announcements were focused on the US stock market (Geske and Roll, 1983; McQueen and Roley, 1993). These studies have been subsequently extended to other developed markets showing the importance of information about the US economy. For example, Nikkinen and Sahlström (2004) examined the impact of US and domestic macroeconomic news on the German and Finnish equity markets. Their study shows a dominant role of information from the US as the volatility on both markets is significantly impacted by US announcements, particularly by information about the unemployment rate and PPI. A wider group of stock markets was considered by Nikkinen et al. (2006), who analysed the impact of US macroeconomic news announcements on 35 stock markets around the world. Among these markets, there were some developed and emerging markets in Europe. Based on data from July 1995 to March 2002, Nikkinen et al. (2006) stated, on the one hand, that unexpected macroeconomic information from the USA affects the volatility on developed stock markets in Europe and Asia. On the other hand, the volatility on emerging CEE markets (including the Czech Republic, Hungary, Poland, Russia and Slovakia) was not significantly impacted by announcements of US macroeconomic indicators. This showed that developed and emerging markets in Europe reacted differently to US macroeconomic news announcements. However, this observation may have been caused by the application of data from the early period of the development of equity markets in the CEE region. This observation is supported by opposite results shown by Gurgul et al. (2012). Based on data from January 2004 to December 2011, Gurgul et al. (2012) pointed out a significant reaction of daily returns of the WIG20 to unexpected news about inflation and industrial production in the United States.

More precise results on the impact of US macroeconomic news on European markets were obtained by applying intraday data. For example, based on the five-minute returns Andersen et al. (2007) analysed the impact of US macroeconomic news on US, German and British stock, bond and foreign exchange markets. High-frequency data was also applied by Harju and Hussain (2011), who investigated the impact of scheduled US macroeconomic announcements on British, French, German, and Swiss stock markets. They proved that announcements of CPI, PPI, retail sales, durable goods orders, unemployment rate and industrial production lead to significant and immediate changes of the volatility and the 5-minute returns of the CAC40, the DAX30, the FTSE100, and the SMI. Similar results, significant and immediate reaction, were presented by Dimpfl (2011), who analysed the 1-minute returns of the DAX from July 2003 to December 2006. Dimpfl (2011) showed that investors on the Frankfurt Stock Exchange react right after a news release and the significant reaction takes place in the first ten minutes.

Gurgul and Wójtowicz (2015) analysed the reaction of the Austrian stock exchange. Applying 1-minute returns of the ATX (= the main index of Vienna Stock Exchange) from 2 January 2007 to 31 December 2013 they proved a significant impact of announcements of 10 US macroeconomic indicators on the returns and the volatility. The strongest reaction was induced by news from the US labour market included in nonfarm payrolls announcements. Gurgul and Wójtowicz (2015) also examined the changes in the strength of the reaction of the ATX to US macroeconomic announcements in subsequent years of the period under consideration. This analysis led to the conclusion that the strongest reaction of investors in Vienna took place during the global financial crisis in 2007-2009. After this period, the reaction of the ATX to news from the US economy was weaker.

Empirical analysis based on intraday data has also been conducted for European emerging markets. Hanousek et al. (2009) investigated the reaction of stock prices in the Czech Republic, Hungary and Poland to US and EU macroeconomic news. On the basis of the five-minute returns from the period June 2, 2003 – December 29, 2006, Hanousek et al. (2009) showed that the Czech and the Hungarian stock markets reacted significantly to macroeconomic news from both the US and EU, while the stock market in Poland was only affected by announcements from the Eurozone. This line of research was continued by Hanousek and Kočenda (2011). Using 5-minute returns of the WIG20, the PX50 and the BUX from the period 2004–2007 they proved that stock markets in the Czech Republic, in Hungary and in Poland mainly reacted to macroeconomic information from the EU, and that macroeconomic data from the United States was not so important.

Opposite conclusions follow from Gurgul and Wójtowicz (2014), who studied the reaction of the Polish stock market to US announcements. Based on the 1-minute returns from 1 April 2007 and 30 August 2013, they showed that the WIG20 reacted immediately and significantly to unexpected news from the US economy. A significant reaction was observed in the first minute after announcements about industrial production, durable goods orders, retail sales and nonfarm payrolls. Additional analysis performed by Gurgul and Wójtowicz (2014) indicated that US macroeconomic announcements did also influence medium and small stock indices of the stock exchange in Warsaw significantly.

In addition to the impact on the stock markets, the impact of macroeconomic news on foreign exchange markets in the CEE countries has been examined. Égert and Kočenda (2014) showed that the Czech, the Hungarian and the Polish currencies significantly react to American macroeconomic information. However, this reaction is different in the pre-crisis (2004-2007) than in the crisis period (2008-2009). The reaction of the foreign exchange markets in these countries to macroeconomic news from the Eurozone and the US was analysed also by Kočenda and Moravcova (2018), who applied intraday data from 2011-2015.

3. Data and methodology

3.1. Announcements

In this paper, we investigate the impact of the announcements of 13 macroeconomic indicators describing various aspects of the US economy. These include: the Consumer Confidence Index (CCI), the Consumer Price Index (CPI), the Durable Goods Orders (DGO), the Existing Home Sales (EHS), the Housing Starts (HS), the Industrial Production (IP), the ISM Manufacturing Index (ISM), the Initial Jobless Claims (IJC), the Nonfarm Payrolls (NFP), the New Home Sales (NHS), the Producer Price Index (PPI), the Real GDP (GDP), and the Retail Sales (RS). In most papers the unemployment situation in the USA is described by the Unemployment Rate. However, as Andersen et al. (2007) show, Nonfarm Payrolls is one of the most significant macroeconomic indicators to describe the US unemployment situation. Similar conclusions follow from the research of Suliga and Wójtowicz (2013).

We chose these indicators because they contain the most current and the most important information for investors. Almost all of these indicators are released on a monthly basis and they describe the economic situation in the USA in the previous (or even in the current) month. The only exception is IJC, which is announced weekly. It contains information from the previous week. Taking into account monthly data ensures a sufficient number of announcements to conduct this study. The second advantage of these indicators is that they have been widely studied in literature. Thus, we can compare the results of this analysis with previous research.

All the indicators under study are released during trading hours on the WSE. Most of them (CPI, DGO, HS, IJC, NFP, PPI, GDP, and RS) are published at 8:30 EST (14:30 CET), CCI, EHS, ISM, and NHS are released at 10:00 EST (16:00 CET) and only values of IP are announced at 9:15 EST (15:15 CET), where EST means Eastern Standard Time and CET means Central European Time.

The announcements are released on different days of the month and different days of the week. The sequence in which US macroeconomic indicator announcements are released may play an important role on how they are perceived by investors. The earlier the indicator is released, the more important it is for investors because it is more probable that it contains new, unexpected information. The value of indicators released later in a month can be forecasted based on the value of earlier indicators. The earliest published indicator is ISM, which is announced in the first few days of a month. Then, it is followed by NFP, which is a part of the Employment Report published by the Bureau of Labour Statistics usually on the first Friday of each month. The majority of the other indicators (CPI, EHS, HS, IP, PPI, and RS) is released mainly in the middle of a month. The rest of them (CCI, DGO, NHS, and GDP) is released in the last few days of a month. However, it should be noted here that values of CCI describe consumers' perception of the economic conditions in the current month.

In this paper, we study the impact of unexpected news related to these US macroeconomic announcements. Thus, for each macroeconomic news release the actual value of the announced indicator is compared with its consensus forecast. All comparisons are performed on the basis of the consensus published by Bloomberg a few days before announcements. It allows us to classify all releases into three clusters: 'above consensus', 'below consensus' and 'in line with consensus'. Because the news in the last cluster is in line with previous investor expectations, our analysis focuses only the first two clusters, which contain unexpected news.

In order to interpret the results of the analysis correctly we divide the announcements according to their meaning rather than simply compare them with the consensus. For most of the indicators, the announcement above the consensus is good news because it is expected to have a positive impact on the stock market. The only exception is publication of CPI, PPI and IJC, where values greater than the forecast are expected to have a negative impact on the stock prices and thus are seen by investors as bad news. Analogously, if the values of CPI, PPI and IJC are lower than the forecast, it is considered good news for the stock market. Based on this consideration we divide all the announcements into two categories of unexpected news: good news and bad news. For these two sets of data, we will perform the empirical analysis.

In addition to analysing the impact of announcements of an individual indicator, we also examine the impact of all good and all bad news. In the set of all good (bad) news, we take into account only monthly announcements, i.e. without IJC announcements released weekly. Additionally, when two or more indicators are announced on the same day, we consider only the first indicator. Subsequent announcements on the same day are excluded from the sample because expectations about their value could be heavily influenced by earlier news, and thus, they might be different from consensus. When two or more announcements are made at the same hour, we consider them only if they do not contain contradictory information, i.e. when each of them is good news or each of them is bad news. The final numbers of the different types of events under study that take place during trading days on the WSE are reported in the second column of Table 1.

3.2. Returns

To describe the impact of US macroeconomic announcements on investors operating on the Warsaw Stock Exchange correctly we study the 5-minute percentage log-returns $R_{i,t}$ of the WIG20 from 2 January 2004 to 31 July 2019:

$$R_{i,t} = 100\left(ln\, P_{i,t} - ln\, P_{i,t-1}\right), \tag{1}$$

where $P_{i,t}$ is the value of the WIG20 at the end of the t-th 5-minute period on day i. The application of the 5-minute intraday returns is a common compromise between accuracy and the negative effects of market microstructure (e.g. Jones et al., 2005; Andersen et al., 2007; Harju and Hussain, 2011). The value of the WIG20 obtained from the WSE is calculated on the basis of stock prices of 20 largest and most liquid companies listed on the WSE. The behaviour of the share prices of WIG20 companies, as well as the perspectives of the companies themselves, are subject to deep analysis by investors. This is why we expect new important information to be included in the WIG20 very quickly. These expectations are also supported by the results of previous studies, for example Gurgul and Wójtowicz (2014).

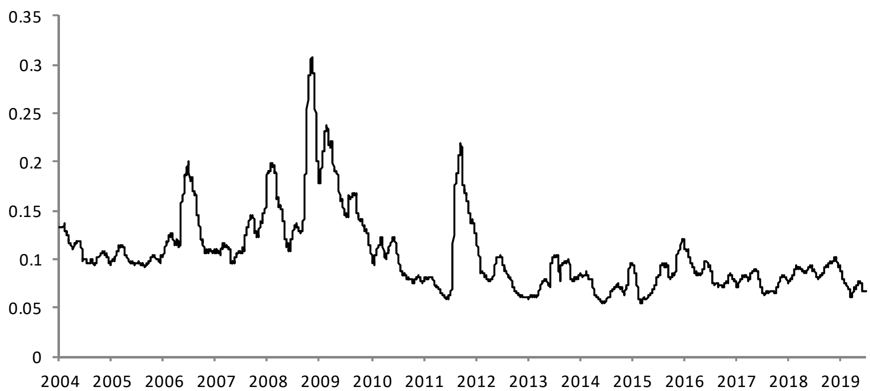

Figure 1. Standard deviation of 5-min percentage WIG20 log-returns in the period 2004-2019

Source: Own elaboration.

The regarded period covers about 15 years characterised by changes in the economic situation in the United States and in the whole world. These changes include various crises that took place in these years. It is well known that the volatility on stock markets increases during such turbulent periods. This phenomenon has also been observed on the WSE. It is also visible on Figure 1, where we present values of the standard deviation S_i computed for each day i based on the 5-min log-returns of the WIG20 from the continuous trading phase of a session from days $i - 20, \dots, i + 20$. Due to these changes in volatility, to compare the strength of the reaction of the WIG20 returns to publications of US macroeconomic indicators in various sub-periods of the main period 2004-2019 we will also consider standardised 5-min returns $SR_{i,t}$ defined as 5-min log-returns $R_{i,t}$ divided by the corresponding standard deviation S_i defined as above for day i. In that case, standardised returns are expressed in terms of return standard deviation.

3.3. Event study

To investigate the impact of US macroeconomic news on the intraday returns of the WIG20 we use the event study methodology. In brief, it includes the analysis of the significance of the abnormal behaviour of returns (abnormal returns) around the event (in the so-called event window). In this paper, the events are defined as the announcements of unexpected macroeconomic news described previously. An event window contains two 5-minute WIG20 returns before the announcement and three returns after the announcement was made.

Abnormal returns are defined as the difference between actual returns and their expected values computed based on data prior to the event window (form the pre-event window). For the i-th event and time t abnormal return AR_{it} is defined as:

$$AR_{it} = R_{it} - E(R_{it}|\Omega), \tag{2}$$

where R_{it} is the 5-minute return and $E(R_{it}|\Omega)$ is the expectation of R_{it} conditional on information set Ω form the pre-event window. In this paper we consider the pre-event window containing 36 values of 5-minute WIG20 returns just before the event window. This choice of the length of event and pre-event window ensures that the pre-event window does not start earlier than 10:25CET (when macroeconomic indicator in announced at 13:30CET) and even for data from before October 2005 (when trading sessions on the WSE started at 10:00CET) it does not contain intraday returns from the initial part of a trading session with increased volatility. To set up notation let us denote the moment of a news release by $t = 0$. Then, the event window includes the 5-minute returns for $t = -1, \dots, 3$, while the pre-event window includes returns for $t = -37, \dots, -2$. It should be noted here that the impact of the i-th news announcement can be observed only for $t \geq 1$.

There are various methods of computing expected values of R_{it}. In this paper, however, we apply the constant mean model where $E(R_{it}|\Omega)$ is equal to the average of returns in the pre-event window. It is a simple but very useful and robust model.

To test the significance of mean abnormal returns in the event window, we apply the nonparametric generalized rank test of Kolari and Pynnonen (2011) with a correction for event-implied volatility. The great advantage of this nonparametric test is that it does not need any assumption about the normality of abnormal returns. The test statistics is constructed as follows.

In the first step of the test procedure, we group events into a cluster. The events are a specific type of announcements, for example the announcements of a given macroeconomic indicator that are good (or bad) news for investors. For each i-th event in the cluster, for $t = -37, \ldots, 3$ we compute abnormal returns AR_{it} from (2) with $E(R_{it}|\Omega)$ computed earlier as the average of returns in the pre-event window ($t = -37, \ldots, -2$). Then, for each event, all abnormal returns in the event and pre-event windows are standardised:

$$SAR_{it} = AR_{it}/S_{AR_i}, \tag{3}$$

where S_{AR_i} is the standard deviation of abnormal returns in the pre-event window. Thisprocedure ensures the comparability of abnormal returns computed based on data from days with high or low volatility.

In order to account for any event-induced increase in volatility observed in the event window (Corrado, 2011; Corrado and Truong, 2008; Kolari, Pynnonen, 2011) we re-standardise the SAR_{it}s in the event window for $t > 0$ by dividing them by the cross-sectional standard deviation:

$$SAR'_{it} = \begin{cases} SAR_{it} & t = -37, \ldots, 0 \\ SAR_{it}/S_{SAR_t} & t = 1, \ldots, 3, \end{cases} \tag{4}$$

where

$$S_{SAR_t} = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(SAR_{it} - \overline{SAR_{it}})^2} \tag{5}$$

is the cross-sectional standard deviation of the standardised abnormal returns, and N is the number of events in the cluster. Under the null hypothesis of no news effect, SAR'_{it}s are zero mean and unit variance random variables.

To study the impact of a news release we test the significance of abnormal returns for each t_0 in the event window separately. Thus, for each $t_0 = -1, \ldots, 3$ the demeaned standardised abnormal ranks of SAR'_{it}s are given by the formula:

$$U_{it} = \frac{rank(SAR'_{it})}{T+1} - 1/2 \tag{6}$$

for $i = 1, \ldots, N$, where $t \in \Theta = \{-37, \ldots, -2, t_0\}$, $T - 1$ is the length of the pre-event window and $rank(SAR'_{it})$ denotes the rank of SAR'_{it} within the vector consisting of standardised abnormal returns from the pre-event window and SAR'_{it_0}. With this notation U_{it_0} denotes the demeaned standardised abnormal rank of SAR'_{it_0} and the null hypothesis of no news effect is equivalent to

$$E(U_{it_0}) = 0. \tag{7}$$

To test this hypothesis we apply the generalised rank t_{grank} test statistic of Kolari-Pynnönen (2011) defined as:

$$t_{grank} = Z\sqrt{\frac{T-2}{T-1-Z^2}}, \tag{8}$$

where $Z = \bar{U}_{t_0}/S_{\bar{U}}$, $S_{\bar{U}} = \sqrt{\frac{1}{T}\sum_{t\in\Theta}\bar{U}_t^2}$ and $\bar{U}_t = \frac{1}{N}\sum_{i=1}^{N} U_{it}$.

Under the null hypothesis of no news effect, the distribution of t_{grank} statistic converges to Student *t* distribution with $T-2$ degrees of freedom when the number of events N in the cluster increases.

It is worth noting here that the above procedure can be applied to standardised returns $SR_{i,t}$ defined in Section 3.2 instead of returns $R_{i,t}$. Then, due to standardisation (3) and application of constant mean model abnormal standardised returns ASR_{it} computed similarly to (2) are equal to standardised returns AR_{it} divided by corresponding standard deviation of 5-minute returns:

$$ASR_{it} = \frac{AR_{it}}{S_i}, \tag{9}$$

where S_i is the standard deviation of 5-minute log returns of the WIG20 from the continuous trading phase of a session from 40-day window around the day of i-th announcement. Additionally, application of standardised returns instead of returns in the Kolari-Pynnönen test gives the same value of test statistic t_{grank}. The more specific applications of described methodology are also given in Gurgul and Suliga (2019).

In the analysis presented in the following section we use event study to test the significance of the announcements. However, the strength of the impact is described by average of abnormal returns $\overline{AR}_t$ or the average of abnormal standardised returns $\overline{ASR}_t$ computed for given time t.

4. Empirical results

4.1. Reaction in the whole period

In the first step of the analysis, we study the reaction of the WIG20 5-minute returns in the whole period 2004-2019. This will provide a background for further, more detailed analysis and comparisons.

Table 1 presents the values of mean abnormal returns $\overline{AR}_t$ computed in the event window for bad (Panel A) or good (Panel B) unexpected news included in announcements of the US macroeconomic indicators described previously. Together with the values of $\overline{AR}_t$ we report results of the Kolari-Pynnönen generalised rank test.

In addition to the means for single indicators in Table 1, we also present the results of the event study analysis for all the bad and all the good news (rows "ALL").

Most of the significant means of abnormal returns are observed in the first five minutes just after a news release ($t = 1$). It indicates that there is an immediate reaction of investors on the WSE to publication of US macroeconomic data. This significant change in the WIG20 is implied by most of the announcements. Only bad news included in announced values of the EHS and the PPI and good news about the DGO, the NHS, the PPI do not induce significant changes in the WIG20 at least at the 10% level.

The strongest reaction, measured by values of the mean $\overline{AR}_1$, is observed particularly after bad news regarding the DGO, the NFP and the GDP were released. In this case, the value of the WIG20 fell in the first five minutes after the announcement by about -0.12% additionally. In the case of good news the announcements of the NPF implied the strongest changes of the WIG20 ($\overline{AR}_1 \approx 0.166\%$). The analysis of the Kolari-Pynnönen test results indicates that the significant changes in the WIG20 are mainly limited to first five minutes after a news release. It confirms that the reaction of investors on the WSE is immediate and only lasts for a very short time.

When we compare the values of $\overline{AR}_1$ after good news and the values of $-\overline{AR}_1$ after bad news it turns out that for most of the indicators they are very close. An additional comparison of the strength of the changes of the WIG20 after different kinds of news shows no reaction asymmetry. To be more precise, for each of the indicators the Mann-Whitney test confirms that there is no significant difference between distributions of AR_1 after good news and the distribution of $-AR_1$ after bad news. These observations do confirm previous results of Gurgul and Wójtowicz (2014).

Table 1. Mean abnormal returns (in %) of the WIG20 in the event window for bad and good news from the US economy

Indicator	Number of events	$t = -1$	$t = 0$	$t = 1$	$t = 2$	$t = 3$
Panel A: bad news						
CCI	50	-0.028	-0.004	-0.086**	-0.011	0.011
CPI	49	-0.007	-0.012	-0.102*	-0.005	-0.003
DGO	91	-0.010	0.000	-0.123***	0.008	-0.003
EHS	54	-0.014	-0.012	-0.041	0.014	0.015
HS	98	0.009*	0.001	-0.024*	-0.004	0.032
IP	90	0.009	0.005	-0.053***	-0.002	0.010
ISM	35	-0.008	-0.003	-0.078**	0.044**	0.036
IJC	360	0.005	0.002	-0.051***	0.004	0.004
NFP	97	-0.009	0.005	-0.123**	-0.002	0.016
NHS	51	-0.017	0.025**	-0.016**	-0.019	0.002
PPI	83	0.012	-0.017	-0.004	0.008	0.017
GDP	80	0.022**	0.015*	-0.124***	-0.031	0.010
RS	92	0.001	-0.010	-0.079**	0.018	0.003
ALL	859	-0.002	0.000	-0.068***	0.000	0.013**

Table 1. Mean abnormal returns (in %) of the WIG20 in the event window for bad and good news from the US economy (cont.)

Indicator	Number of events	$t = -1$	$t = 0$	$t = 1$	$t = 2$	$t = 3$
Panel B: good news						
CCI	52	0.017*	-0.007	0.037**	0.000	-0.009
CPI	60	0.010	-0.003	0.060*	0.014	0.022
DGO	85	-0.004	0.006	0.063	-0.005	-0.013*
EHS	47	-0.002	0.007	0.035**	-0.015	0.001
HS	85	0.009	0.004	0.059***	0.019	-0.009
IP	73	0.018	0.027***	0.053**	0.021	-0.015
ISM	51	-0.022	-0.012	0.084**	-0.012	-0.013
IJC	391	-0.004	-0.007	0.048***	0.009	-0.007
NFP	78	0.006	0.013	0.166***	-0.007	0.005
NHS	50	-0.008	0.019	0.046	0.011	-0.029
PPI	78	0.007	0.000	0.008	0.021	-0.001
GDP	67	0.000	0.005	0.096**	-0.002	-0.026
RS	76	0.002	0.014*	0.144***	-0.002	0.002
ALL	806	0.002	0.005***	0.071***	0.003	-0.007*

*, **, *** indicate significance of a mean at 10%, 5% and 1%, respectively, resulting from the Kolari-Pynnönen rank test.

Source: Own work.

4.2. Changes in reaction strength

The main part of the empirical study in this paper is dealing with changes in the strength of investors' reaction to US macroeconomic news in the last 15 years. To do this we compare the results of event study analysis carried out in various sub-periods. These sub-periods should be long enough to include a suitable number of macroeconomic announcements. On the other hand, these sub-periods should be as short as possible to give more accurate results. Finally, as a compromise, we perform an event study analysis in 5-year windows that are shifted every quarter. The first of these windows starts in January 2004 and ends in December 2008, while the last window is a little shorter and begins in October 2014 and ends in July 2019.

As previously mentioned, a correct comparison of the strength of the WIG20 changes after news announcements in various sub-periods may be biased by changes in the volatility. To overcome this problem we will also consider the standardised 5-minute returns $SR_{i,t}$ defined in Section 3.2. We note once again that the application of $SR_{i,t}$ instead of the returns does not change the results of the Kolari-Pynnönen test.

The procedure described above is flexible enough to provide us with an appropriate description of changes in the reaction to US macroeconomic news announcements. In addition to that, due to the symmetry of the reaction of the WIG20 and to increase the number of events in each window we consider both types of unexpected news (good and bad news) together. To do this, we multiply the abnormal returns (and abnormal standardised returns) corresponding to bad news by -1. As a result, all the abnormal results should move in the same direction after news announcements.

Table 2. Means of abnormal standardised returns of the WIG20 in the event window

	2004-2019		2004-2008		2009-2013		2014-2019		End of significant impact
	N	$\overline{ASR}_1$	N	$\overline{ASR}_1$	N	$\overline{ASR}_1$	N	$\overline{ASR}_1$	
CCI	102	0.574^{***}	-	-	35	0.917^{***}	60	0.315	Dec 2017
CPI	109	0.663^{***}	42	1.016^{***}	33	0.132	31	0.599	Sep 2012
DGO	176	0.747^{***}	58	1.076^{***}	57	1.064^{***}	56	0.144	Jun 2016
EHS	101	0.377^{*}	-	-	37	0.816^{**}	59	0.097	Dec 2015
HS	183	0.373^{***}	60	0.372^{*}	58	0.637^{***}	59	0.096^{*}	Mar 2018
IP	163	0.403^{***}	51	0.280^{*}	53	0.968^{***}	53	-0.140	Mar 2017
ISM	86	0.794^{***}	-	-	29	1.668^{***}	52	0.318	Jun 2017
IJC	751	0.416^{***}	243	0.379^{***}	243	0.765^{***}	242	0.171^{**}	Jul 2019
NFP	175	1.364^{***}	57	1.095^{*}	54	2.937^{***}	57	0.260	Jun 2017
NHS	101	0.335^{***}	-	-	35	0.636^{***}	60	0.035	Jun 2016
PPI	161	0.033	53	0.451	55	0.013	49	-0.343	Insignif. t
GDP	147	1.030^{***}	45	1.335^{***}	52	1.339^{**}	45	0.368^{*}	Sep 2017
RS	168	0.916^{***}	59	0.612^{***}	53	1.506^{***}	49	0.657	Dec 2017
ALL	1665	0.620^{***}	418	0.780^{***}	543	1.054^{***}	640	0.160^{**}	-

*, **, *** indicate significance of a mean at 10%, 5% and 1%, respectively, resulting from the Kolari-Pynnönen rank test.

Source: Own work.

Due to the large number of the results of the analysis in the sub-periods, we do not report here all of them. In Table 2, we present the results of the empirical study for three disjoint windows: January 2004–December 2008, January 2009–December 2013, and October 2014–July 2019. More precisely, we report only the values of means of abnormal standardised returns $\overline{ASR}_1$ in the first five minutes after macroeconomic news announcements are made. As a background, we also present the values of $\overline{ASR}_1$ computed for the whole period considered.

The comparison of the results in Table 2 clearly shows that the strongest reaction to macroeconomic news from the United States was observed in first two presented windows that include data from the period of the global financial crisis or from the period just after the crisis. However, for most of the indicators higher values of $\overline{ASR}_1$ are in the post-crisis period (2009-2013). In the case of NPF, $\overline{ASR}_1$ is close to 3 indicating that the average of abnormal returns in the first five minutes after NFP announcements was three times the standard deviation of usual 5-minute returns.

In addition, the results of the Kolari-Pynnönen test indicate a very strong and highly significant impact of US macroeconomic news in this post-crisis period.

The comparison of the last column of Table 2 with the rest of the results indicates a very strong decrease in the strength of impact of US macroeconomic data announcements on investors on the WSE in recent years. Only in a few cases, changes in the WIG20 implied by news under study are significant. Moreover, they are significant at most at the 5% level. Such a situation can also be observed when all the announcements are joined together. The insignificance of the reaction is accompanied by very low values of $\overline{ASR_1}$. For example, in the case of the NPF the average falls from 2.94 in 2009-2013 to insignificant value of 0.26 in 2014-2019.

To describe the changes in the reaction of the WIG20 to publications of US data during the whole period 2004-2019 more precisely we present the values of $\overline{ASR_1}$ computed for all events in each of the 5-year windows in Figure 2. As a comparison, we do also present the values of $\overline{ASR_1}$ from 2-year windows shifted also by one quarter. The results of the analysis in the 2-year window are more flexible and better describe the changes in the strength of the impact. However, it can be applied only to study the impact of announcements of all indicators together. In the case of a single indicator a 2-year window does not contain enough data to provide reliable results.

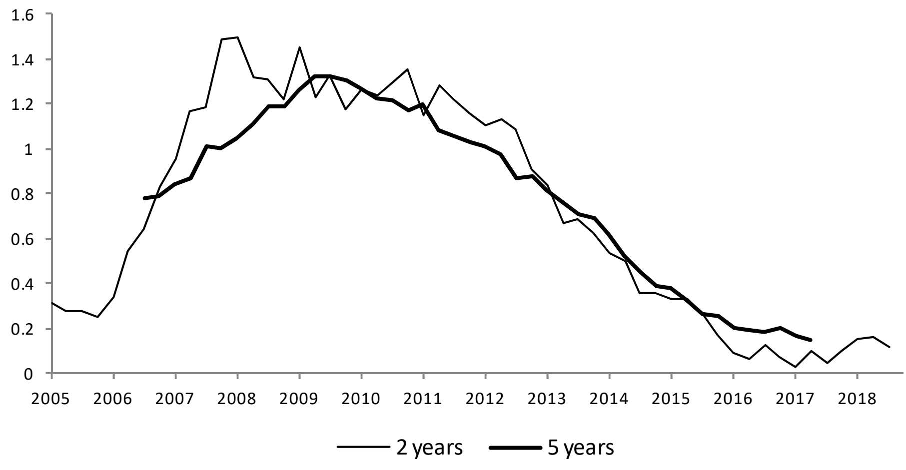

Figure 2. Averages of the abnormal standardised returns of the WIG20 over 2-year and 5-year windows

Source: Own work.

The results presented in Figure 2 confirm the conclusion already drawn based on Table 2. They clearly indicate a very strong impact of US macroeconomic news announcements on investors on the Warsaw Stock Exchange during the global financial crisis. In the pre-crisis periods, the values of $\overline{ASR}_1$ in a 2-year period are on the level of 0.3. However, when the crisis begins and the windows begin to include data from it, the average $\overline{ASR}_1$ begins to grow rapidly up to about 1.4. The strong impact of news from the United States lasts until late 2012 when the averages slowly begin to decrease. From 2015-2016 we may observe a stabilisation of $\overline{ASR}_1$ about 0.1. A similar behaviour of $\overline{ASR}_1$ may be observed in the case of 5-year window. However, in this case the changes are slower and much smoother.

Despite this observed decreasing tendency in $\overline{ASR}_1$ it should be pointed out that according to the results of the Kolari-Pynnönen test even in the last window the changes in the WIG20 induced by US macroeconomic news are significant. In fact, they are significant in every 5-year window. However, in some cases it is probably due to the large number of events in the cluster increasing the power of the test.

The last column of Table 2 shows that changes of the WIG20 only after IJC announcements remain significant during the whole period. The rest of indicators become insignificant earlier. For example, the NFP announcements, which showed the strongest impact, are significant until July 2012–June 2017.

5. Conclusions

In this paper, we analyse the changes in the impact of US macroeconomic news on investors on the Warsaw Stock Exchange. We examine the behaviour of 5-minute returns of the WIG20 in a short period after the announcements of 13 macroeconomic indicators describing the US economy were made. These indicators characterise inflation, industrial production, retail sales, the housing market, the labour market, and the GDP, among other things. Based on intraday returns from a 15-year period from January 2004 to July 2019 we are able to compare the strength of the impact of US macroeconomic news on the WIG20.

When the whole period is taken into account, the WIG20 reacts significantly to announcements of most of the indicators considered. This reaction is immediate and it is usually limited to the first 5-minute returns. The strongest impact is observed after NFP announcements.

The analysis in sub-periods leads to the conclusion that, in general, US macroeconomic news announcements induced the highest averages of abnormal returns during the global financial crisis (2007-2009) and in the first few years after the crisis. In later years, the impact of information from the United States was notably weaker. This change in the impact of US macroeconomic data was probably caused by the end of the crisis and by stabilising the economic situation in the United States. Additionally, new crises in various parts of the world attracted the attention of investors.

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Agu-Eghwerido distribution, regression model and applications

Friday Ikechukwu Agu[1], **Joseph Thomas Eghwerido**[2]

ABSTRACT

Modelling lifetime data with simple mathematical representations and an ease in obtaining the parameter estimate of survival models are crucial quests pursued by survival researchers. In this paper, we derived and introduced a one-parameter distribution called the Agu-Eghwerido (AGUE) distribution with its simple mathematical representation. The regression model of the AGUE distribution was also presented. Several basic properties of the new distribution, such as reliability measures, mean residual function, median, moment generating function, skewness, kurtosis, coefficient of variation, and index of dispersion, were derived. The estimation of the proposed distribution parameter was based on the maximum likelihood estimation method. The real-life applications of the distribution were illustrated using two real lifetime negatively and positively skewed data sets. The new distribution provides a better fit than the Pranav, exponential, and Lindley distributions for the data sets. The simulation results showed that the increase in parameter values decreases the mean squared error value. Similarly, the mean estimate tends towards the true parameter value as the sample sizes increase.

Key words: AGUE distribution, AGUE regression model, moment generating function, means residual function, hazard rate function, survival rate function.

1. Introduction

Introducing one-parameter distributions is a continuous concern for distribution theory and survival researchers. Thus, the researchers desire to introduce mathematically tractable and flexible lifetime probability models which can represent the random behaviour of real-world lifetime situations without difficulty.

In the statistical literature, many one-parameter, as well as two or more parameters probabilistic models have been introduced by researchers for modelling lifetime situations. Some of these probability models provide good results for various life situations. However, some of these developed models do not give good results for real-life scenarios. This might be the case arising from physical sciences, medical sciences, biological sciences, agricultural science, engineering among others. Similarly, some of these models require quite a complex and time-consuming algorithm for their parameter estimation.

Lindley (1958) introduced a parameter Lindley distribution. Ghitany, Atieh, and Nadarajah (2008) applied the mathematical treatment to a parameter Lindley distribution. Shanker

[1]Department of Statistics, University of Calabar, Calabar, Nigeria, E-mail: epsilomanalyst@gmail.com. ORCID: https://orcid.org/0000-0002-2367-4732.

[2]Department of Statistics, Federal University of Petroleum Resources, Effurun, Delta State, Nigeria. E-mail: eghwerido.joseph@fupre.edu.ng. ORCID: https://orcid.org/0000-0001-8986-753X.

(2015a, 2016) proposed one-parameter Akash and Aradhana distributions. Shanker and Shukla (2017) proposed one-parameter Ishita distribution. Shanker (2016) proposed one-parameter Sujatha distribution. Shukla (2018) proposed one-parameter Pranav distribution. Odom and Ijomah (2019) proposed one-parameter Odoma distribution. However, their regression models were not explored. These distributions yielded good fit over Lindley and exponential distributions for various data sets. Furthermore, the Pranav distribution by Shukla was applied on three real lifetime data sets and provides a better fit to the data sets than Lindley, Sujatha, Akash, and Ishita distributions respectively. However, the Odoma distribution was applied to strength data of glass of the aircraft window and provides a better fit than Pranav, Sujatha, Aradhana, Akash, Lindley, and exponential distributions respectively. The mathematical properties of these probability models, as well as parameter estimation, has been studied. Most of these probability models provide a good fit in some real-life situations and performed poorly in others. Due to the shortfall of some of these models, for instance the constant hazard rate of the exponential distribution could limit its wider applications to increasing hazard rate lifetime situations. Hence, some of these distributions have been extended, generalized, and applied in the literature by the addition of an extra parameter by researchers including Zakerzadeh and Dolati (2009) generalized Lindley distribution. Nadarajah et al. (2011) extended exponential distribution. Gómez and Calderín (2011) derived the discrete Lindley distribution, properties, and applications. Bakouch et al. (2012) obtained exponentiated exponential binomial distribution. Shanker and Mishra (2013) obtained a sized-biased Quasi Poisson-Lindley distribution. Agu and Onwukwe (2019) proposed exponentiated Laplace distribution as the extension of the Laplace distribution. Eghwerido et al. (2020) proposed the alpha power Gompertz distribution. Shanker and Amanuel (2013) obtained a new quasi Lindley distribution. Ghitany et al. (2013) proposed the Power Lindley distribution. Merovci (2013) extends the Rayleigh distribution. Agu and Runyi (2018) studied the goodness of fit tests for normal distribution. Warahena and Pararai (2014) proposed the generalized power Lindley distribution. Oluyede and Yang (2014) generalized the inverse Weibull distribution among others. However, the extensions do not guarantee high confidence in the model reliability. These extensions could lead to complex mathematical representation, complex and time-consuming algorithm, and difficulty in the estimation of parameters resulting in unreliable results.

In this respect, this paper is motivated to introduce a heavy-tailed simple mathematically structured one-parameter probability model with non-decreasing survival and hazard functions, derive its regression model, require less time and simple algorithm, and ease in the parameter estimation based on the test statistics performance results obtained from real-life scenarios that are more reliable. The proposed model was generated by making use of gamma and exponential distributions because of their inherent uniform base and constant failure rate respectively. However, in statistical modelling, the Weibull, Gompertz, exponential and Lindley models are very popularly used in modelling compared to the lognormal and gamma distributions. This is as a result of the inability to express their survival functions in a closed form. Although, the exponential function has one-parameter with a constant hazard rate, the Lindley has one-parameter monotonic decreasing hazard rate with a mixture component of a gamma model with a shape parameter 2 and exponential distribution. Its mixture proportions are $\frac{\lambda}{\lambda+1}$ and $\frac{1}{\lambda+1}$ for a scale parameter λ. Also, the Akash model tries to improve the Lindley distribution by using a mixture of exponential and gamma with a shape parameter 3 and mixing proportions $\frac{\lambda^2}{\lambda^2+2}$ and $\frac{2}{\lambda^2+1}$. Thus, to improve the flexibility of the Lindley, Pranav, exponential, and Akash distributions, this article introduces a parsimonious tractable distribution model called the AGUE distribution.

Let us consider two components mixture of one-parameter gamma distribution having shape parameter 3 and scale parameter λ and exponential distribution with scale parameter λ with their mixing proportions $\frac{\lambda^6}{\lambda^6+8}$ and $\frac{8}{\lambda^6+8}$ respectively. This idea would be used to develop the one-parameter distribution with simple mathematical representation.

The rest of the paper is structured as follows: Section 2 introduced the new distribution. Section 3 explored the reliability measures of the new distribution. Section 4 explored the moment generating function for the new distribution. Section 5 discussed the parameter estimation for the new distribution. Section 5.1 introduced the regression model for the new distribution. In Section 6, two real lifetime data set were adopted to illustrate the behaviour of the new distribution. Section 7 provided the concluding remarks.

2. The AGUE distribution

This section introduces the proposed model. It also, examines some potential properties. Let x be a random variable. Then, the new lifetime density function is introduced as

$$P(x) = \frac{\lambda^3}{\lambda^6+8}\left[\lambda^4 + 4x^2\right] exp(-\lambda x), \tag{1}$$

where $\lambda > 0,\ x > 0$. We would call the probability density function (pdf) of the one-parameter life time distribution in (1) "AGU-EGHWERIDO (AGUE) distribution". See the proof of the density function in Appendix. It follows from (1) that

$$\frac{dP(x)}{dx} = \frac{\lambda^3}{\lambda^6+8}\left[8x - \lambda^5 - 4\lambda x^2\right] exp(-\lambda x).$$

For

1. $\lambda < 1$, $\frac{dP(x)}{dx} = 0$, this implies that $P(x)$ is maximized at x_0.
2. $\lambda \geq 1$, $\frac{dP(x)}{dx} \leq 0$, this implies that $P(x)$ is decreasing at x.

The cumulative distribution function (cdf) of the AGUE distribution in (1) is given as

$$\frac{dP(x)}{dx} = 1 - \frac{1}{\lambda^6+8}\left[\lambda^6 + 4\lambda^2 x^2 + 8(\lambda x + 1)\right] exp(-\lambda x), \tag{2}$$

where $\lambda > 0, x > 0$.

3. Reliability measures

In this section, some reliability properties of the AGUE distribution were examined and investigated.

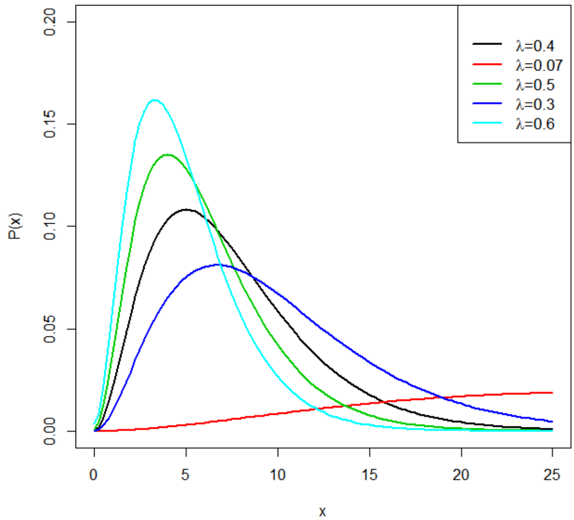

Figure 1: Pdf of AGUE distribution with different parameter cases

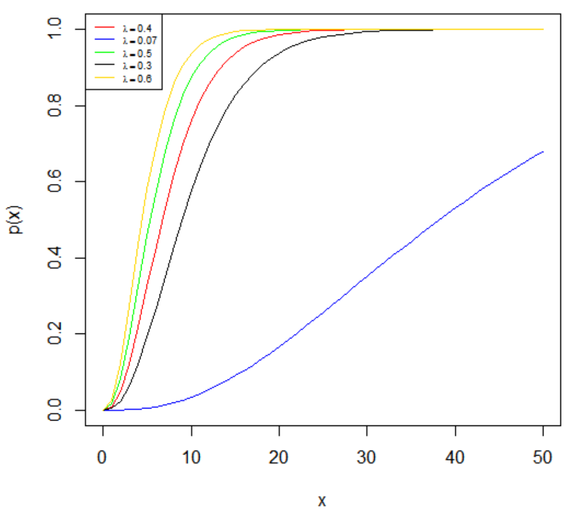

Figure 2: Cdf plot of the AGUE distribution with different parameter cases

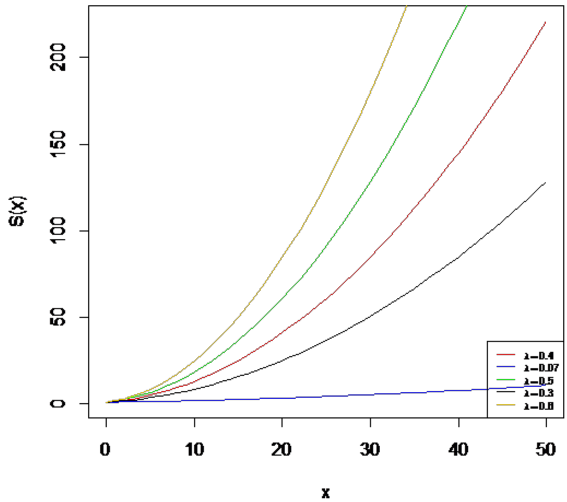

Figure 3: Survival plot of the AGUE distribution with different parameter cases

Let X be a continuous random variable with pdf $P(x)$ and cdf $p(x)$. Then, the survival rate function is expressed as

$$S(x) = 1 - \frac{1}{\lambda^6 + 8}\left[\lambda^6 + 4\lambda^2 x^2 + 8(\lambda x + 1)\right] exp(-\lambda x). \tag{3}$$

The survival rate plot for different parameter values is shown in Figure 3.

3.1. The failure rate function

The failure rate function is given as

$$H(x) = \lim_{\triangle x \longrightarrow 0} \frac{v(X < x + \triangle x | R > x)}{\triangle x} = \frac{P(x;\lambda)}{S(x;\lambda)}.$$

Thus, we have the AGUE failure rate function as

$$H(x) = \frac{\lambda^3\left[\lambda^4 + 4x^2\right]}{\lambda^6 + 4\lambda^2 x^2 + 8(\lambda x + 1)}. \tag{4}$$

However, for

1. $H(0) = \frac{\lambda^7}{\lambda^6 + 8}$.
2. $H(x)$ is an increasing function in x, λ and $\frac{\lambda^7}{\lambda^6 + 8} < H(x) < \lambda$.

The hazard rate plot for different parameter values is shown in Figure 4.

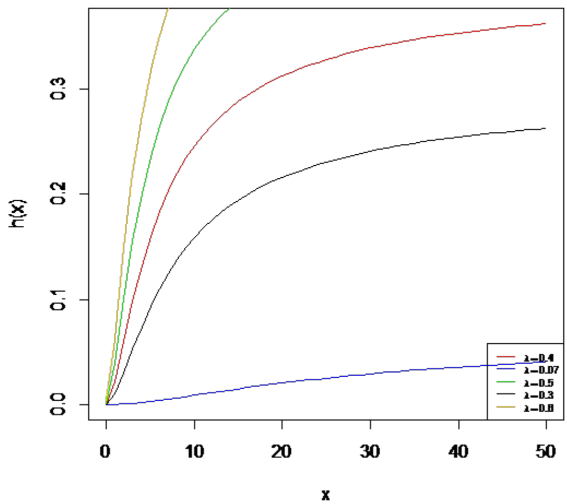

Figure 4: Hazard plot of the AGUE distribution with different parameter cases

3.2. The mean residual function

Let be a random variable with pdf and cdf defined in (1) and (9) respectively. Thus, the mean residual function of X is defined as

$$m(x) = E[X - x|X > x] = \frac{1}{p(x)} \lim_{c\to\infty} \int_x^c [1 - p(t)]dt.$$

However, for $1 - p(x) > 0$.

$$m(x) = \frac{\lambda^5 + 4\lambda x^2 + 16x + \frac{24}{\lambda}}{\lambda^6 + 4\lambda^2 x^2 + 8(\lambda x + 1)}, \tag{5}$$

Moreover, for

1. $m(0) = \frac{\lambda^5 + \frac{24}{\lambda}}{\lambda^6 + 8}$.
2. $m(x)$ is a decreasing function in x and λ.

The mean residual plot for different parameter values is shown in Figure 5.

3.3. The median

The median of a random variable X is expressed as

$$m_2(x) = \lim_{c\to\infty} \int_0^c |x - K|P(x)dx,$$

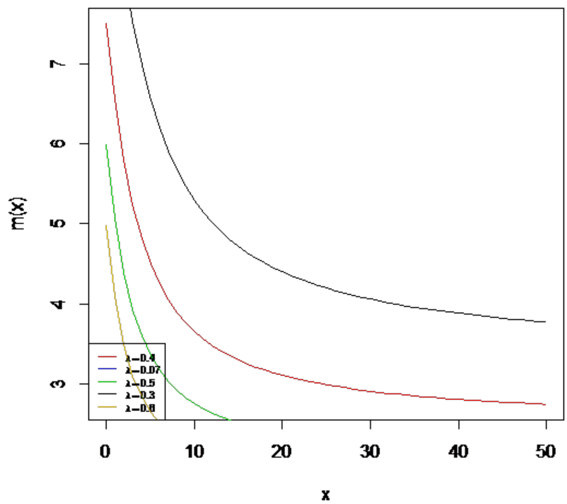

Figure 5: Mean residual plot of the AGUE distribution with different parameter cases

where $\mu = E(X)$ and K = Median (X). The above measures can be calculated using the relationship defined as

$$E(|X-k|) = \int_0^k (k-x)P(x)dx + \lim_{c\to\infty}(x-k)P(x)dx = 2\int_0^k (k-x)P(x)dx. \tag{6}$$

Thus, the median of the AGUE distribution can be given as

$$m_2(x) = 2\left[K - \frac{\lambda^5 + \frac{24}{\lambda}}{\lambda^6+8} - \frac{1}{\lambda^6+8}\left[exp(-\lambda K)(4\lambda K^2 + 16K + \lambda^5 + \frac{24}{\lambda}) + \lambda^6 K + 8K - \lambda^5 - \frac{24}{\lambda}\right]\right].$$

3.4. The quantile function

The quantile function of the AGUE distribution is obtained using the Lambert W function with W function defined as the solution to the equation $W(x)exp(W(x)) = x \in [-1,\infty)$, where W_0 is the principal branch of the Lambert function. The solution of the Lambert W function of $W(x)exp(W(x)) = x$, with $W_0(0) = 0$ and $W_0(x)$ increases with increase in x. Thus, for $x > 0$, $u^* = (\lambda^6+8)(1-u)(\lambda^6+4x^2\lambda^2+(9\lambda x+8))$, $\lambda < 0$, and $u \in (0,1)$. Thus, the $x_u = W_0(exp(u^*)) - \frac{\lambda^3-2}{\lambda}$.

A simulation is obtained using the quantile function obtained above. The values of the parameter are chosen as 0.5, 1.0 and 2.5 for sample size 5, 10, 50, 100, 150, 250, 350 and 500. The sample size is replicated 5000 times. Table 1 shows the results.

Table 1. Simulation results for AGUE distribution

n	Parameter	Average Estimate	Bias	MSE
5	λ =0.5	0.9950	0.0150	0.1929
	λ=1.0	1.2008	0.2008	0.5558
	λ=2.5	2.7462	0.5349	0.5071
10	λ =0.5	0.8516	0.0847	0.1077
	λ =1.0	1.1847	0.1838	0.4671
	λ =2.5	2.6838	0.1921	0.4014
50	λ =0.5	0.6664	0.0972	0.1005
	λ =1.0	1.0072	0.1583	0.3273
	λ =2.5	2.5949	0.1049	0.2702
100	λ =0.5	0.4754	0.1625	0.0071
	λ =1.0	1.0025	0.1348	0.1929
	λ =2.5	2.4712	0.0881	0.1052
150	λ =0.5	0.4544	0.1109	0.0045
	λ =1.0	1.0009	0.0181	0.0824
	λ =2.5	2.4519	0.0324	0.0841
250	λ =0.5	0.5013	0.2733	0.0024
	λ =1.0	1.0008	0.0112	0.0053
	λ =2.5	2.5030	0.0030	0.0070
350	λ =0.5	0.5002	0.4814	0.0015
	λ =1.0	1.0002	0.0088	0.0034
	λ =2.5	2.5003	0.0018	0.0055
500	λ =0.5	0.5001	0.4948	0.0011
	λ =1.0	1.0005	0.3651	0.0041
	λ =2.5	2.5001	0.2476	0.0006

In Table 1, increase in parameter values decreases the MSE. Also, the mean estimate tends to the true parameter value as the sample sizes increase.

4. Moment generating function

The AGUE moment generating function is expressed as

$$\begin{aligned} m_R(t) =& \frac{\lambda^3}{\lambda^6+8} \lim_{c\to\infty}\left[\lambda^6 \sum_{m=0}^{c} (\frac{t}{\lambda})^m + \frac{8}{\lambda} \sum_{m=0}^{c} \binom{m+2}{m} (\frac{t}{m})^m\right] \\ =& \lim_{c\to\infty}\left[\frac{t^m\left[\lambda^6+(m+2)(m+4)\right]}{\lambda^m(\lambda^6+8)}\right]. \end{aligned} \tag{7}$$

Thekth moment about origin μ_k' obtained as coefficient of $\frac{t^k}{k!}$ in $M_X(t)$ of the AGUE distribution is given by

$$\mu'_k = E(X^k) = \frac{k!\left[\lambda^6 + (k+2)(k+4)\right]}{\lambda^k(\lambda^6+8)} \quad k = 1,2,3,\cdots \tag{8}$$

In particular, we can obtain the first four moments of the AGUE distribution as

$$\mu'_1 = \frac{\lambda^6+15}{\lambda(\lambda^6+18)}, \ \mu'_2 = \frac{2(\lambda^6+24)}{\lambda^2(\lambda^6+8)}, \ \mu'_3 = \frac{6(\lambda^6+35)}{\lambda^3(\lambda^6+8)}, \ \textit{and} \ \mu'_4 = \frac{24(\lambda^6+48)}{\lambda^4(\lambda^6+8)}.$$

Using the relationship between the raw moment and the central moment or the moment about mean, we obtain the central of the AGUE distribution (1) as

$$\mu_k = E(X-\mu)^k = \sum_{m=0}^{k} \binom{k}{m} \mu'_m (-\mu)^{k-m}, \quad m = 0,1,2,\cdots,k.$$

In particular, we have

$$\mu_0 = 1, \quad \mu_1 = 0.$$

$$\mu_2 = \mu'_2 - \mu^2 = \frac{\lambda^{12}+34\lambda^6+159}{\lambda^2(\lambda^6+8)^2}.$$

$$\mu_3 = \mu'_3 - 3\mu\mu'_2 + 2\mu^3 = \frac{2\lambda^{18}+114\lambda^{12}+62\lambda^6+2910}{\lambda^3(\lambda^6+8)^3}.$$

$$\mu_4 = \mu'_4 + 6\mu^2\mu'_2 - 4\mu\mu'_3 - 3\mu^4 = \frac{9\lambda^{24}+1194\lambda^{18}+16002\lambda^{12}+31164\lambda^6+96963}{\lambda^4(\lambda^6+8)^4}.$$

The coefficient of skewness $\sqrt{\beta_1}$, coefficient of kurtosis β_2 , coefficient of variation $C.V$, and index of dispersion γ of the AGUE distribution (1) are obtained as

$$C.V = \frac{\sqrt{\lambda^{12}+34\lambda^6+159}}{\lambda^6+15}, \quad \sqrt{\beta_1} = \frac{2\lambda^{18}+114\lambda^{12}+62\lambda^6+2910}{(\lambda^{12}+34\lambda^6+159)^{\frac{3}{2}}}$$

$$\beta_2 = \frac{9\lambda^{24}+1194\lambda^{18}+16002\lambda^{12}+3116\lambda^6+96963}{(\lambda^{12}+34\lambda^6+159)^2}, \ \textit{and} \ \gamma = \frac{\lambda^{12}+34\lambda^6+159}{\lambda(\lambda^6+8)(\lambda^6+15)}.$$

Table 2 shows the variance, mean, skewness, kurtosis, coefficients of variation, and index of dispersion for different parameter values for the AGUE distribution. The results in Table 2 shown that the parameter values increases with decrease in the variance values.

Table 2. Coefficients of variation, mean, variance, skewness, kurtosis and index of dispersion for the AGUE distribution.

(λ)	μ	σ^2	CV	$\sqrt{\beta_1}$	β_2	γ
0.4	4.6864	15.5250	0.8408	1.4545	3.8292	3.1567
0.01	187.500	24843.750	0.8406	1.4514	3.8354	132.33
0.6	3.1165	6.8894	0.8422	1.4410	3.7669	2.0684
0.5	3.7466	9.9319	0.8412	1.4519	3.8110	2.4999
0.016	117.188	9704.59	0.8406	1.4514	3.8354	82.647
2	0.5486	0.3101	1.0151	1.9281	5.9972	4.0703

4.1. Stochastic Orderings

The comparative behaviour of continuous random variables can be evaluated using stochastic ordering.

A random variable X is said to be smaller than a random variable Y (Shaked 1994) if

$$X \leq_L Y \Rightarrow X \leq_{hr} \Rightarrow X \leq_m Y$$

$$\Downarrow$$

$$X \leq_{st} Y$$

- Hazard rate order $(X \leq_{hr} Y)$ if $P_x(x) \geq P_y(x)$ for all x
- Stochastic order $(X \leq_{st} Y)$ if $G_X(x) \geq G_Y(x)$ for all x
- Mean residual life order $(X \leq_m Y)$ if $m_X(x) \leq m_Y(x)$ for all x
- Likelihood ratio order $(X \leq_L Y)$ if $\frac{P_x(x)}{P_y(x)}$ decreases in x

Theorem 4.1 *Let X and Y follow the AGUE distribution with λ_1 and λ_2 respectively. If $\lambda_1 \geq \lambda_2$, then $X \leq_L Y$. Hence $X \leq_{hr} Y$, $X \leq_m Y$ and $X \leq_{st} Y$.*

Proof

The AGUE distribution will be ordered based on the strongest likelihood ratio ordering as established in Shaked (1994).

$$\frac{P_x(x;\lambda_1)}{P_y(x;\lambda_2)} = \frac{\frac{\lambda_1^3 exp(-\lambda_1 x)\left[\lambda_1^4+4x^2\right]}{(\lambda_1^6+8)}}{\frac{\lambda_2 exp(-\lambda_2 x)\left[\lambda_2^4+4x^2\right]}{(\lambda_2^6+8)}}. \tag{9}$$

$$\frac{d}{dx}\log\frac{P_x(x;\lambda_1)}{P_y(x;\lambda_2)} = \frac{-2(\lambda_1^4-\lambda_2^4)}{(\lambda_1^4+4x^2)(\lambda_2^4+4x^2)} - (\lambda_1-\lambda_2). \tag{10}$$

Thus, for $\lambda_1 > \lambda_2$, $\frac{d}{dx}\log\frac{P_x(x;\lambda_1)}{P_y(x;\lambda_2)} < 0 \Rightarrow X \leq_L Y$, for $\lambda_1 < \lambda_2$, $\frac{d}{dx}\log\frac{P_x(x;\lambda_1)}{P_y(x;\lambda_2)} > 0$ and if $\lambda_1 = \lambda_2$, $\frac{d}{dx}\log\frac{P_x(x;\lambda_1)}{P_y(x;\lambda_2)} = 0$. Therefore $X \leq_{hr} Y$,$X \leq_m Y$ and $X \leq_{st} Y$.

5. Parameter estimation of the AGUE distribution

Let $x_1, x_2, x_3, \cdots, x_k$ be a random sample of size k observations sampled from the AGUE distribution. Then, for $\bar{x}$, the sample mean, the log-likelihood function of the AGUE distribution can be derived as

$$InL = 3kIn\lambda - kIn(\lambda^6 + 8) + \sum_{m=1}^{k} In(\lambda^6 + 4x_m^2) - k\lambda \sum_{m=1}^{k} x_m. \tag{11}$$

Thus, taking the partial derivative, we have

$$\frac{dInL}{d\lambda} = \frac{3k}{\lambda} - \frac{k6\lambda^5}{(\lambda^6+8)} + \sum_{m=1}^{k}\left[\frac{6\lambda^5}{(\lambda^6+4x_m^2)}\right] - k\bar{x}. \tag{12}$$

The estimate of the parameter λ can be obtained by equating to zero. The Newton-Raphson algorithm Software like R, MATLAB, MAPLE and others are used to obtain the estimate.

5.1. The AGUE distribution regression model

In this section, we introduce the regression model for the AGUE distribution. In the literature, numerous researchers have introduced regression form of probability model intending to improve on the model flexibility and also make predictions easier using such a regression model such as the Bivariate exponentiated-exponential geometric regression model (Famoye, 2019), Exponentiated-exponential geometric regression model (Famoye and Carl, 2016), Bivariate Weibull regression model based on censored samples (Hanagal, 2006), Transmuted Burr Type X distribution regression model (Khan, King, and Hudson, 2019), and Transmuted Log-logistic regression model (Granzotto and Louzada, 2015), among others.

However, the Granzotto and Louzada (2015) approach is adopted for the AGUE model.

Consider Z to be a random variable with pdf of the AGUE distribution and $g(x) = \lambda$ as a parameter depending on the covariate vector $X = (1, x_m, \cdots, x_k)^T$, where $m = 1, 2, \cdots, k$ and $g(x) = \lambda = \theta_0 + \theta_m x_m + \cdots + \theta_k x_k$. Hence, the pdf of the AGUE distribution can be redefined as

$$P(z|g(x)) = \frac{(g(x))^3\left[(g(x))^4 + 4z^2\right]exp(-g(x)z)}{(g(x))^6+8}, \tag{13}$$

where $g(x)$ is a regression model. The corresponding survival $S(z|g(x))$ and hazard $h(z|g(x))$ rate functions at period z of (13) can be written as

$$S(z|g(x)) = \frac{\left[(g(x))^6 + 4(g(x)z)^2 + 8(g(x)z+1)\right]exp(-g(x)z)}{(g(x))^6+8}. \tag{14}$$

$$h(z|g(x)) = \frac{(g(x))^3\left[(g(x))^4 + 4z^2\right]}{(g(x))^6 + 4(g(x)z)^2 + 8(g(x)z+1)}. \tag{15}$$

Let $p_m, \cdots, p_k$ be a sample of size from the AGUE distribution, and $X_m = (1, x_{1m}, \cdots, x_{km})^T$, be an m^{th} vector of covariates, $m = 0, 1, \cdots, k$ and $g(x) = \lambda = \theta_0 + \theta_1 x_1 + \theta_2 x_2 + \cdots + \theta_k x_k$. Also, for convenience notation, let set $x_0 = 1$, then $g(x) = \theta X^T$, where $\theta = (\theta_0, \theta_1, \cdots, \theta_k) \in \mathbb{R}^{1\times(n+1)}$ and

$$X = \begin{pmatrix} x_0 \\ x_1 \\ . \\ . \\ . \\ x_k \end{pmatrix} \in \mathbb{R}^{1\times(n+1)}.$$

Thus, the log-likelihood function can be written as

$$L = InL(g(x)|p,x) = 3k\sum_{m=0}^{k} In\big((g(x))^6 + 8\big) + \sum_{m=0}^{k}\big((g(x))^6 + 4p_m^2\big) - k\sum_{m=0}^{k} g(x)p_m. \quad (16)$$

The maximum likelihood estimates of the parameters of $g(x)$, which maximizes (16), must satisfy the equations

$$\frac{dL}{d\theta_0} = \frac{3k}{\sum_{m=0}^{k} g(x)} - \frac{6k\sum_{m=0}^{k}(g(x))^5}{\sum_{m=0}^{k}\left[(g(x))^6 + 8\right]} + 6\sum_{m=0}^{k}(g(x))^5 - k\sum_{m=0}^{k} y_m = 0.$$

$$\frac{dL}{d\theta_m} = \frac{3k\sum_{m=0}^{k} x_m}{\sum_{m=0}^{k} g(x)} - \frac{6k\sum_{m=0}^{k} x_m(g(x))^5}{\sum_{m=0}^{k}\left[(g(x))^6 + 8\right]} + 6\sum_{m=0}^{k} x_m(g(x))^5 - k\sum_{m=0}^{k} x_m p_m = 0.$$

6. Real-life data applications

The numerical applications of the one-parameter AGUE distribution are demonstrated using two data sets.

Data set I is a data set report consisting of 63 observations of the strengths of 1.5cm glass fibers. The data set has previously been analyzed in Sharma et al.(2016), Oguntunde et al.(2017), Abdal-hameed et al. (2018), Eghwerido et al. (2021), Oguntunde et al.(2018), Eghwerido and Agu (2021), Eghwerido, Agu and Ibidoja (2021a) and Khaleel, Al-Noor and Abdal-Hameed (2020).

Data set II is a data set of about 346 nicotine measurements collected by the Federal Trade Commission. [http://www.ftc.gov/ reports/tobacco or https: // pw1.netcom.com/ rdavis2/ smoke. html.] used in Handique and Chakraborty (2016).

Tables 3a, 3b, 4a, and 4b present the parameter estimate and values of the test statistics for the fitted models on the data sets. In addition, the model parameter (Par.) and their corresponding Log-likelihood (LL), standard errors (Str. Error), and confidence intervals (CI) are presented. R-programming was used to obtain the results. However, in Tables 3a, 3b, 4a, and 4b, the AGUE model test statistics are the lowest among all fitted models for the data sets. Thus, the AGUE model is chosen as the best model among others for these data sets.

Figures 6, 7, 8, and 9 show the plots of the empirical estimated densities and density of the models considered for the data sets.

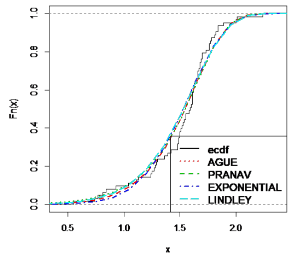

Figure 6: The plots of the estimated ecdf for the AGUE distribution for data set I

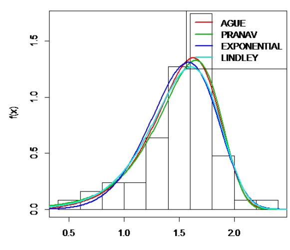

Figure 7: The plots of the estimated density of the AGUE distribution for data set

Table 3a. Parameter estimates of the strengths of 1.5 cm glass fibers data.

Model	Par.	Est.	Str. Error	LL	CI (95%)	
					Upper	Lower
AGUE	$\hat{\lambda}$	4.590	0.225	-237.634	5.031	4.149
Exponential	$\hat{\lambda}$	0.664	0.084	-88.830	0.829	0.499
Pranav	$\hat{\lambda}$	1.561	0.079	-90.481	1.716	1.406
Lindley	$\hat{\lambda}$	0.996	0.095	-81.278	1.182	0.800

Table 3b. The test statistics values of the strengths of 1.5 cm glass fibers data.

Model	AIC	CAIC	BIC	HQIC
AGUE	-473.268	-473.2024	-471.125	-472.425
Exponential	-179.661	-179.726	-181.804	-180.504
Pranav	-182.963	-183.028	-185.106	-183.806
Lindley	-164.557	-164.623	-166.700	-165.390

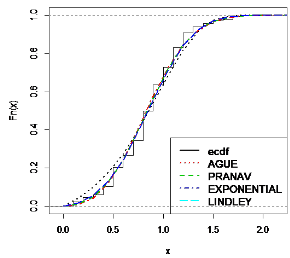

Figure 8: The plots of the estimated ecdf for the AGUE distribution for data set II

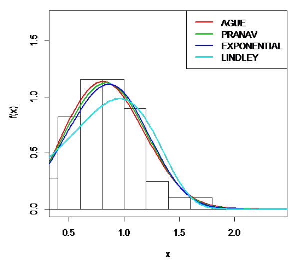

Figure 9: The plots of the estimated density of AGUE distribution for data set II

Table 4a. Parameter estimates of several brands of cigarettes data.

Model	Par.	Est.	Str. Error	LL	CI (95%)	
					Upper	Lower
AGUE	$\hat{\lambda}$	8.207	0.167	-2677.47	8.534	7.870
Exponential	$\hat{\lambda}$	1.173	0.063	-290.826	1.296	1.049
Pranav	$\hat{\lambda}$	2.056	0.052	-345.079	2.157	1.955
Lindley	$\hat{\lambda}$	1.620	0.068	-269.796	1.755	1.486

Table 4b. The test statistics values of several brands of cigarettes data.

Model	AIC	CAIC	BIC	HQIC
AGUE	-5352.941	-5352.929	-5349.094	-5351.409
Exponential	-583.651	-583.663	-587.498	-585.183
Pranav	-692.157	-692.169	-696.004	-693.689
Lindley	-541.593	-541.605	-545.439	-543.125

7. Conclusions

We introduced one-parameter distribution called the AGUE distribution with its mathematical representation and parameter estimation in this study. The regression model and basic statistical properties such as the index of dispersion and others were explored. The

AGUE parameter is estimated using the method of maximum likelihood estimation. The lifetime applications of the AGUE distribution was illustrated using two-lifetime data sets. The characteristic of the introduced model for larger sample size is examined via simulation study. The AGUE distribution has the lowest value of test statistics. Thus, it provides the best fit and more flexible than Pranav, exponential, and Lindley distributions for the data sets. Ultimately, the AGUE distribution can serve as an alternative model to Pranav, exponential, and Lindley distributions in the literature. A further research question is how the applications of the regression model of the AGUE distribution can be explored on real lifetime data.

7.1. Appendix: Probability density function

Let us write as Mood, Graybill and Boes (1974) stated that any function $P(.)$ is defined to be a pdf if and only if the following conditions are satisfied

1. $P(x) \geq 0$ for all x and
2. $\lim_{c\to\infty} \int_{-\infty}^{c} P(x)dx = 1$.

It is easy to see that the first property is satisfied for all $x > 0$. The second property is shown as follows. Firstly,

$$\begin{aligned}\lim_{c\to\infty}\int_0^c P(x)dx &= \int_0^\infty \frac{\lambda^3}{\lambda^6+8}\left[\lambda^4+4x^2\right]exp(-\lambda x)dx \\ &= \frac{\lambda^3}{\lambda^6+8}\lim_{c\to\infty}\left[\int_0^c \lambda^4 exp(-\lambda x)dx + \int_0^c 4x^2 exp(-\lambda x)dx\right].\end{aligned}$$

By performing integration by parts, we obtained

$$\frac{\lambda^3}{\lambda^6+8}\left[\lambda^3+\frac{8}{\lambda^3}\right] = 1.$$

Therefore, equation (1) is a pdf.

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Conflıct of ınterest

The authors state that there is no conflict of interest related to this study.

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A new extension of Odd Half-Cauchy Family of Distributions: properties and applications with regression modeling

Subrata Chakraburty[1], Morad Alizadeh[2], Laba Handique[3], Emrah Altun[4], G. G. Hamedani[5]

ABSTRACT

The paper proposes a new family of continuous distributions called the extended odd half Cauchy-G. It is based on the $T-X$ construction of Alzaatreh et al. (2013) by considering half Cauchy distribution for T and the exponentiated $G(x;\boldsymbol{\xi})$ as the distribution of X. Several particular cases are outlined and a number of important statistical characteristics of this family are investigated. Parameter estimation via several methods, including maximum likelihood, is discussed and followed up with simulation experiments aiming to asses their performances. Real life applications of modeling two data sets are presented to demonstrate the advantage of the proposed family of distributions over selected existing ones. Finally, a new regression model is proposed and its application in modeling data in the presence of covariates is presented.

Key words: $T-X$ method; regression; simulation; estimation

1. Introduction

Following the $T-X$ construction of Alzaatreh et al. (2013), Cordeiro et al. (2017) proposed a new generator of continuous probability distribution by considering Half-Cauchy for T and exponentiated G (Lehmann alternative-I) for X. They called the family generalized odd Half-Cauchy (GOHC-G$(\alpha,\boldsymbol{\xi})$) and investigated its properties and applications.
In the present paper we introduce a new ganerator called extended half Cauchy family of distribution following the same construction by considering exponentiated G (Lehmann alternative-II) for X and T following Half-Cauchy with probability density function (pdf) $q(t)=\frac{2}{\pi(1+t^2)}$, $t>0$, where $G(x;\boldsymbol{\xi})$ is the cumulative distribution function (cdf) of the baseline distribution with parameter vector $\boldsymbol{\xi}$. Now, following Alzaatreh et al. (2013) we define

[1]Department of Statistics, Dibrugarh University, Dibrugarh-786004, India. E-mail: subrata_stats@dibru.ac.in, ORCID: https://orcid.org/0000-0002-6405-1486.
[2]Department of Statistics, Faculty of Intelligent Systems Engineering and Data Science, Persian Gulf University, Bushehr, 75169, Iran. E-mail: m.alizadeh@pgu.ac.ir, ORCID: https://orcid.org/0000-0001-6638-2185.
[3]Department of Mathematics and Statistics, Golaghat Commerce College, Dibrugarh University, India. E-mail: handiquelaba@gmail.com, ORCID: https://orcid.org/0000-0001-9255-2918.
[4]Department of Mathematics, Bartin University, Bartin 74100, Turkey. E-mail: emrahaltun@bartin.edu.tr, ORCID: https://orcid.org/0000-0001-5065-2523.
[5]Department of Mathematical and Statistical Sciences Marquette University, USA, USA. E-mail: gholamhoss.hamedani@marquette.edu, ORCID: https://orcid.org/0000-0001-7976-1088.

the proposed extended odd Half-Cauchy-G with the cdf

$$F(x;\alpha,\boldsymbol{\xi}) = \int_0^{\frac{1-\bar{G}(x;\boldsymbol{\xi})^\alpha}{\bar{G}(x;\boldsymbol{\xi})^\alpha}} \frac{2}{\pi(1+t^2)}dt = \frac{2}{\pi}\arctan\left[\frac{1-\bar{G}(x;\boldsymbol{\xi})^\alpha}{\bar{G}(x;\boldsymbol{\xi})^\alpha}\right], \tag{1}$$

where $x \in \boldsymbol{R}$ and $\alpha > 0$ is a parameter. The proposed family is denoted as shortly EOHC-G$(\alpha,\boldsymbol{\xi})$.

The pdf corresponding to (1) is given by

$$f(x;\alpha,\boldsymbol{\xi}) = \frac{2\alpha\, g(x;\boldsymbol{\xi})\,\bar{G}(x;\boldsymbol{\xi})^{-\alpha-1}}{\pi\left[1+\left\{1-\bar{G}(x;\boldsymbol{\xi})^{-\alpha}\right\}^2\right]} = \frac{2\alpha\, g(x;\boldsymbol{\xi})\,\bar{G}(x;\boldsymbol{\xi})^{\alpha-1}}{\pi\left[\bar{G}(x;\boldsymbol{\xi})^{2\alpha}+\left\{1-\bar{G}(x;\boldsymbol{\xi})^{\alpha}\right\}^2\right]}, \tag{2}$$

where $g(x;\boldsymbol{\xi}) = \frac{d}{dx}G(x;\boldsymbol{\xi})$ is the baseline pdf. Henceforth, a random variable X with density function (2) is denoted by $X \sim$ EOHC-G$(\alpha,\boldsymbol{\xi})$.

It should be noted that for $\alpha = 1$ both GOHC-G$(\alpha,\boldsymbol{\xi})$ and EOHC-G$(\alpha,\boldsymbol{\xi})$ reduce to the *odd half-Cauchy* (*OHC*) family. Otherwise for $\alpha < 1$, EOHC-G$(\alpha,\boldsymbol{\xi}) >_{st}$ GOHC-G$(\alpha,\boldsymbol{\xi})$ and for $\alpha > 1$ EOHC-G$(\alpha,\boldsymbol{\xi}) <_{st}$ GOHC-G$(\alpha,\boldsymbol{\xi})$. As such the two families give rise to diffrent sets of distributions as special case for $\alpha \neq 1$.

For convenience we shall use $G(x) = G(x;\boldsymbol{\xi})$, $f(x) = f(x;\alpha,\boldsymbol{\xi})$, etc.

The EOHC-G family is related to some distributions as stated below. Let $X \sim$EOHC-G$(\alpha,\boldsymbol{\xi})$. Then, we have the following results.

1. If $Y = \bar{G}(X;\boldsymbol{\xi})^{-\alpha}$, then $F_Y(y) = \frac{2}{\pi}\arctan(y-1)$ and $f_Y(y) = \frac{2}{\pi}\frac{1}{1+(1-y)^2}, \quad y > 1.$

2. If $Y = \bar{G}(X;\boldsymbol{\xi})^{-\alpha} - 1$, then $Y \sim HC(0,1)$ with pdf $f_Y(y) = \frac{2}{\pi}\frac{1}{1+y^2}, \quad y > 0.$

3. If $Y = \bar{G}(X;\boldsymbol{\xi})^{\alpha}$, then $F_Y(y) = \frac{2}{\pi}\arctan(\frac{1-y}{y}), \quad 0 < y < 1.$

The hazard rate function (hrf) of X is

$$h(x;\alpha,\boldsymbol{\xi}) = \frac{2\alpha\, g(x;\boldsymbol{\xi})\,\bar{G}(x;\boldsymbol{\xi})^{\alpha-1}}{\pi\left[\bar{G}(x;\boldsymbol{\xi})^{2\alpha}+\left\{1-\bar{G}(x;\boldsymbol{\xi})^{\alpha}\right\}^2\right]\left[1-\frac{2}{\pi}\arctan\left\{\bar{G}(x;\boldsymbol{\xi})^{-\alpha}-1\right\}\right]}. \tag{3}$$

1.1. Useful relation with the exponentiated class

Based on the following result of Gradshtyn and Ryzhik (2007) page 61, for $x > 0$,

$$\arctan(x) = \frac{\pi}{2} - \sum_{i=0}^{\infty}\frac{(-1)^i}{(2i+1)x^{2i+1}}.$$

We can derive the following mixture representation of the cdf and pdf of EOHC-G:

$$
\begin{aligned}
F(x) &= \frac{2}{\pi}\arctan\left[\frac{1-\bar{G}(x)^{\alpha}}{\bar{G}(x)^{\alpha}}\right] \\
&= 1-\frac{2}{\pi}\sum_{i=0}^{\infty}\frac{(-1)^{i}\,\bar{G}(x)^{\alpha(2i+1)}}{(2i+1)(1-\bar{G}(x)^{\alpha})^{2i+1}} \\
&= 1-\frac{2}{\pi}\sum_{i,j=0}^{\infty}\frac{(-1)^{i+j}\binom{-2i-1}{j}\bar{G}(x)^{\alpha(2i+1)+\alpha j}}{2i+1}. \qquad (4)
\end{aligned}
$$

Hence $F(x)$ can be expressed as an infinite mixture of the exponentiated $G(x)$ (Lehmann alternative-II). Again

$$
\begin{aligned}
F(x) &= 1-\frac{2}{\pi}\sum_{i,j,k=0}^{\infty}\frac{(-1)^{i+j+k}\binom{-2i-1}{j}\binom{\alpha(2i+1)+\alpha j}{k}G(x)^{k}}{2i+1} \\
&= 1-\sum_{k=0}^{\infty}\gamma_k\,G(x)^{k}=\sum_{k=0}^{\infty}\nu_k\,G(x)^{k}, \qquad (5)
\end{aligned}
$$

where

$$
\gamma_k=\frac{2}{\pi}\sum_{i,j=0}^{\infty}\frac{(-1)^{i+j+k}\binom{-2i-1}{j}\binom{\alpha(2i+1)+\alpha j}{k}}{2i+1},
$$

$\nu_0=1-\gamma_0$ and $\nu_k=-\gamma_k$ for $k\geq 1$

Thus $F(x)$ is seen as an infinite mixture of $G(x)^k$, which is the exponentiated $G(x)$ distribution. Consequently, it is easy to verify that

$$
\begin{aligned}
f(x) &= \sum_{k=0}^{\infty}\nu_k(k+1)g(x)\,G(x)^{k} \\
&= \sum_{k=0}^{\infty}\nu_k h_{k+1}(x), \qquad (6)
\end{aligned}
$$

where $H_{k+1}(x)=G(x)^{k+1}$, $h_{k+1}(x)=\frac{d}{dx}H_{k+1}(x)=(k+1)\,G(x)^{k}\,g(x)$ and $h_1(x)=g(x)$.

The rest of the paper is organized as follows. A few special cases are presented in Section 2. Important properties like quantile function (qf), moments and moment generating function (mgf) are presented in Section 3. In Section 4 maximum likelihood estimation and its performance assessment via simulation is presented. Some other estimation methods and their performance through simulation is presented in Section 5. In Section 6, a new regression model is presented. In Section 7, data modelling applications with and without covariate are presented. The paper ends with a concluding section.

2. Sub-models of EOHC-G family

2.1. The EOHC-Burr XII (EOHC-BXII) distribution

Considering the BurrXII distribution (Burr, 1942) with pdf and cdf given by $g(x) = \lambda\beta x^{\lambda-1}(1+x^{\lambda})^{-\beta-1}$, $x>0$ and $G(x) = 1-(1+x^{\lambda})^{-\beta}$, $\lambda>0$ and $\beta>0$ the pdf and cdf of EOHC-BXII distribution are given respectively by

$$f^{\text{EOHC\_BXII}}(x;\alpha,\lambda,\beta) = \frac{2\alpha\lambda\beta \mathrm{x}^{\lambda-1}\left(1+x^{\lambda}\right)^{-\alpha\beta-1}}{\pi\left[\left(1+x^{\lambda}\right)^{-2\alpha\beta}+\left\{1-\left(1+x^{\lambda}\right)^{-\alpha\beta}\right\}^{2}\right]}, x>0, \tag{7}$$

$$F^{\text{EOHC\_BXII}}(x;\alpha,\lambda,\beta) = \frac{2}{\pi}\arctan\left[\left(1+x^{\lambda}\right)^{\alpha\beta}-1\right], x\geq 0. \tag{8}$$

Figure 1 shows the plots of the pdf and hazard of EOHC-BXII distribution for selected parameter values.

2.2. The EOHC-Fr (EOHC-Fr) distribution

Let $g(x)$ and $G(x)$ be the pdf and cdf of the Frechet distribution, given as $g(x) = \beta\theta^{\beta}x^{-\beta-1}\exp(-(\theta/x)^{\beta})$ and $G(x) = \exp(-(\theta/x)^{\beta})$, $x\geq 0$, $\beta>0, \theta>0$ respectively. Then, the pdf and cdf of the EOHC-Fr distribution are

$$f^{\text{EOHC\_F}}(x;\alpha,\beta,\theta) = \frac{2\alpha\beta\theta^{\beta}x^{-\beta-1}\exp(-(\theta/x)^{\beta})\left[1-\exp(-(\theta/x)^{\beta})\right]^{\alpha-1}}{\pi\left[\left[1-\exp(-(\theta/x)^{\beta})\right]^{2\alpha}+\left[1-\left[1-\exp(-(\theta/x)^{\beta})\right]^{\alpha}\right]\right]}, \tag{9}$$

and

$$F^{\text{EOHC\_F}}(x;\alpha,\beta,\theta) = \frac{2}{\pi}\arctan\left[\left[1-\exp(-(\theta/x)^{\beta})\right]^{-\alpha}-1\right], x\geq 0, \tag{10}$$

respectively.

Figure 2 shows the plots of pdf and hazartd of EOHC-Fr distribution for some selected parameters.

3. Properties of EOHC-G family

3.1. Quantile function and random sample generation

For a $U\sim$ Uniform(0,1) we can generate $X\sim$ EOHC-G by inverting (1) as

$$x = Q_G\left\{\frac{\left[1+\tan(\frac{\pi u}{2})\right]^{\frac{1}{\alpha}}-1}{\left[1+\tan(\frac{\pi u}{2})\right]^{\frac{1}{\alpha}}};\xi\right\}, \tag{11}$$

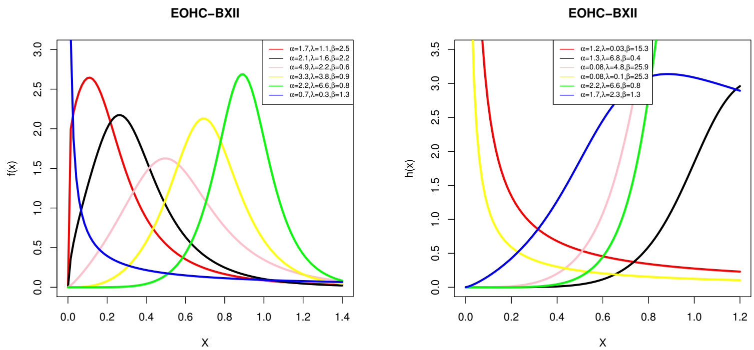

Figure 1: Plots of pdf and hazard for EOHC-BXII.

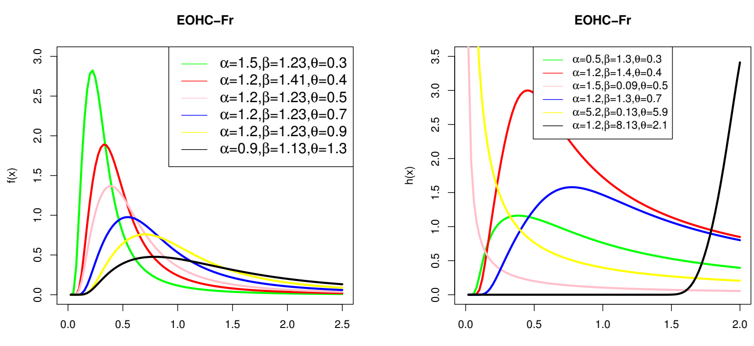

Figure 2: Plots of pdf and hazard for EOHC-Fr.

where $Q_G(\cdot) = G^{-1}(\cdot)$ is the baseline qf. The quantiles of the EOHC-G distributions for any baseline distribution can be obtained by (11). For instance, when $u = 1/2$, we obtain the median of the baseline distribution. Additionally, we can generate random variables from any baseline distribution using the given quantile function, in (11).

3.2. Moments

Let $Y_{k+1} \sim$exp-G$(k+1)$ with pdf $h_{k+1}(x) = (k+1)\,g(x)\,G(x)^k$. An expression for the nth moment of X can be obtained using equation (6) as

$$\mu_n' = E(X^n) = \sum_{k=0}^{\infty} \nu_k\, E(Y_{k+1}^n). \tag{12}$$

Another expression for μ_n' can be derived from equation (6) using the qf $Q_G(u)$ of the baseline distribution G as

$$\mu_n' = \sum_{k=0}^{\infty} (k+1)\,\nu_k\,\tau_{n,k}\ , \tag{13}$$

where $\tau_{n,k} = \int_{-\infty}^{\infty} x^n\, G(x)^k\, g(x)\, dx = \int_0^1 Q_G(u)^n\, u^k du.$ $\tau_{n,k}$ is the (n,k)th probability weighted moment (PWM) of G. Thus, the moments of the EOHC-G distribution can be expressed in terms of the PWMs of G.

For integer values of n, let $\mu_n^{'} = E(X^n)$ and $\mu = \mu_1^{'} = E(X)$, then one can also find the n^{th} central moment of the EOHC-BXII distribution as

$$\mu_n = E(X-\mu)^n = \sum_{i=0}^{n} \binom{n}{i} \mu_i^{'}(-\mu)^{n-i}. \tag{14}$$

Using the first four moments of the EOHC-BXII distribution, we obtain the skewness and kurtosis of the EOHC-BXII distribution. Figure 3 shows the behaviour of skewness and kurtosis of the EOHC-BXII distribution.

3.3. Moment generating function

Lemma 1: The condition for $F(x)$ to have a mgf is that $G(x)$ also has a mgf.

Proof: Let $m = \inf\{x|G(x) \geq 0.5\}$, then

$$\begin{aligned}
M_X(t) &= \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{tx}}\, \mathrm{f(x)dx} = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{tx}}\, \frac{2}{\pi}\, \frac{\mathrm{g(x)}\,\bar{\mathrm{G}}(\mathrm{x})^{\alpha-1}}{\bar{\mathrm{G}}(\mathrm{x})^{2\alpha} + [1-\bar{\mathrm{G}}(\mathrm{x})^{\alpha}]^2}\, \mathrm{dx} \\
&\leq \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{tx}}\, \frac{2}{\pi}\, \frac{\mathrm{g(x)}}{\bar{\mathrm{G}}(\mathrm{x})^{2\alpha} + [1-\bar{\mathrm{G}}(\mathrm{x})^{\alpha}]^2}\, \mathrm{dx} \\
&= \int_{-\infty}^{m} \mathrm{e}^{\mathrm{tx}}\, \frac{2}{\pi}\, \frac{\mathrm{g(x)}}{\bar{\mathrm{G}}(\mathrm{x})^{2\alpha} + [1-\bar{\mathrm{G}}(\mathrm{x})^{\alpha}]^2}\, \mathrm{dx} + \int_{\mathrm{m}}^{\infty} \mathrm{e}^{\mathrm{tx}}\, \frac{2}{\pi}\, \frac{\mathrm{g(x)}}{\bar{\mathrm{G}}(\mathrm{x})^{2\alpha} + [1-\bar{\mathrm{G}}(\mathrm{x})^{\alpha}]^2}\, \mathrm{dx}.
\end{aligned}$$

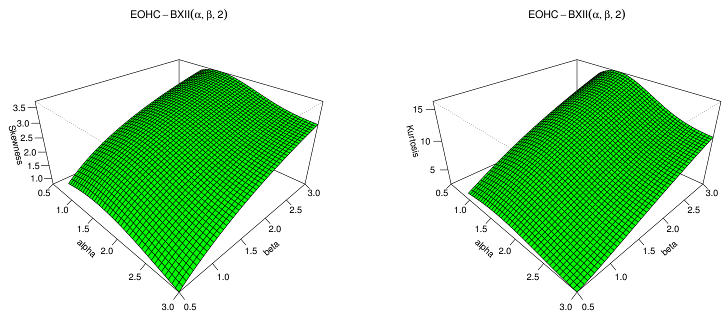

Figure 3: Skewness and Kurtosis for EOHC-BXII.

The second integral above is finite and the first integral is not greater than

$$\int_{m}^{\infty} \mathrm{e}^{\mathrm{t}\,\mathrm{x}}\, \frac{2}{\pi}\, \frac{\mathrm{g(x)}}{\bar{\mathrm{G}}(\mathrm{x})^{2\alpha}}\, \mathrm{dx}.$$

For $x < m$, we have $\bar{G}(x) \geq 0.5$, so that

$$\int_{m}^{\infty} \mathrm{e}^{\mathrm{t}\,\mathrm{x}}\, \frac{2}{\pi}\, \frac{\mathrm{g(x)}}{\bar{\mathrm{G}}(\mathrm{x})^{2\alpha}}\, \mathrm{dx} < \frac{2^{2\alpha+1}}{\pi} \int_{\mathrm{m}}^{\infty} \mathrm{e}^{\mathrm{t}\,\mathrm{x}}\, \mathrm{g(x)}\, \mathrm{dx} < \infty.$$

Thus, $M_X(t) < \infty$.

Corollary 1: Using (6), the mgf of $M(t) = E[\exp(tX)]$ of X is

$$M(t) = \sum_{k=0}^{\infty} \nu_k\, M_{k+1}(t), \tag{15}$$

where $M_{k+1}(t)$ is the mgf of $Y_{k+1} \sim exp-G(k+1)$. Alternatively, using equation (15) we can write

$$M(t) = \sum_{k=0}^{\infty} (k+1)\, \nu_k\, \rho(t,k), \tag{16}$$

where

$$\rho(t,k) = \int_{-\infty}^{\infty} \mathrm{e}^{t\,x}\, G(x)^{k}\, g(x) dx = \int_{0}^{1} \exp\{t\, Q_G(u)\}\, u^{k} du.$$

4. Maximum likelihood estimation

Let $x = (x_1, x_2, ..., x_r)$ be a random sample from EOHC-G family with parameter vector $\eta = (\alpha, \xi)$. The log-likelihood function is

$$\ell = \ell(\eta) = r \log \frac{2\alpha}{\pi} + \sum_{i=1}^{r} \log[g(x_i, \xi))] + (\alpha - 1) \sum_{i=1}^{r} \log[\bar{G}(x_i, \xi)]$$
$$- \sum_{i=1}^{r} \left[\log \bar{G}(x_i, \xi)^{2\alpha} - \{1 - \bar{G}(x_i, \xi)^{\alpha}\}^2\right].$$

The simultaneous solution of the partial derivatives of the log-likelihood gives the maximum likelihood estimators (MLEs) of the parameter of the EOHC-G family for a given baseline distribution. Unfortunately, it is not possible because of the non-linear structures of these derivatives. In this case, we prefer to maximize the log-likelihood function using the iterative algorithms. It can be done by statistical software such as R, Matlab or Python. Here, we use the R software to do this. The standard errors of the parameters are obtained based on the observed information matrix.

4.1. Performance evaluation of MLE

The MLEs of the parameters of the EOHC-BXII distribution are investigated based on the simulation study. The selected true parameter values are $(\alpha, \lambda, \beta) = (1.5, 2, 1)$. The used sample size is from $n = 20$ to $n = 100$. The simulation is replicated $r = 200$ times. The MLEs are obtained as $(\widehat{\alpha}_i, \widehat{\lambda}_i, \widehat{\beta}_i)$. We compute the biases and mean squared errors for each sample size by using the below equations

$$Bias_r\widehat{\theta} = \frac{1}{r}\sum_{i=1}^{r}(\widehat{\theta}_i - \theta_i) \,\text{and}\, MSE_r\widehat{\theta} = \frac{1}{r}\sum_{i=1}^{r}(\widehat{\theta}_i - \theta_i)^2, \text{for}\, \theta = (\alpha, \lambda, \beta).$$

The simulation results are plotted in Figures 4 and 5, which shows that the biases and mean square errors are near the zero for all parameters. These results confirms that the MLEs of the parameters of the EOHC-BXII distributions are unbiased and consistent.

5. The other estimation methods

Several estimation methods can be used to estimate the unknown model parameters. Here, we focus on four different estimation methods. These are briefly summarized in the rest of this section. See Dey et al. (2018) for detailed information on these estimation methods. Note that, $\{t_{i:n}; i = 1, 2, ..., n\}$ are order statistics and F is the distribution function of EOHC-BXII.

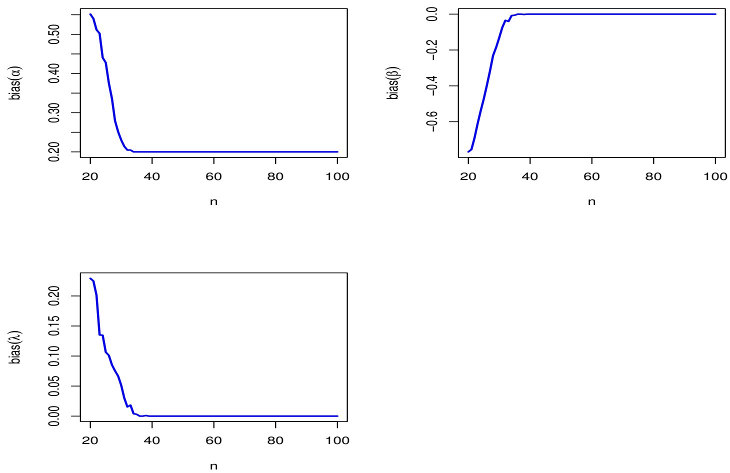

Figure 4: Bias of $\hat{\alpha}, \hat{\beta}, \hat{\lambda}$ versus r for EOHC-BXII when $(\alpha, \beta, \lambda) = (1.5, 1, 2)$.

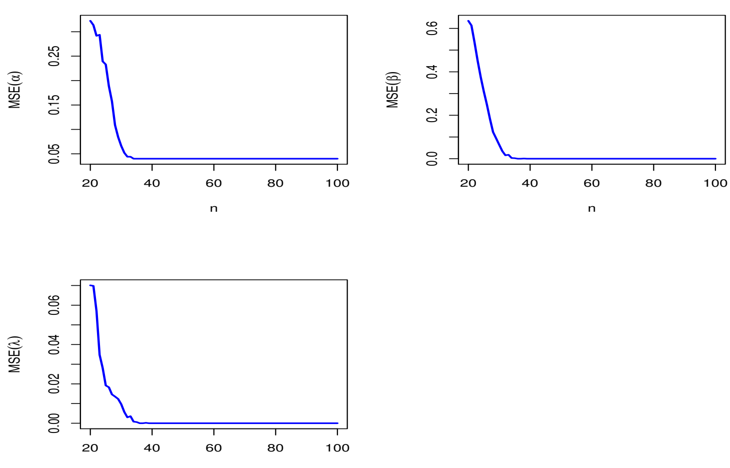

Figure 5: MSE of $\hat{\alpha}, \hat{\beta}, \hat{\lambda}$ versus r for EOHC-BXII when $(\alpha, \beta, \lambda) = (1.5, 1, 2)$.

5.1. Least square and weighted least square estimators

Swain et al. (1988) introduced the estimation methods for least square (LSE) and weighted least square estimators (WLSE). These estimators are easily obtained by minimizing the following functions:

$$S_{\text{LSE}}(\alpha,\xi)=\sum_{i=1}^{n}\left(F(t_{i:n};\alpha,\xi)-\frac{i}{n+1}\right)^2$$

and

$$S_{\text{WLSE}}(\alpha,\xi)=\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left(F(t_{i:n};\alpha,\xi)-\frac{i}{n+1}\right)^2$$

5.2. Cramér–von–Mises estimator

Choi and Bulgren (1968) introduced the method for the Cramér-von-Mises Estimator (CME), which is obtained by minimizing the following function

$$S_{\text{CME}}(\alpha,\xi)=\frac{1}{12n}+\sum_{i=1}^{n}\left(F(t_{i:n};\alpha,\xi)-\frac{2i-1}{2n}\right)^2.$$

5.3. Anderson-Darling and right-tailed Anderson-Darling estimators

Anderson-Darling estimators (ADEs) and right-tailed Anderson Darling estimators, shortly denoted as (RTADEs), were introduced by Anderson and Darling (1952) and Macdonald (1971), respectively. The ADEs for the EOHC-BXII distribution can be obtained by minimizing the below function

$$S_{\text{ADE}}(\alpha,\xi)=-n-\frac{1}{n}\sum_{i=1}^{n}(2i-1)\{\log F(t_{i:n};\alpha,\xi+\log\overline{F}(t_{i:n+1-i};\alpha,\xi\},$$

where $\overline{F}(\cdot)=1-F(\cdot)$.

5.4. Simulation

Again, EOHC-BXII distribution is used to investigate the difference between the estimation methods given in the above section. The true parameter vector is $(\alpha,\lambda,\beta)=(1.5,2,1)$ and the sample size is $n=30,35,\cdots,300$. The simulation is replicated $r=100$ times. The results are plotted in Figure 6.

The following results are obtained.

- For estimating α, AD method has the minimum amount of bias.
- For estimating λ, with small sample size, CVM method and for large sample size, AD has the minimum amount of bias.
- For estimating β, AD method has the minimum amonut of bias.

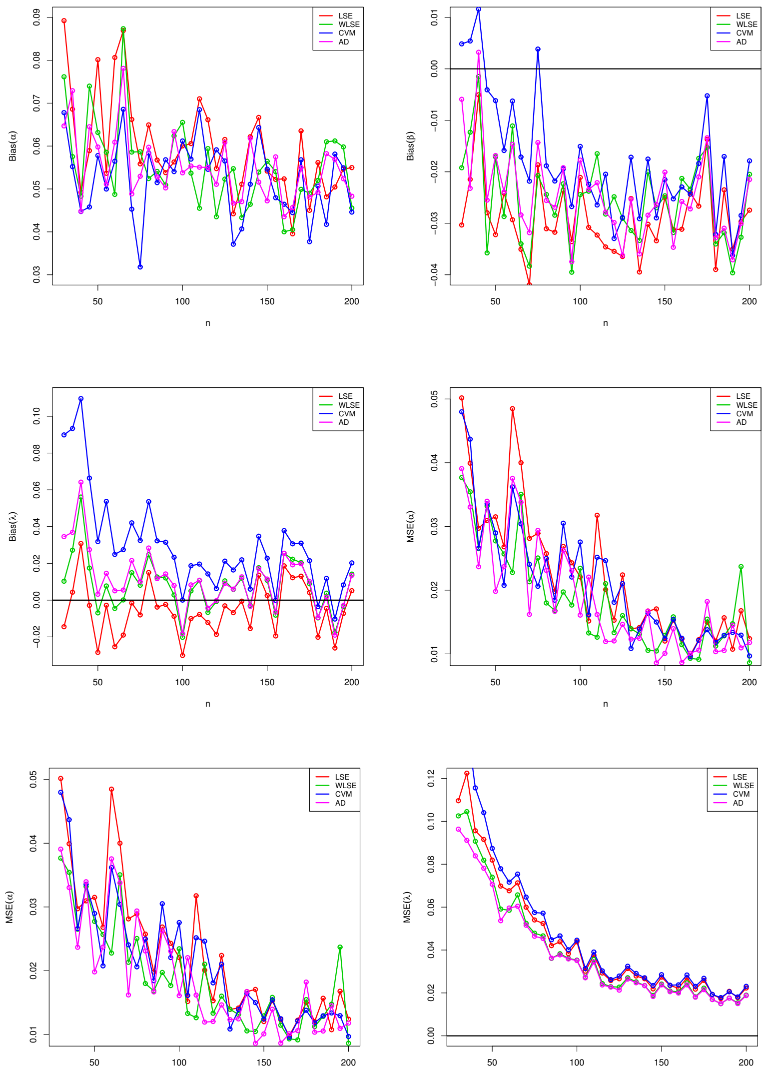

Figure 6: Bias of $\hat{\alpha}$ versus n when $\alpha = 1.5$; Bias of $\hat{\beta}$ versus n when $\beta = 1$; Bias of $\hat{\lambda}$ versus n when $\lambda = 2$; MSE of $\hat{\alpha}$ versus n when $\alpha = 1.5$; MSE of $\hat{\beta}$ versus n when $\beta = 1$; MSE of $\hat{\lambda}$ versus n when $\alpha = 2$

- For estimating α, with small sample size, CVM method and for large sample size, LSE has the minimum amount of MSE.
- For estimating λ, with small sample size, CVM method and for large sample size, AD has the minimum amount of MSE.
- For estimating β, with small sample size, CVM method and for large sample size, AD has the minimum amount of MSE.

6. The log-EOHC-Fr regression model

Consider the EOHC-Fr distribution with three parameters given in (9) and let X be a random variable with EOHC-Fr distribution. Using the transformation $Y = \log(X)$ and the re-parametrizations $\beta = 1/\sigma$ and $\theta = \exp(\mu)$, the pdf of Y is

$$f(y) = \frac{\frac{2\alpha}{\sigma}\exp\left\{-\left(\frac{y-\mu}{\sigma}\right)\right\}\exp\left\{-\exp\left\{-\left(\frac{y-\mu}{\sigma}\right)\right\}\right\}\left(1-\left[\exp\left\{-\exp\left\{-\left(\frac{y-\mu}{\sigma}\right)\right\}\right\}\right]\right)^{-\alpha-1}}{\pi\left[1+\left\{1-\left(1-\left[\exp\left\{-\exp\left\{-\left(\frac{y-\mu}{\sigma}\right)\right\}\right\}\right]\right)^{-\alpha}\right\}^{2}\right]}, \tag{17}$$

where $y \in \Re$. The parameter $\mu \in \Re$ represents the location of Y and the parameter $\sigma > 0$ is treated as a scale parameter and $\alpha > 0$ is the shape parameter. The density in (17) is referred as the Log-EOHC-Fr (LEOHC-Fr) distribution and denoted as $Y \sim$ LEOHC-Fr(α, μ, σ). The survival function of (17) is

$$S(y) = 1 - \frac{2}{\pi}\arctan\left[\frac{1-\left(1-\left[\exp\left\{-\exp\left\{-\left(\frac{y-\mu}{\sigma}\right)\right\}\right\}\right]\right)^{\alpha}}{\left(1-\left[\exp\left\{-\exp\left\{-\left(\frac{y-\mu}{\sigma}\right)\right\}\right\}\right]\right)^{\alpha}}\right], \tag{18}$$

Now, we introduce a new parametric regression model to analyze the lifetimes of individuals with covariates. To do this, the identity link function is used to link the covariates to location of the response variable. Let y_i be a response variable that follows the density in (17) and $\mathbf{v}_i^{\intercal} = (v_{i1}, \ldots, v_{ip})$ be a explanatory variable vector. We consider the below location-scale regression model

$$y_i = \mathbf{v}_i^{\intercal}\beta + \sigma z_i,\ i = 1, \ldots, n, \tag{19}$$

where y_i has density function (17), $\beta = (\beta_1, \ldots, \beta_p)^{\intercal}$, and $\sigma > 0$, $\alpha > 0$ are unknown parameters.

The unknown parameters of the LEOHC-Fr are obtained by means of MLE method. The response variable is defined as $y_i = \min\{\log(x_i), \log(c_i)\}$. The quantities $\log(x_i)$ and $\log(c_i)$ represent the log-lifetimes and log-censoring times, respectively. We define two sets to represents the log-lifetimes and log-censoring times. These are F and C. The set F contains the log-lifetimes and C contains the log-censoring times. The general equation for the log-likelihood function on the model in (19) is given by

$$l(\Theta) = \sum_{i \in F}\log[f(y_i)] + \sum_{i \in C}\log[S(y_i)]$$

where $\Theta = (\alpha, \sigma, \beta^{\intercal})$, $l_i(\tau) = \log[f(y_i)]$ and $l_i^{(c)}(\Theta) = \log[S(y_i)]$, $f(y_i)$. Replacing $f(y_i)$ with (17) and $S(y_i)$ with (18) in the above equation, the log-likelihood function of the LEOHC-Fr regression model is

$$\begin{aligned}\ell(\Theta) &= r\log\left(\tfrac{2\alpha}{\sigma}\right) - \sum_{i\in F} z_i - \sum_{i\in F}\exp(z_i)\\ &+(-\alpha-1)\sum_{i\in F}\log\left(1-[\exp\{-\exp\{-z_i\}\}]\right)\\ &-\sum_{i\in F}\log\pi\left[1+\left\{1-\left(1-[\exp\{-\exp\{-z_i\}\}]\right)^{-\alpha}\right\}^2\right]\\ &+\sum_{i\in C}\log\left(1-\tfrac{2}{\pi}\arctan\left[\tfrac{1-(1-[\exp\{-\exp\{-z_i\}\}])^{\alpha}}{(1-[\exp\{-\exp\{-z_i\}\}])^{\alpha}}\right]\right),\end{aligned} \tag{20}$$

where $z_i = (y_i - \mu_i)/\sigma$, and r is the number of uncensored observations. The MLE of the parameter vector, $\ell(\Theta)$ is obtained by direct maximization of (20) using the optim function of R software.

6.1. Residual analysis

Two types of the residuals are considered to study the residual analysis of the LEOHC-Fr regression model.

6.1.1 Martingale residual

The martingale residuals for LEOHC-Fr model is (see Fleming and Harrington, 1994, for details)

$$r_{M_i} = \begin{cases} 1+\log\left(1-\tfrac{2}{\pi}\arctan\left[\tfrac{1-(1-[\exp\{-\exp\{-z_i\}\}])^{\alpha}}{(1-[\exp\{-\exp\{-z_i\}\}])^{\alpha}}\right]\right) & \text{if } i\in F,\\ \log\left(1-\tfrac{2}{\pi}\arctan\left[\tfrac{1-(1-[\exp\{-\exp\{-z_i\}\}])^{\alpha}}{(1-[\exp\{-\exp\{-z_i\}\}])^{\alpha}}\right]\right) & \text{if } i\in C,\end{cases} \tag{21}$$

where $z_i = (y_i - \mu_i)/\sigma$.

6.1.2 Modified deviance residual

The interpretation of the martingale residuals is not easy since it is not symmetrically distributed around zero. Therefore, the modified deviance residual was proposed by Therneau et al. (1990) to remove the lack of the martingale residuals. The modified deviance residual for LEOHC-Fr model is

$$r_{D_i} = \begin{cases} sign(r_{M_i})\{-2[r_{M_i}+\log(1-r_{M_i})]\}^{1/2}, & \text{if } i\in F,\\ sign(r_{M_i})\{-2r_{M_i}\}^{1/2}, & \text{if } i\in C,\end{cases} \tag{22}$$

where r_{M_i} is the martingale residual.

7. Real life applications

7.1. Modelling without covariates

Two different real-life data sets are considered here to study the suitability of the distributions from EOHC-G(α,ξ) family in comparison with some existing distributions taking BurrXII distribution as the baseline distribution. We have used AIC (Akaike Information Criterion), BIC (Bayesian Information Criterion), CAIC (Consistent Akaike Information Criterion) and HQIC (Hannan-Quinn Information Criterion) for selecting the best model. The figures of fitted densities and the fitted cdf's presented alongside the corresponding observed histograms and ogives for visual checking.

Here we have considered Burr-XII as the baseline distribution in the EOHC-G family and compared it with the following important extensions of Burr-XII model including GOHC-BXII.

1. BXII distribution:

$$f(x) = \lambda\beta x^{\lambda-1}\left(1+x^{\lambda}\right)^{-\beta-1}, \lambda > 0, \beta > 0, x > 0.$$

2. MOBXII distribution (Arwa Y. Al-Saiari et al., 2014):

$$f(x) = \frac{\lambda\beta\alpha x^{\lambda-1}\left(1+x^{\lambda}\right)^{-\beta-1}}{\left[1-(1-\alpha)\left(1+x^{\lambda}\right)^{-\beta}\right]^{2}}, \ \alpha > 0, \lambda > 0, \beta > 0, x > 0.$$

3. TLBXII distribution (Hesham and Soha, 2017):

$$\begin{aligned} f(x) &= 2\alpha\lambda\beta x^{\lambda-1}\left(1+x^{\lambda}\right)^{-2\beta-1}\left[1-\left(1+x^{\alpha}\right)^{-2\beta}\right]^{\alpha-1}, \\ &\quad \alpha > 0, \lambda > 0, \beta > 0, x > 0. \end{aligned}$$

4. KwBXII distribution (Paranaiba et al., 2013):

$$\begin{aligned} f(x) &= \frac{ab\lambda\beta x^{\lambda-1}}{\left(1+x^{\lambda}\right)^{\beta+1}}\left[1-\left(1+x^{\lambda}\right)^{-\beta}\right]^{a-1} \times \\ &\quad \left\{1-\left[1-\left(1+x^{\lambda}\right)^{-\beta}\right]^{a}\right\}^{b-1}, \\ &\quad a > 0, b > 0, \lambda > 0, \beta > 0, x > 0. \end{aligned}$$

5. BBXII distribution (Paranaiba et al., 2011):

$$\begin{aligned} f(x) &= \frac{\lambda\beta}{B(a,b)}x^{\lambda-1}\left(1+x^{\lambda}\right)^{-\beta(b+1)}\left[1-\left(1+x^{\lambda}\right)^{-\beta}\right]^{a-1}, \\ &\quad a > 0, b > 0, \lambda > 0, \beta > 0, x > 0. \end{aligned}$$

6. BEBXII distribution (Mead, 2014):

$$f(x) = \frac{\lambda \beta \alpha}{B(a,b)} x^{\alpha-1} \left(1+x^{\lambda}\right)^{-\beta-1} \left[1-\left(1+x^{\lambda}\right)^{-\beta}\right]^{a\alpha-1} \times \left\{1-\left[1-\left(1+x^{\lambda}\right)^{-\beta}\right]^{\alpha}\right\}^{b-1},$$

$$a>0, b>0, \alpha>0, \lambda>0, \beta>0, x>0.$$

7. FBBXII distribution (Paranaiba et al., 2011):

$$f(x) = \frac{\lambda \beta \alpha^{-\lambda}}{B(a,b)} x^{\lambda-1} \left[1+\left(\frac{x}{\alpha}\right)^{\lambda}\right]^{-\beta b-1} \left\{1-\left[1+\left(\frac{x}{\alpha}\right)^{\lambda}\right]^{-\beta}\right\}^{a-1},$$

$$a>0, b>0, \alpha>0, \lambda>0, \beta>0, x>0.$$

8. FKwBXII distribution (Paranaiba et al., 2013):

$$f(x) = \frac{a b \lambda \beta x^{\lambda-1}}{\left[1+\left(\frac{x}{\alpha}\right)^{\lambda}\right]^{\beta+1}} \left[1-\left(1+\left(\frac{x}{\alpha}\right)^{\lambda}\right)^{-\beta}\right]^{a-1} \times \left\{1-\left[1-\left(1+\left(\frac{x}{\alpha}\right)^{\lambda}\right)^{-\beta}\right]^{a}\right\}^{b-1},$$

$$a>0, b>0, \alpha>0, \lambda>0, \beta>0, x>0.$$

9. GOHC-BXII distribution (Cordeiro et al., 2017):

$$f(x;\alpha,\lambda,\beta) = \frac{2\alpha\lambda\beta x^{\lambda-1}\left(1+x^{\lambda}\right)^{-\beta-1}\left[1-\left(1+x^{\lambda}\right)^{-\beta}\right]^{\alpha-1}}{\pi\left[\left[1-\left(1+x^{\lambda}\right)^{-\beta}\right]^{2\alpha\beta}+\left\{1-\left[1-\left(1+x^{\lambda}\right)^{-\beta}\right]^{\alpha}\right\}^{2}\right]},$$

$$\alpha>0, \lambda>0, \beta>0, x>0.$$

In the first application, we work with the survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli, observed and reported by Bjerkedal (1960). It is used also by Shibu and Irshad (2016). The second data set is obtained from Hinkley, (1977). It consists of thirty successive values of March precipitation (in inches) in Minneapolis/St Paul. We have presented the descriptive statistics of the data sets I, and II in Table 1. Findings of the data fitting in Tables 2, 3,4, 5. The total time on test (TTT) plot proposed by Aarset (1987) is drawn to get information about the shape of the hazard of a given data set. If the resulting shape of the TTT plot is a straight diagonal line, is of convex shape and concave shape then the corresponding hazard is constant, decreasing and increasing respectively. The TTT plots for the data sets considered here are presented in Figure 7 and

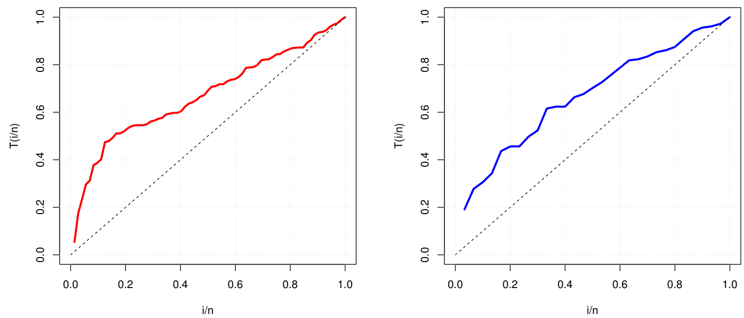

Figure 7: TTT Plots of data I and II from right to left

Table 1: Descriptive Statistics for the data set I, and II

n	Min.	1st Qu.	Third Qu.	Mean	Median	Max.	Variance	Skewness	Kurtosis
72	0.1	1.08	2.30	1.85	1.56	7.00	1.44	1.79	4.16
30	0.32	0.92	2.09	1.67	1.47	4.75	1.00	1.03	0.93

indicate that the all the three data sets are increasing hazard rate.

In Tables 2-5 the MLEs with standard errors of the parameters for all the fitted the models, their AIC, BIC, CAIC and HQIC for the data sets I and II are presented respectively. From these tables it is evident that for both the data sets considered here the EOHC-BXII distribution with lowest AIC, BIC, CAIC, HQIC turned out to be the best model. Moreover, the plots of estimated pdf against the observed histograms and the estimated cdf of EOHC-BXII against empirical cdfs in Figures 8 and 9 reveal that the proposed distribution provides closest fit to both the data sets. It may be mentioned that the proposed three parameter distribution has even outperformed the four and five parameter extensions considered here.

7.2. Modelling with covariates

Yousof et al. (2018) introduced the Log-odd log-logistic-Fréchet (LOLL-Fr) regression model and analysed the Stanford heart transplant data set. The same data set was also analyzed by Korkmaz et al. (2020). Now, we use the same data set to illustrate the importance of the LEOHC-Fr regression model and compare its performance with a regression model of Yousof et al. (2018), LOLL-Fr regression. The data set can be found in an R package, **p3state.msm**. The sample and censoring rate are 103 and 27%, respectively. The aim of the study is to analyze the survival times of individuals, say (y_i) with covariates: v_1- year of acceptance to the program; v_2- age of patient (in years); v_3- previous surgery status $(1 = yes, 0 = no)$; v_4-transplant indicator $(1 = yes, 0 = no)$. The model in (23) is considered and fitted by two models: LEOHC-Fr and LOLL-Fr regression models.

$$y_i = \boldsymbol{\beta_0} + \boldsymbol{\beta_1}\mathrm{v}_{i1} + \boldsymbol{\beta_2}\mathrm{v}_{i2} + \boldsymbol{\beta_3}\mathrm{v}_{i3} + \boldsymbol{\beta_4}\mathrm{v}_{i4} + \sigma z_i \,, \tag{23}$$

Table 2: MLEs, standard errors, confidence interval (in parentheses) for the data set I

Model	$\hat{\alpha}$	$\hat{a}$	$\hat{b}$	$\hat{\lambda}$	$\hat{\beta}$
BXII (λ,β)	···	···	···	3.102	0.465
				(0.538)	(0.077)
				(2.05,4.16)	(0.31,0.62)
MOBXII (α,λ,β)	8.989	···	···	2.259	1.533
	(4.587)			(0.864)	(0.907)
	(0,17.97)			(0.57,3.95)	(0,3.31)
TLBXII (α,λ,β)	1.796	···	···	2.393	0.488
	(0.915)			(0.907)	(0.244)
	(0.002,3.59)			(0.62,4.17)	(0,0.97)
KwBXII (a,b,λ,β)	···	14.105	7.424	0.525	2.274
		(10.805)	(11.850)	(0.279)	(0.990)
		(0,35.28)	(0,30.65)	(0,1.07)	(0.33,4.21)
BBXII (a,b,λ,β)	···	2.555	6.058	1.800	0.294
		(1.859)	(10.391)	(0.955)	(0.466)
		(0,6.28)	(0,26.42)	(0,3.67)	(0,1.21)
BEBXII $(\alpha,a,b,\lambda,\beta)$	0.572	1.876	2.991	1.780	1.341
	(0.325)	(0.094)	(1.731)	(0.702)	(0.816)
	(0,1.21)	(1.69,2.06)	(0,6.38)	(0.40,3.16)	(0,2.94)
FBBXII $(\alpha,a,b,\lambda,\beta)$	1.655	0.621	0.549	3.398	1.381
	(0.436)	(0.541)	(1.011)	(2.785)	(2.312)
	(0.81,4.48)	(0,1.68)	(0,2.53)	(0,8.86)	(0,5.91)
FKwBXII $(\alpha,a,b,\lambda,\beta)$	1.475	0.588	0.308	3.399	2.131
	(0.361)	(0.442)	(0.314)	(2.082)	(1.833)
	(0.76,2.18)	(0,1.42)	(0,0.92)	(0,7.47)	(0,5.72)
GOHCBXII (α,λ,β)	1.828	···	···	1.981	0.987
	(1.170)			(0.899)	(0.643)
	(0,4.12)			(0.21,3.74)	(0,2.24)
EOHCBXII (α,λ,β)	0.491	···	···	3.126	0.998
	(0.017)			(0.462)	(0.136)
	(0.45,0.52)			(2.22,4.03)	(0.73,1.26)

Table 3: AIC, BIC, CAIC, HQIC values for the data set I

Model	AIC	BIC	CAIC	HQIC
BXII (λ,β)	209.60	214.15	209.77	211.40
MOBXII (α,λ,β)	209.74	216.56	210.09	212.44
TLBXII (α,λ,β)	211.80	218.63	212.15	214.52
KwBXII (a,b,λ,β)	208.76	217.86	209.36	212.38
BBXII (a,b,λ,β)	210.44	219.54	211.03	214.06
BEBXII $(\alpha,a,b,\lambda,\beta)$	212.10	223.50	213.00	216.60
FBBXII $(\alpha,a,b,\lambda,\beta)$	206.80	218.20	207.71	211.30
FKwBXII $(\alpha,a,b,\lambda,\beta)$	206.50	217.90	207.41	211.00
GOHCBXII (α,λ,β)	206.66	213.50	207.01	209.36
EOHCBXII (α,λ,β)	205.96	212.80	206.31	208.66

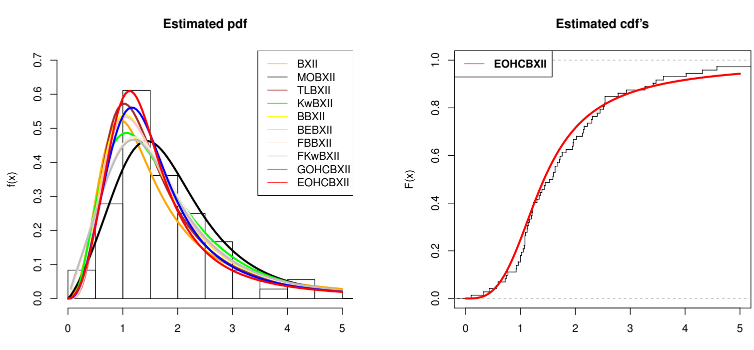

Figure 8: Plots of the observed histogram and estimated pdfs for the BXII, MOBXII, ,TL-BXII, KwBXII, BBXII,BEBXII, FBBXII, FKwBII and EOHCBXII and observed ogive and estimated cdf EOHCBXII for data set I from right to left

Table 4: MLEs, standard errors, confidence interval (in parentheses) for the data set II

Model	$\hat{\alpha}$	$\hat{a}$	$\hat{b}$	$\hat{\lambda}$	$\hat{\beta}$
BXII (λ,β)	···	···	···	3.255 (0.645) (1.99,4.52)	0.687 (0.137) (0.41,0.95)
MOBXII (α,λ,β)	5.205 (8.599) (0,22.05)	···	···	2.121 (1.041) (0.08,4.16)	1.666 (1.295) (0,4.20)
TLBXII (α,λ,β)	3.949 (5.685) (0,15.09)	···	···	1.420 (1.019) (0,3.42)	0.977 (0.904) (0,2.75)
KwBXII (a,b,λ,β)	···	34.377 (109.591) (0,249.17)	30.999 (63.785) (0,156.01)	0.292 (0.368) (0,1.01)	3.006 (4.298) (0,11.43)
BBXII (a,b,λ,β)	···	39.029 (6.983) (0,25.34)	15.796 (10.693) (0,36.75)	0.389 (0.028) (0.33,0.44)	1.645 (0.176) (1.30,1.98)
BEBXII $(\alpha,a,b,\lambda,\beta)$	1.000 (1.232) (0,3.41)	15.563 (22.109) (0,58.89)	7.818 (13.299) (0,33.88)	0.617 (0.421) (0,1.44)	1.388 (0.667) (0,2.69)
FBBXII $(\alpha,a,b,\lambda,\beta)$	26.693 (9.938) (7.21,46.17)	3.925 (4.717) (0,13.17)	58.407 (11.969) (34.94,81.86)	0.889 (0.521) (0,1.91)	0.925 (0.237) (0.46,1.39)
FKwBXII $(\alpha,a,b,\lambda,\beta)$	1.929 (0.668) (0.62,3.24)	0.612 (0.144) (0.32,0.89)	0.771 (0.823) (0,2.38)	3.344 (1.396) (0.61,6.08)	2.532 (1.077) (0.42,4.64)
GOHCBXII (α,λ,β)	4.641 (8.491) (0,21.28)	···	···	1.124 (0.951) (0,2.98)	2.177 (2.345) (0,6.77)
EOHCBXII (α,λ,β)	0.432 (0.174) (0.09,0.77)	···	···	2.730 (0.570) (1.61,3.84)	1.347 (0.098) (1.15,1.53)

Table 5: AIC, BIC, CAIC, HQIC values for the data set II

Model	AIC	BIC	CAIC	HQIC
BXII (λ,β)	88.50	91.30	88.94	89.38
MOBXII (α,λ,β)	87.28	91.48	88.20	88.60
TLBXII (α,λ,β)	86.62	90.82	87.54	87.94
KwBXII (a,b,λ,β)	86.16	91.76	87.76	87.92
BBXII (a,b,λ,β)	87.14	92.74	88.74	88.90
BEBXII $(\alpha,a,b,\lambda,\beta)$	87.26	94.26	89.76	89.46
FBBXII $(\alpha,a,b,\lambda,\beta)$	87.36	94.36	89.86	89.56
FKwBXII $(\alpha,a,b,\lambda,\beta)$	87.14	94.14	89.64	89.34
GOHCBXII (α,λ,β)	84.78	88.98	85.70	86.12
EOHCBXII (α,λ,β)	84.42	88.62	85.34	85.74

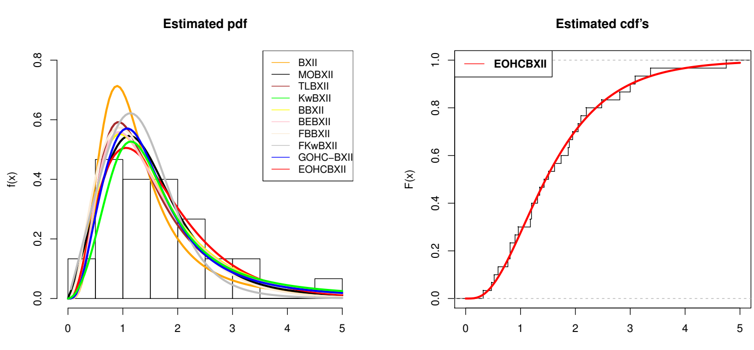

Figure 9: Plots of the observed histogram and estimated pdfs for the BXII, MOBXII, ,TL-BXII, KwBXII, BBXII,BEBXII, FBBXII, FKwBII and EOHCBXII and observed ogive and estimated cdf EOHCBXII for data set II from right to left

Table 6: MLEs of the parameters to Stanford Heart Transplant Data for LOLL-Fr and LEOHC-Fr regression models with corresponding SEs, p-values and $-\ell$, AIC and BIC statistics.

	Models					
	LOLL-Fr			LEOHC-Fr		
Parameters	Estimate	S.E.	p-value	Estimate	S.E.	p-value
α	2.078	0.790	-	24.344	49.796	-
σ	2.886	0.954	-	5.728	2.952	-
β_0	1.252	0.561	0.025	9.661	6.964	0.165
β_1	0.181	0.096	0.061	0.204	0.094	0.031
β_2	-0.047	0.018	0.010	-0.052	0.018	0.004
β_3	-0.151	0.501	0.763	0.206	0.484	0.670
β_4	0.551	0.268	0.039	0.437	0.365	0.230
$-\ell$	160.932			158.965		
AIC	335.865			331.931		
BIC	354.308			350.374		

The results of the fitted regression models including the estimated parameters, standard errors and corresponding p-values as well as model selection criteria such as AIC and BIC values are given in Table 6. As seen from the reported values of AIC and BIC, the LEOHC-Fr regression model has lower values of these statistics than those of the LOLL-Fr regression model. Therefore, we conclude that the LEOHC-Fr regression model is more appropriate than the LOLL-Fr regression model for the data used. Additionally, the regression parameters β_1 and β_2 are statistically significant since the p-values of these parameters are less than 5% significance level.

7.2.1 Residual Analysis of LEOHC-Fr model

Figure 10 displays the residuals analysis results of the LEOHC-Fr model. These figures reveal the applicability and accuracy of the fitted LEOHC-Fr model. Since all residuals are in the plotted envelopes, there is no possible outlier.

8. Conclusion

T-X method is used to generate a new family of continuous distributions. Important statistical properties are investigated. Different estimation methods are discussed to estimate the unknown model parameters via comprehensive simulation studies. Applications of data modelling with distribution fitting and regression modelling have shown favourable results for distributions belonging to the proposed family.

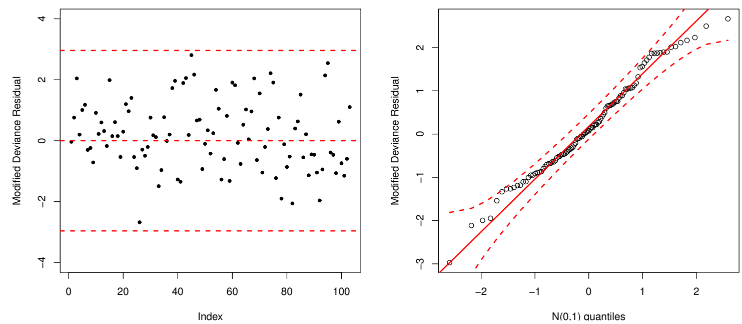

Figure 10: The results of residual analysis: (left) plot of the modified deviance residuals and (right) its quantile-quantile plot

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A study on exponentiated Gompertz distribution under Bayesian discipline using informative priors

Muhammad Aslam[1], **Mehreen Afzaal**[2], **M. Ishaq Bhatti**[3]

ABSTRACT

The exponentiated Gompertz (EGZ) distribution has been recently used in almost all areas of human endeavours, starting from modelling lifetime data to cancer treatment. Various applications and properties of the EGZ distribution are provided by Anis and De (2020). This paper explores the important properties of the EGZ distribution under Bayesian discipline using two informative priors: the Gamma Prior (GP) and the Inverse Levy Prior (ILP). This is done in the framework of five selected loss functions. The findings show that the two best loss functions are the Weighted Balance Loss Function (WBLF) and the Quadratic Loss Function (QLF). The usefulness of the model is illustrated by the use of real-life data in relation to simulated data. The empirical results of the comparison are presented through a graphical illustration of the posterior distributions.

Key words: exponentiated Gompertz distribution, loss functions, informative priors, Bayes estimators, posterior risks.

1. Introduction

The Gompertz distribution was named after Benjamin Gompertz. It is an exponentially increasing, continuous probability distribution. Exponentiated Gompertz (EGZ) distribution is basically a truncated extreme value distribution (Johnson et al. 1994 and Chaturvedi et al. 2012) which ranges from zero to positive infinity. In early days EGZ was used in the area of insurance to measure life expectancy and human mortality rates with a range of 0 to $\cong$ 100. However, recently these distributions have been used in a wide range of other applications in various areas of human endeavours, including risk management, economic and finance, cancer treatment, medical and biological sciences and demography. Recently, the various

1 Department of Mathematics and Statistics, Riphah International University, Pakistan. E-mail: m.aslam@riphah.edu.pk. ORCID: https://orcid.org/0000-0003-3355-2330.

2 Department of Mathematics and Statistics, Riphah International University, Pakistan. E-mail: mehreenafzaal12@gmail.com. ORCID: https://orcid.org/0000-0002-1550-3415.

3 Corresponding author: La Trobe Business School, La Trobe University, Australia. E-mail: I.Bhatti@latrobe.edu.pk. ORCID: https://orcid.org/0000-0002-5027-7871.

applications and properties of EGZ distribution have been studied by Chaturvedi et al. (2000), Abu-Zinadah et al. (2017), Hoseinzadeh et al. (2019), Mazucheli et al. (2019), Alrajhi et al. (2020), Dey et al. (2018), Leren et al. (2019), Anis and De (2020), Anis (2020), Jha et al. (2020), Shrivastava et al. (2019) and Obeidat et al. (2020), among others. For example, Abu-Zinadah et al. (2017), Dey et al. (2018) developed some theoretical properties of EGZ, which are being used by economists, financiers and practitioners. For example, Jha et al. (2020) applied EGZ distribution in reliability whereas Hoseinzadeh et al. (2019) employed in financial markets and risk management. Moreover, Mazucheli et al. (2019) introduced the unit-Gompertz (UG) distribution and studied some important properties. Anis and De (2020) pointed out a flaw of an error term in Mazucheli et al.'s (2019) paper and proposed a new type of UG-distribution with additional interesting properties. Moreover, Alraihi et al. (2020) tackled hybrid censored sample issue of complexity in a fuzzy system and artificial intelligence, whereas Leren et al. (2020) applied EGZ model-based distribution to bladder cancer patient's data and observed interesting properties in bioinformatics.

In some early studies, El-Gohary et al. (2013) suggested EGZ's interesting properties. Sherpiency et al. (2013) introduced a new distribution called bivariate generalized Gompertz (BGG) distribution, whose marginals are generalized Gompertz distributions (GGD) and discussed some of its properties. Zinadah and Oufi (2014) studied the EGZ distribution and its properties like, quantiles, median, mode, mean residual lifetime, mean deviations, Rényi entropy, density, survival and hazard functions were derived. Zinadah (2014a) derived the expressions for reliability and failure rate functions of the EGZ distribution. Saraçoğlu et al. (2014) considered the Maximum Likelihood Estimators (MLE) and Bayes Estimators (BE) for unknown parameters of GGD. Moreover, Zinadah (2014) also worked on three goodness of fit test statistics, namely Kolmogorov Smirnov (KS), Anderson Darling (AD) and Cramer Von Mises (CVM) for EGZ distribution utilizing complete and type-II censored data. Jafari et al. (2014) introduced a new four parameter generalized version of Gompertz distribution called Beta-Gompertz (BG) distribution. Zinadah (2014) examined the EGZ distribution, for estimating the shape parameter θ, considering five different estimation methods. Namely Maximum Likelihood method, method of Moments, method of Percentiles, Least Square method and Weighted Least Square method. Damcese et al. (2015) demonstrated a new lifetime model called Odd Generalized Exponential Gompertz (OGE-G) distribution.

Furthermore, an important work of Zinadah and Oufi (2016) on the four estimators, namely: ML, Least Squares (LS), Weighted Least Squares (WLS), and Percentiles (PC) for the EGZ distribution is some extra contribution in the literature. Cordeiro et al. (2016) investigated a new distribution called Exponentiated Gompertz Generated (EGG) distribution. Bassiouny et al. (2017) proposed a new model, namely

Exponentiated Generalized Weibull-Gompertz (EGWG) distribution. Ade et al. (2017) developed a distribution known as Generalized Exponentiated Gompertz Makeham (EGGM) distribution, consisting of five parameters. Bakouch et al. (2017) introduced a new distribution called the Weighted Gompertz (WGO) distribution.

From the above studies one can see that the literature review revealed that none of the authors have worked on attaining BEs and PRs of the EGZ distribution and have only studied its properties. Hence, this paper is an attempt to fill this gap in the literature. It attempts to analyze the unknown shape parameter of the EGZ distribution. The rest of the paper is organized as follows. In Section 2, we define the pdf of EGZ Distribution and derive its likelihood function. In Section 3, analysis is done on the unknown shape parameter of EGZ Distribution when the rest of the parameters are known. We have derived its posterior distribution, Bayes Estimators and PRs utilizing various loss functions. It considers GP and ILP which are contemplated as informative priors to acquiring the posterior distribution. In Section 4, simulation study is conducted, and comparison of the estimates is presented along with graphical illustration. A real life data set is considered for the analysis purpose in Section 5 and its results are discussed and compared with that of simulation using tabulation, and graphics of the posterior distribution are demonstrated to show that the best loss function is the WBLF followed by QLF. The final section contains some concluding remarks.

2. EGZ distribution and its likelihood function

The pdf of EGZ of variable X is given as:

$$f(x,\lambda,\alpha,\theta)=\theta\lambda e^{\alpha x}e^{\frac{-\lambda}{\alpha}(e^{\alpha x}-1)}\{1-e^{\frac{-\lambda}{\alpha}(e^{\alpha x}-1)}\}^{\theta-1},\quad x,\lambda,\alpha,\theta>0. \tag{1}$$

EGZ has the following likelihood function for random sample x = $x_1,\ldots,x_n$:

$$L(x,\alpha,\lambda,\theta)=(\theta\lambda)^n\prod_{i=1}^{n}e^{\alpha x_i}\prod_{i=1}^{n}e^{\frac{-\lambda}{\alpha}(e^{\alpha x_i}-1)}\prod_{i=1}^{n}\{1-e^{\frac{-\lambda}{\alpha}(e^{\alpha x_i}-1)}\}^{\theta-1},$$

$$L(x,\alpha,\lambda,\theta)=(\theta\lambda)^n\prod_{i=1}^{n}e^{\alpha x_i}\prod_{i=1}^{n}e^{\frac{-\lambda}{\alpha}(e^{\alpha x_i}-1)}e^{(\theta-1)\sum_{i=1}^{n}\ln\{1-e^{\frac{-\lambda}{\alpha}(e^{\alpha x_i}-1)}\}}, \tag{2}$$

Then (2) becomes:

$$L(x,\alpha,\lambda,\theta)=(\theta\lambda)^n\prod_{i=1}^{n}e^{\alpha x_i}\prod_{i=1}^{n}e^{\frac{-\lambda}{\alpha}(e^{\alpha x_i}-1)}e^{(\theta-1)m_1},$$

The likelihood function of EGZ with known scale parameter α, known shape parameter λ and unknown shape parameter θ is:

$$L(x,\theta)\propto\theta^n e^{-\theta(-m_1)}, \tag{3}$$

where, $m_1 = \sum_{i=1}^{n} \ln\{1 - e^{\frac{-\lambda}{\alpha}(e^{\alpha x_i} - 1)}\}.$

3. Analysis of shape parameter of EGZ distribution

3.1 Posterior distribution using informative priors

Here, in this subsection we consider GP and ILP which are contemplated as informative priors to acquiring the posterior distribution.

3.1.1 Gamma prior

The gamma prior of θ with hyperparameters 'v' and 'w' is given by:

$$p(\theta) = \frac{w^v}{\Gamma v}\theta^{v-1}e^{-w\theta}, \qquad v, w, \theta > 0. \tag{4}$$

The posterior distribution using equations (3) and (4) is given by:

$$p(\theta|\mathbf{x}) \propto \theta^{v-1}e^{-w\theta}\theta^n e^{-\theta(-m_1)},$$

$$p(\theta|\mathbf{x}) \propto \theta^{\Phi_2-1}e^{-\theta\Psi_2},$$

where, $\Phi_2 = v + n$ and $\Psi_2 = w - m_1$.

which is the density kernel of gamma distribution having parameters Φ_2 and Ψ_2.

Hence, the posterior distribution $\theta|\mathbf{x}$ is Gamma (Φ_2, Ψ_2).

3.1.2 Inverse levy prior

The inverse Levy prior of θ with hyperparameter 'c' is given by:

$$p(\theta) = \sqrt{\frac{c}{2\pi}}\theta^{-\frac{1}{2}}e^{\frac{-c\theta}{2}}, \qquad c, \theta > 0. \tag{5}$$

The posterior distribution using equations (3) and (5) is given by:

$$p(\theta|\mathbf{x}) \propto \theta^{\frac{-1}{2}}e^{\frac{-c\theta}{2}}\theta^n e^{-\theta(-m_1)},$$

$$p(\theta|\mathbf{x}) \propto \theta^{\Phi_3-1}e^{-\theta\Psi_3},$$

where, $\Phi_3 = n + \frac{1}{2}$ and $\Psi_3 = \frac{c}{2} - m_1$.

which is the density kernel of gamma distribution having parameters Φ_3 and Ψ_3.

Hence, the posterior distribution $\theta|\mathbf{x}$ is Gamma (Φ_3, Ψ_3).

3.2. BEs and PRs under different loss functions

The general expressions for loss functions along with the expressions of their Bayes Estimators and PRs are given as follows.

3.2.1 Squared error loss function

The expression for squared error loss function is given as:

$$L(\theta,\theta^*) = (\theta-\theta^*)^2, \tag{6}$$

The Bayes estimator and posterior risk of SELF are:

$$\theta^* = E_{\theta|\mathbf{x}}(\theta), \qquad \rho(\theta^*) = E_{\theta|\mathbf{x}}(\theta)^2 - \{E_{\theta|\mathbf{x}}(\theta)\}^2. \tag{7}$$

3.2.2 Weighted squared error loss function

The expression for weighted squared error loss function is given as:

$$L(\theta,\theta^*) = \frac{(\theta-\theta^*)^2}{\theta}, \tag{8}$$

The Bayes estimator and posterior risk of WSELF are:

$$\theta^* = \{E_{\theta|\mathbf{x}}(\theta^{-1})\}^{-1}, \qquad \rho(\theta^*) = E_{\theta|\mathbf{x}}(\theta) - \{E_{\theta|\mathbf{x}}(\theta^{-1})\}^{-1}. \tag{9}$$

3.2.3 Precautionary loss function

The expression for precautionary loss function is given as:

$$L(\theta,\theta^*) = \frac{(\theta-\theta^*)^2}{\theta^*}, \tag{10}$$

The Bayes estimator and Posterior risk of PLF are:

$$\theta^* = \sqrt{E_{\theta|\mathbf{x}}(\theta^2)}, \qquad \rho(\theta^*) = 2\{\sqrt{E_{\theta|\mathbf{x}}(\theta^2)} - E_{\theta|\mathbf{x}}(\theta)\}. \tag{11}$$

3.2.4 Weighted balance loss function

The expression for weighted balance loss function is given as:

$$L(\theta,\theta^*) = \left(\frac{\theta-\theta^*}{\theta^*}\right)^2, \tag{12}$$

The Bayes estimator and posterior risk of WBLF are:

$$\theta^* = \frac{E_{\theta|\mathbf{x}}(\theta^2)}{E_{\theta|\mathbf{x}}(\theta)}, \qquad \rho(\theta^*) = 1 - \frac{\{E_{\theta|\mathbf{x}}(\theta)\}^2}{E_{\theta|\mathbf{x}}(\theta^2)}. \tag{13}$$

3.2.5 Quadratic loss function

The expression for quadratic loss function is:

$$L(\theta,\theta^*)=\left(\frac{\theta-\theta^*}{\theta}\right)^2, \tag{14}$$

The Bayes estimator and posterior risk of QLF are:

$$\theta^* = \frac{E_{\theta|\mathbf{x}}(\theta^{-1})}{E_{\theta|\mathbf{x}}(\theta^{-2})}, \qquad \rho(\theta^*)=1-\frac{\{E_{\theta|\mathbf{x}}(\theta^{-1})\}^2}{E_{\theta|\mathbf{x}}(\theta^{-2})}. \tag{15}$$

3.3. Expressions for BEs and PRs under different loss functions

This section derives and summarize the expressions for BEs and PRs under SELF in the presence of priors based on GP and ILP distributions. This is done in tabular form below in Tables 3.1. to 3.5., which represent five loss functions, respectively.

Table 3.1. Expressions for BEs and PRs under SELF

Priors	BEs	PRs
GP	$(v+n)\left[w-\sum_{i=1}^{n}\ln\left\{1-e^{\frac{-\lambda}{\alpha}(e^{\alpha x_i}-1)}\right\}\right]^{-1}$	$(v+n)\left\{w-\sum_{i=1}^{n}\ln[1-e^{\frac{-\lambda}{\alpha}(e^{\alpha x_i}-1)}]\right\}^{-2}$
ILP	$(2n+1)\left[c-2\sum_{i=1}^{n}\ln\left\{1-e^{\frac{-\lambda}{\alpha}(e^{\alpha x_i}-1)}\right\}\right]^{-1}$	$2(2n+1)\left\{c-2\sum_{i=1}^{n}\ln[1-e^{\frac{-\lambda}{\alpha}(e^{\alpha x_i}-1)}]\right\}^{-2}$

Table 3.2. Expressions for BEs and PRs under WSELF

Priors	BEs	PRs
GP	$(v+n-1)\left[w-\sum_{i=1}^{n}\ln\left\{1-e^{\frac{-\lambda}{\alpha}(e^{\alpha x_i}-1)}\right\}\right]^{-1}$	$\left[w-\sum_{i=1}^{n}\ln\left\{1-e^{\frac{-\lambda}{\alpha}(e^{\alpha x_i}-1)}\right\}\right]^{-1}$
ILP	$(2n-1)\left[c-2\sum_{i=1}^{n}\ln\left\{1-e^{\frac{-\lambda}{\alpha}(e^{\alpha x_i}-1)}\right\}\right]^{-1}$	$(2)\left[c-2\sum_{i=1}^{n}\ln\left\{1-e^{\frac{-\lambda}{\alpha}(e^{\alpha x_i}-1)}\right\}\right]^{-1}$

Table 3.3. Expressions for BEs and PRs under PLF

Priors	Bes	PRs
GP	$\left(\sqrt{(v+n)(v+n+1)}\right)\left[w-\sum_{i=1}^{n}\ln\left\{1-e^{\frac{-\lambda}{\alpha}(e^{\alpha x_i}-1)}\right\}\right]^{-1}$	$2\left(\sqrt{(v+n)(v+n+1)}-(v+n)\right)\left[w-\sum_{i=1}^{n}\ln\left\{1-e^{\frac{-\lambda}{\alpha}(e^{\alpha x_i}-1)}\right\}\right]^{-1}$
ILP	$\left(\sqrt{(2n+1)(2n+3)}\right)\left[c-2\sum_{i=1}^{n}\ln\left\{1-e^{\frac{-\lambda}{\alpha}(e^{\alpha x_i}-1)}\right\}\right]^{-1}$	$2\left(\sqrt{(2n+1)(2n+3)}-(2n+1)\right)\left[c-2\sum_{i=1}^{n}\ln\left\{1-e^{\frac{-\lambda}{\alpha}(e^{\alpha x_i}-1)}\right\}\right]^{-1}$

Table 3.4. Expressions for BEs and PRs under WBLF

Priors	Bes	PRs
GP	$(v+n+1)\left[w-\sum_{i=1}^{n}\ln\left\{1-e^{\frac{-\lambda}{\alpha}(e^{\alpha x_i}-1)}\right\}\right]^{-1}$	$(v+n+1)^{-1}$
ILP	$(2n+3)\left[c-2\sum_{i=1}^{n}\ln\left\{1-e^{\frac{-\lambda}{\alpha}(e^{\alpha x_i}-1)}\right\}\right]^{-1}$	$2(2n+3)^{-1}$

Table 3.5. Expressions for BEs and PRs under QLF

Priors	Bes	PRs
GP	$(v+n-2)\left[w-\sum_{i=1}^{n}\ln\left\{1-e^{\frac{-\lambda}{\alpha}(e^{\alpha x_i}-1)}\right\}\right]^{-1}$	$(v+n-1)^{-1}$
ILP	$(2n-3)\left[c-2\sum_{i=1}^{n}\ln\left\{1-e^{\frac{-\lambda}{\alpha}(e^{\alpha x_i}-1)}\right\}\right]^{-1}$	$2(2n-1)^{-1}$

4. Simulation study

In this section, we conduct a simulation study using the expression of the loss functions from above tables, which are obtained by the BEs and PRs for the shape parameter θ, using two informative priors, namely: GP and ILP, under **five** different loss functions, namely SELF, WSELF, PLF, WBLF and QLF. Various sample sizes such as 20, 30,100, 300, 500, 1000 are used for simulation purposes, taking 10,000 replications in 'R'. Several values of the scale and shape parameters are considered. α is taken as 2, λ is taken as 1 and 3 and θ is taken as 1, 2 and 3. The estimated values of the parameters for all BEs and PRs are tabulated in Table 4.1. below. It is important to note that corresponding to selected samples of GP's and ILP's values are given. BEs are without parenthesis while estimates of PRs are enclosed in parenthesis for each prior and loss function under different sample sizes.

Note from the above table, as the sample size, n, increases then the GP and ILP of BE also goes up but PRs decreases. Similarly, once we change the value of θ=1 to θ=2 GP of BE and ILP follow the similar pattern with one exception. The same pattern is observed from the graphics demonstration via Bayes estimates and posterior risks for selected values of α, λ, *w*, *v*, and θ given below in graphs Figures 4.1. to fig. 4.5. and figures 5.1. to 5.3. Rest of the graphs are not presented due to similar observations.

Table 4.1. BEs and PRs under SELF using various priors

Priors	α =2 , λ=1, θ=1, v =1, $w=1$, $c=1$					
N	20	30	100	300	500	1000
GP	0.97626 (0.056405)	0.97803 (0.03374)	0.98101 (0.012509)	1.02273 (0.00316)	1.08835 (0.00192)	1.12402 (0.00095)
ILP	0.82894 (0.10345)	0.94924 (0.02252)	0.979703 (0.00896)	0.99385 (0.00362)	1.04427 (0.00197)	1.45631 (0.00095)
Priors	α =2 , λ=1, θ=2, v =1, $w=1$, $c=1$					
GP	2.34054 (0.26086)	2.22234 (0.15931)	2.08343 (0.04102)	2.03552 (0.01181)	1.91313 (0.007305)	1.88565 (0.00433)
ILP	2.12046 (0.159907)	2.09176 (0.11592)	1.97831 (0.04474)	1.89975 (0.01201)	1.88035 (0.00781)	1.81055 (0.00437)
Priors	α =2 , λ=1, θ=3 , v =1, $w=1, c=1$					
GP	2.65453 (0.33554)	2.67769 (0.23129)	2.84175 (0.08995)	2.99406 (0.02682)	3.01425 (0.01858)	3.05168 (0.00895)
ILP	2.62072 (0.39717)	2.85344 (0.27311)	2.86699 (0.10028)	2.88619 (0.02285)	2.93718 (0.01642)	3.17466 (0.00862)
Priors	α =2 , λ=3, θ=1, v =1, $w=1, c=1$					
GP	0.74424 (0.02637)	0.84441 (0.0230009)	0.858404 (0.00729)	1.01196 (0.00352)	1.03065 (0.00217)	1.04442 (0.00102)
ILP	0.65127 (0.04646)	0.92084 (0.013906)	0.94216 (0.00843)	0.97597 (0.00295)	1.02088 (0.00208)	1.03285 (0.00106)
Priors	α =2 , λ=3, θ=2, v =1, $w=1$, $c=1$					
GP	1.720105 (0.22553)	1.85295 (0.09544)	1.98521 (0.03399)	1.99369 (0.01309)	2.03254 (0.00824)	2.17628 (0.00397)
ILP	1.83514 (0.17915)	1.90751 (0.16993)	1.91639 (0.03351)	2.00648 (0.01411)	2.05984 (0.00726)	2.27661 (0.00402)
Priors	α =2 , λ=3, θ=3, v =1, $w=1, c=1$					
GP	2.49229 (0.35855)	2.74402 (0.20037)	2.79417 (0.077301)	2.94542 (0.03222)	3.11463 (0.02063)	3.215501 (0.00866)
ILP	2.23795 (0.364107)	2.727302 (0.16421)	2.73206 (0.15021)	3.00286 (0.03603)	3.29053 (0.01486)	3.88547 (0.00901)
Priors	α =2 , λ=1, θ=1 , v =1, w =2, $c=2$					
GP	0.87387 (0.04876)	0.90072 (0.02463)	0.90842 (0.00817)	0.95581 (0.00269)	0.98606 (0.00182)	1.01198 (0.00097)
ILP	1.20682 (0.06767)	1.17785 (0.03656)	1.05607 (0.01449)	0.98644 (0.00311)	0.98438 (0.00193)	0.96682 (0.00097)

Simulation Results' Graphs

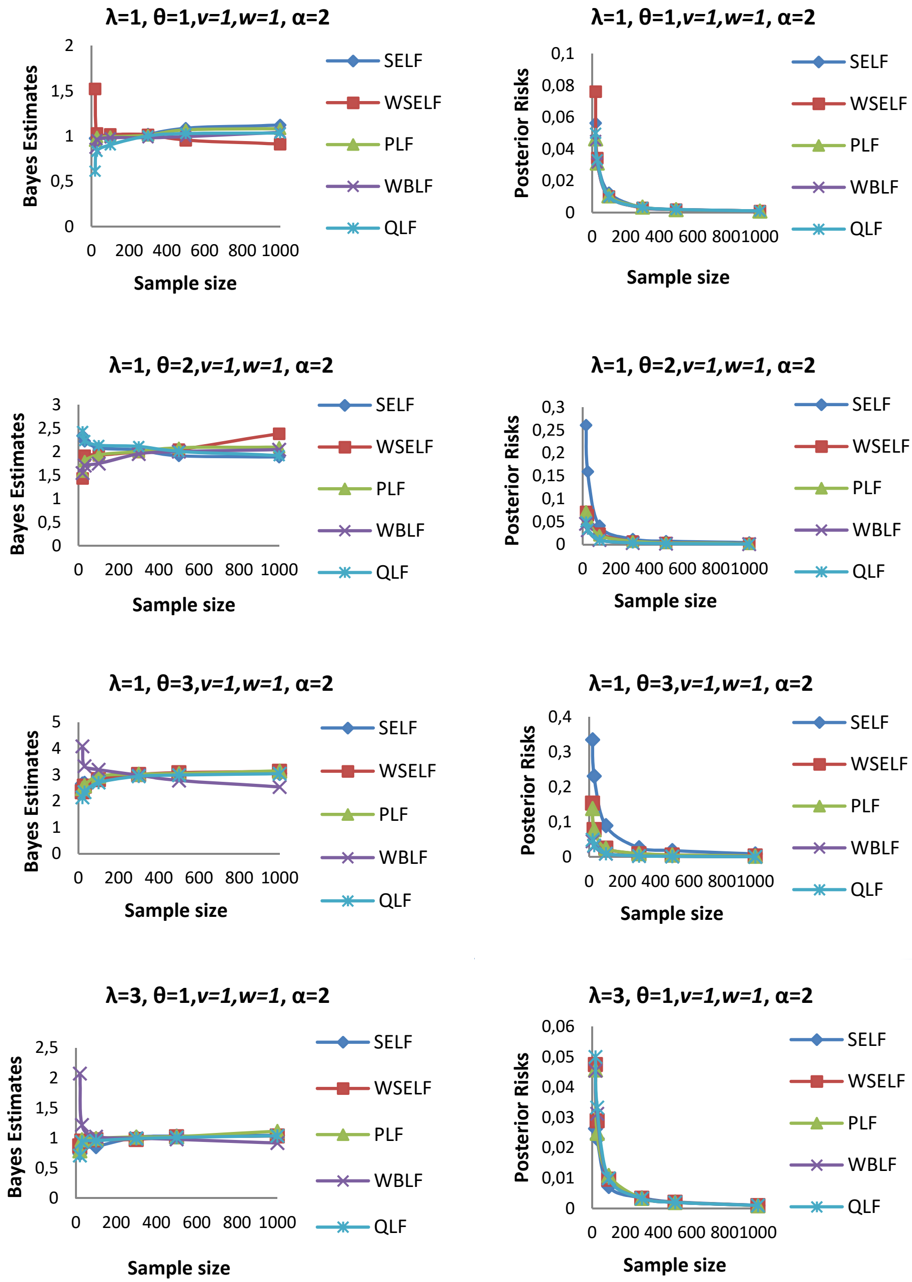

Figure 4.1. Graphs of BEs and PRs for GP

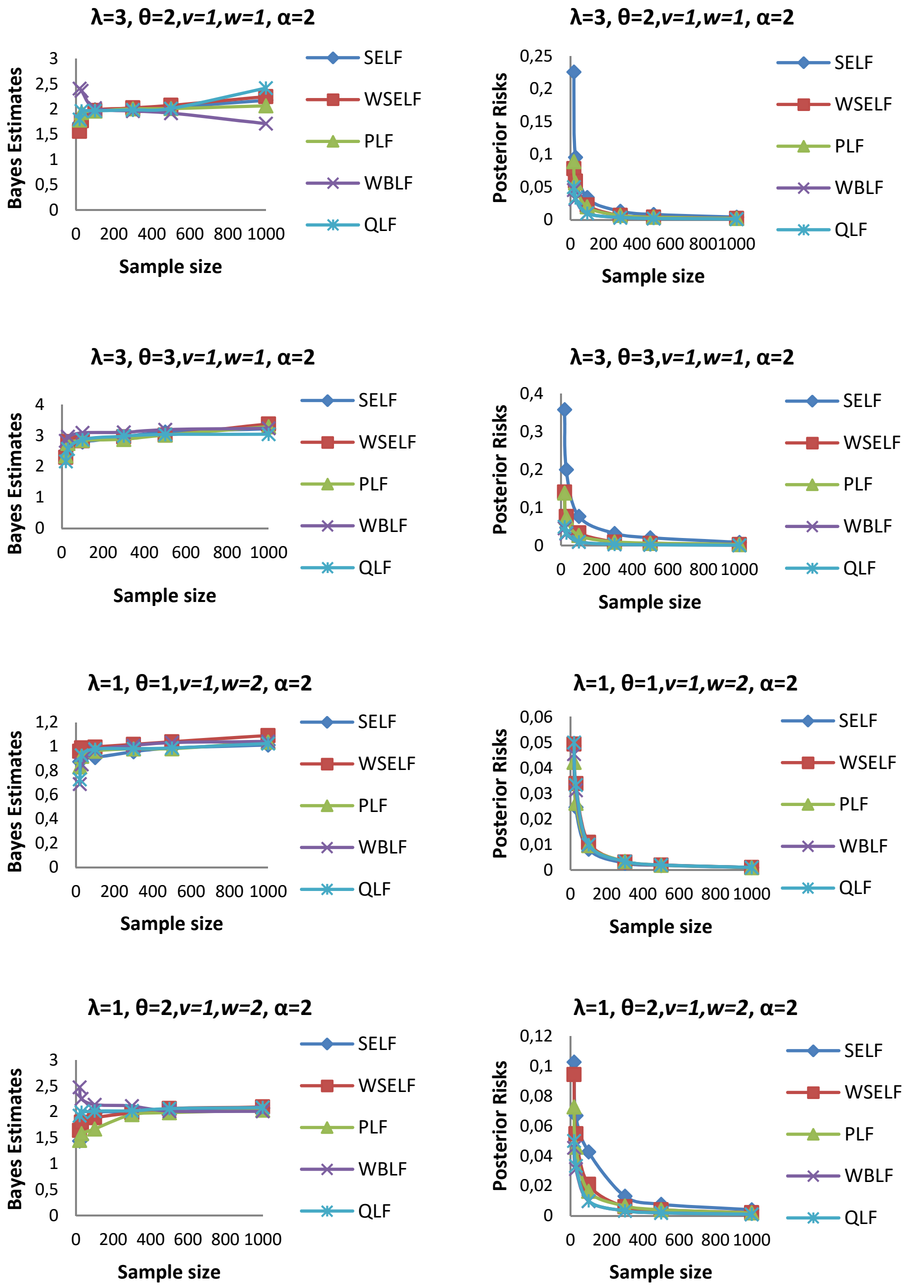

Figure 4.2. Graphs of BEs and PRs for GP

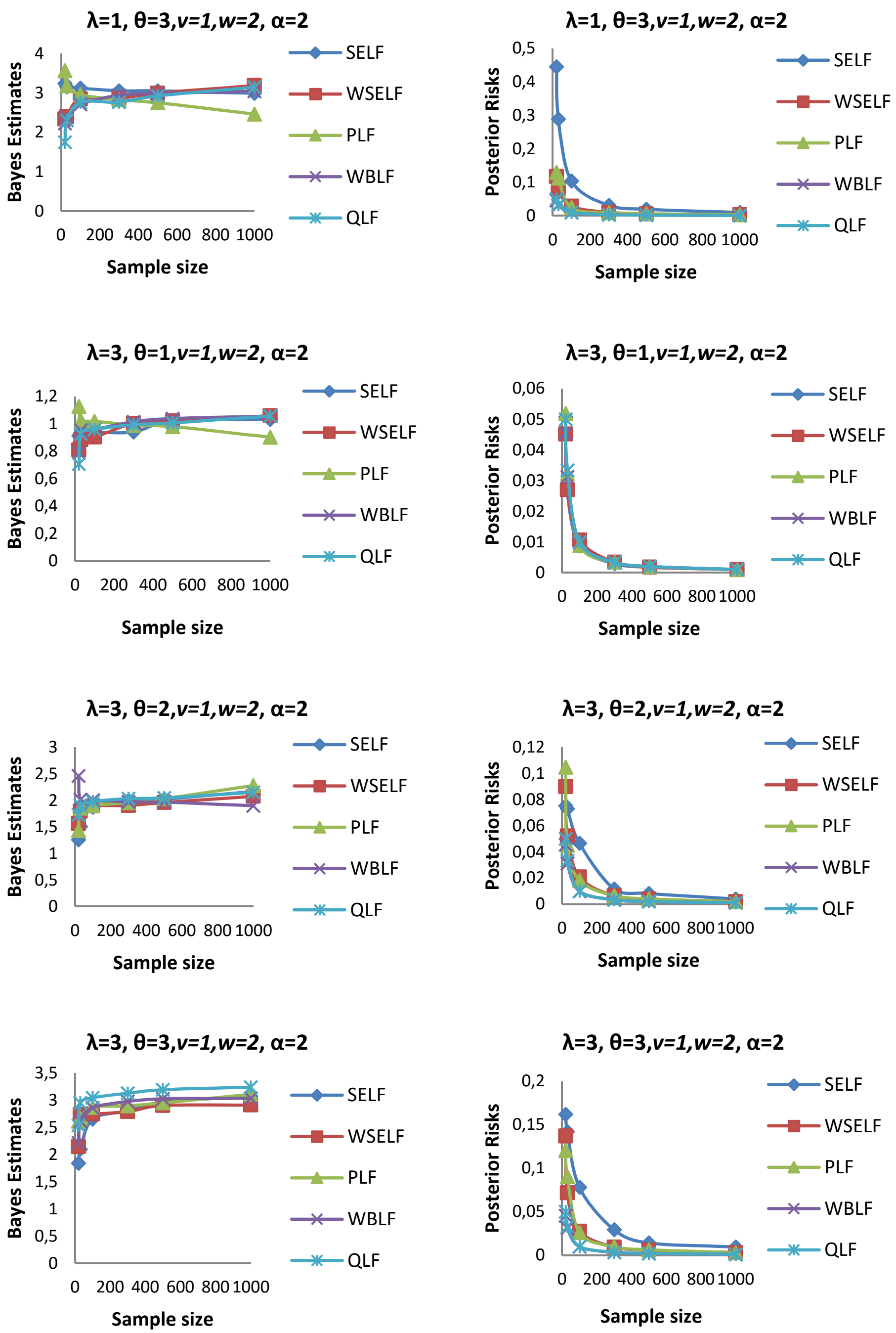

Figure 4.3. Graphs of BEs and PRs for GP

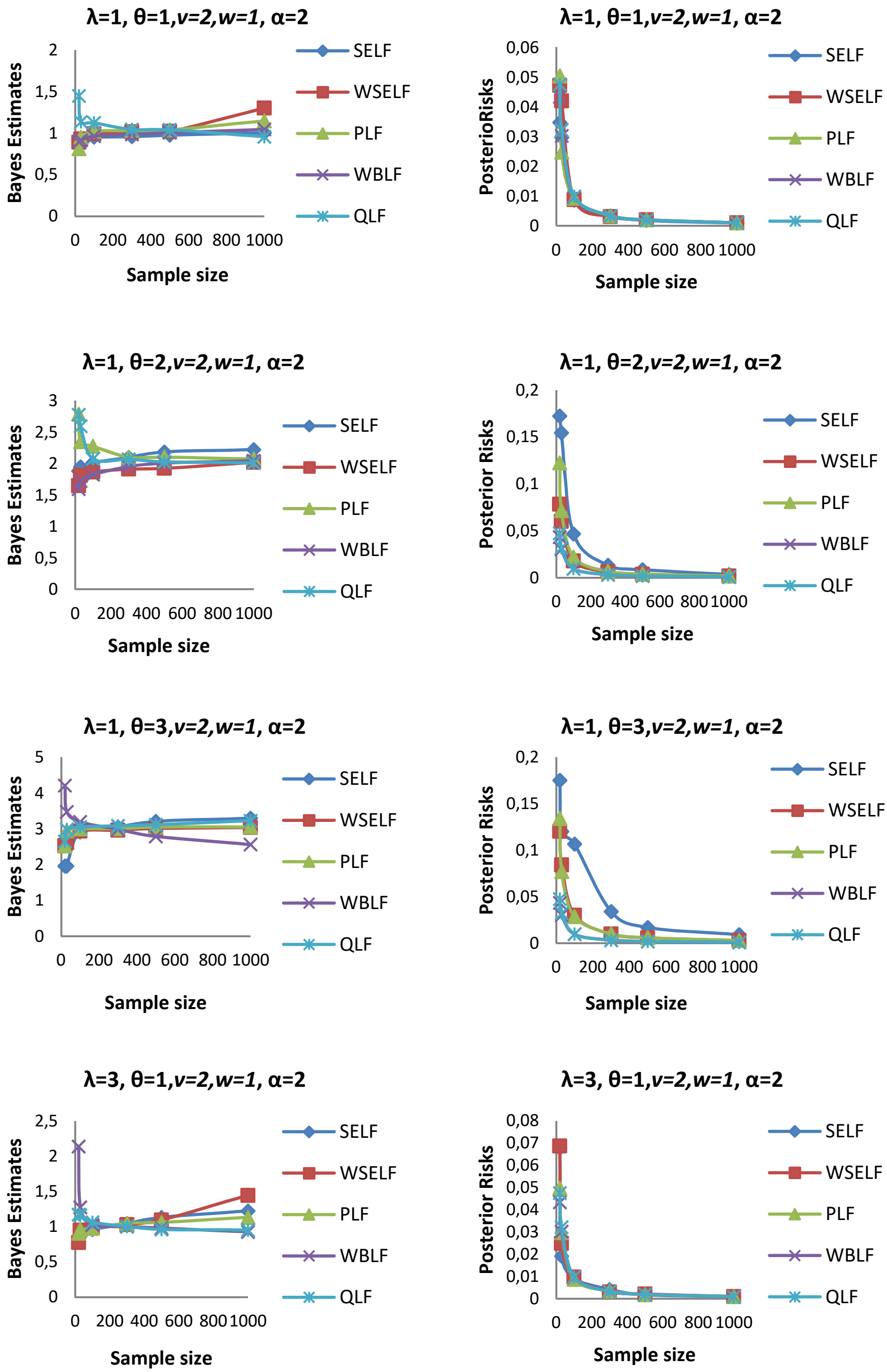

Figure 4.4. Graphs of BEs and PRs for GP

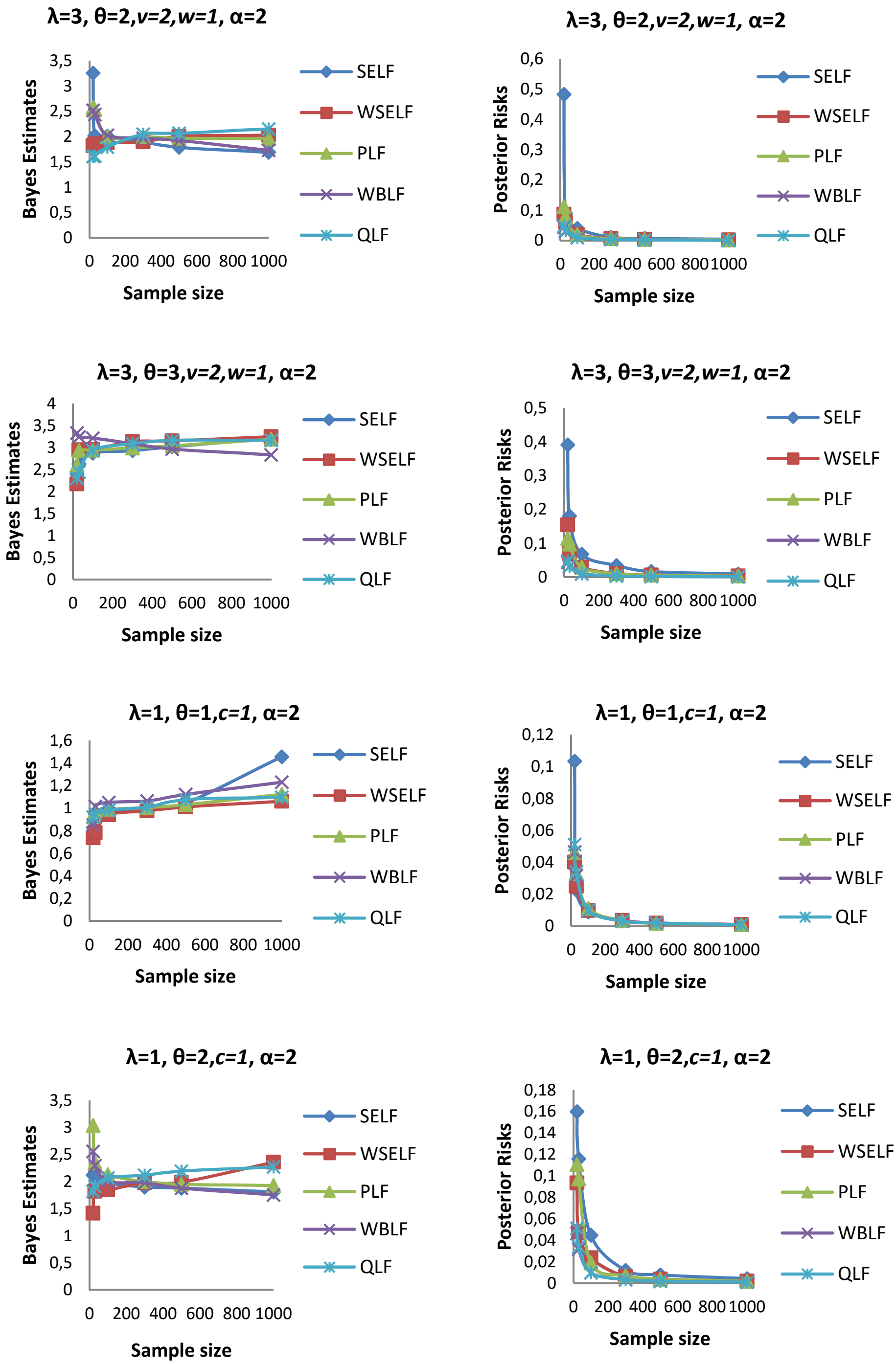

Figure 4.5. Graphs of BEs and PRs for GP

5. Examining a real-life data set

The data set consists of 50 observations of lifetimes of devices as given in Gohary et al. (2013). These are given below for the ready reference of the readers.

0.1, 0.2, 1, 1, 1, 1, 1, 2, 3, 6, 7, 11, 12, 18, 18, 18, 18, 18, 21, 32, 36, 40, 45, 46, 47, 50, 55, 60, 63, 63, 67, 67, 67, 67, 72, 75, 79, 82, 82, 83, 84, 84, 84, 85, 85, 85, 85, 85, 86, 86.

The various values of the BEs, PRs for parameter θ for selected values of $\alpha, \lambda, v, w\ and\ c$ are tabulated in Tables 5.1. to 5.6. for two priors and five loss functions. The WBLF loss function highlighted in bold indicate the prefer priors.

Table 5.1. BEs and PRs for parameter θ when $\alpha = 2, \lambda = 1, v = 1, w = 1, c = 1$

Prior		Loss Functions				
		SELF	WSELF	PLF	WBLF	QLF
GP	BEs	10.22408	10.02361	10.32383	**10.42455**	9.82313
	PRs	2.04964	0.200472	0.19949	0.01923	0.02000
ILP	BEs	11.25167	11.02886	11.36252	**11.47447**	10.80606
	PRs	2.50693	0.222805	0.22171	0.01941	0.020202

Table 5.2. BEs and PRs for parameter θ when $\alpha = 2, \lambda = 1, v = 1, w = 2, c = 2$

Prior		Loss Functions				
		SELF	WSELF	PLF	WBLF	QLF
GP	BEs	8.51671	8.34972	8.59980	**8.683709**	8.18272
	PRs	1.42224	0.16699	0.16618	0.01923	0.02000
ILP	BEs	10.12384	9.92337	10.22359	**10.32432**	9.72289
	PRs	2.02954	0.20047	0.19948	0.01941	0.020202

Table 5.3. BEs and PRs for parameter θ when $\alpha = 2, \lambda = 1, v = 2, w = 1, c = 3$

Prior		Loss Functions				
		SELF	WSELF	PLF	WBLF	QLF
GP	BEs	10.42455	10.22408	10.52431	**10.62502**	10.02361
	PRs	2.08983	0.20047	0.199517	0.01886	0.019607
ILP	BEs	9.20151	9.01931	9.29217	**9.38372**	8.837102
	PRs	1.67659	0.182208	0.18131	0.01941	0.020202

Table 5.4. BEs and PRs for parameter θ when $\alpha = 2, \lambda = 3, v = 1, w = 1, c = 1$

Prior	Loss Functions					
		SELF	WSELF	PLF	WBLF	QLF
GP	BEs	17.49828	17.15517	17.669	**17.84138**	16.81207
	PRs	6.00372	0.343103	0.34143	0.01923	0.02000
ILP	BEs	20.91467	20.50052	21.12073	**21.32883**	20.08637
	PRs	8.66185	0.41415	0.41212	0.01941	0.020202

Table 5.5. BEs and PRs for parameter θ when $\alpha = 2, \lambda = 3, v = 1, w = 2, c = 2$

Prior	Loss Functions					
		SELF	WSELF	PLF	WBLF	QLF
GP	BEs	13.02824	12.77279	13.15535	**13.2837**	12.51733
	PRs	3.32813	0.25545	0.25421	0.01923	0.02000
ILP	BEs	17.32673	16.98362	17.49744	**17.66983**	16.64052
	PRs	5.94486	0.343103	0.34142	0.01941	0.020202

Table 5.6. BEs and PRs for parameter θ when $\alpha = 2, \lambda = 3, v = 2, w = 1, c = 3$

Prior	Loss Functions					
		SELF	WSELF	PLF	WBLF	QLF
GP	BEs	17.84138	17.49828	18.01212	**18.18448**	17.15517
	PRs	6.12144	0.343103	0.34146	0.01886	0.019607
ILP	BEs	14.78955	14.49669	14.93527	**15.08241**	14.20383
	PRs	4.331304	0.29286	0.29142	0.01941	0.020202

5.1. Analysis of real-life data set

From the above tables we can see that WBLF is the best and most preferable loss function under all the priors since it has the lowest PRs followed by QLF, both the loss functions having a minute difference in their PRs. Also, SELF has the highest PRs making it the least preferable loss function. Moreover, the PRs are minimum for the values of hyperparameters (1, 2) of GP making it the most preferable prior and pair to be used. Additionally, WBLF and QLF have same PRs for all values of α, λ. The only difference comes for GP for the values of hyperparameters (2,1), where the value of hyper-parameter 'v' changes and gives a different risk. It can also be noticed that the results of simulation and the results for real life data are identical. These are demonstrated in the graphs below in Fig. 5.1. to 5.3. for various values of v and w.

Posterior distribution graphs

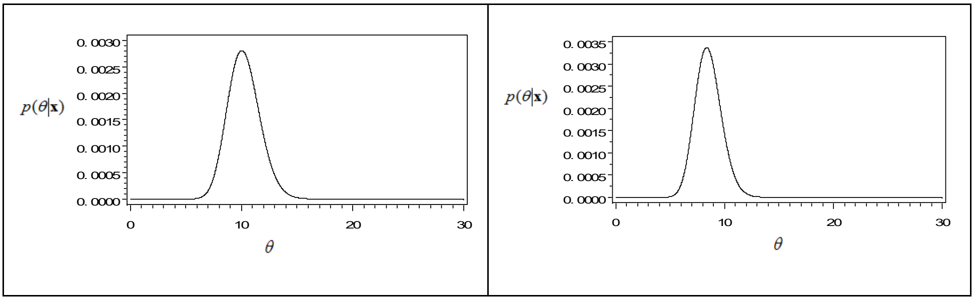

Figure 5.1. Graph of posterior dist. For GP when $v = 1, w = 1$ and $v = 1, w = 2$

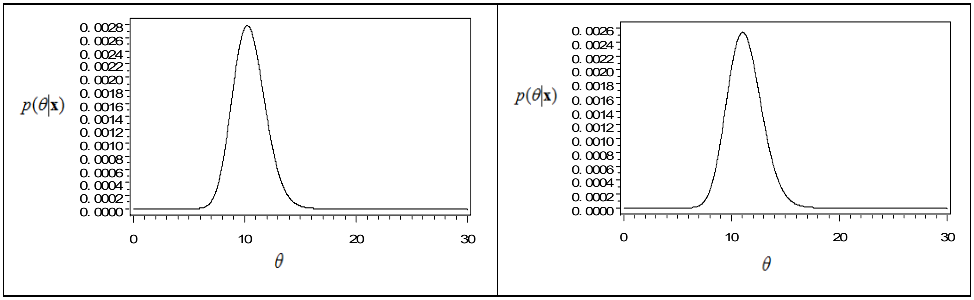

Figure 5.2. Graph of posterior dist. For GP when $v = 2, w = 1$ and ILP for $c = 1$

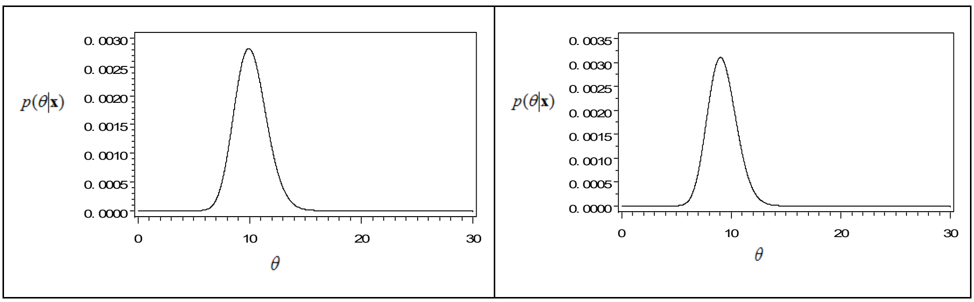

Figure 5.3. Graph of posterior dist. using ILP when $c = 2$ and $c = 3$

6. Concluding remarks

This paper explores important properties of the EGZ distribution under Bayesian using two informative priors: GP and ILP. This is done under five selected loss functions. We observe that the best loss functions are WBLF followed by QLF. The simulated study and real-life data were used for various sample sizes with 10,000 replications. Several values of the scale and shape parameters are considered. α is taken as 2, λ is taken as 1 and 3 and θ is taken as 1, 2 and 3. We observe that as the sample size is extended, PR declines and BE comes nearest to the true value of shape parameter

θ. Also, at various places over-estimation of parameters is noted. We also noted that the loss functions WBLF and QLF have similar PRs for all values of θ, α, λ under both the informative priors: GP and ILP. The only difference comes for GP for the values of hyperparameters (2,1), where the value of hyper-parameter 'v' changes and gives a different risk. In addition, WBLF has minimum PRs as compared to other loss functions followed by QLF under all the priors. Moreover, note that GP is the best prior as compared to all other priors and works best for the pair of hyperparameters (1, 2) since it has minimum PRs. The results for real life data and simulation are identical.

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The problem of statistical assessment of the potential for the development of regional integration processes

Oleksandr Osaulenko[1], **Olena Bulatova**[2], **Olha Zakharova**[3], **Natallia Reznikova**[4]

ABSTRACT

The present article illustrates the use of integrated indices to evaluate the potential for the development of regional integration processes. The study examines a new research and methodological approach, which involves the use of an integral index of the potential for the development of integration processes, proposed with regard to the intensity of the influence of internal and external factors on integrative relations development. The application of the above-mentioned integrated index in a comparative analysis of the potential for the development of integration processes allows a comprehensive and quantitative description of the current regional integration processes taking place in the modern economy under certain regional models.

Key words: integrated index, regional integration, EU, NAFTA, ASEAN, APTA.

1. Introduction

Modern processes of economic integration development taking place at the global and regional level are of complicated and contradictory nature and differ in depth and rate. However, today all the countries of the world economy are involved in this process regardless of the level of economic development achieved.

Modern research conducted by international organizations like the World Bank, the World Trade Organization, the UN, regional banks of development and others provide sufficient statistical data and methodology that allow defining the scale, intensity and peculiarities of regional integration processes development. To find more information about research conducted by the UN, see Statistics Database COMTRADE

[1] Doctor of Public Administration, Professor, National Academy of Statistics, Accounting and Audit, Ukraine. E-mail: O.Osaulenko@nasoa.edu.ua. ORCID: https://orcid.org/0000-0002-7100-7176.

[2] Doctor of Economics, Professor, Mariupol State University, Ukraine. E-mail: olena_bulatova@yahoo.com. ORCID: https://orcid.org/0000-0001-7938-7874.

[3] Candidate of Economics, Associate Professor, Mariupol State University, Ukraine. E-mail: o.zakharova@mdu.in.ua. ORCID: https://orcid.org/0000-0002-8718-0205.

[4] Doctor of Economics, Professor, Taras Shevchenko National University of Kyiv, Ukraine. E-mail: reznikovanataliia@gmail.com. ORCID: https://orcid.org/0000-0003-2570-869X.

with integration associations of the world included: http://www.unstats.un.org; comparative research of regional integration conducted under the program of the UN University (Regional Integration Knowledge System): www.cris.unu.edu/riks.

A number of regional organizations suggest a well-developed system of indicators that allow evaluating the level of development of regional integration processes using numerous best practices. The most widespread are as follows: indicators suggested by the European Commission (see http://ec.europa.eu); resource database of NAFTA Secretariat, which is used for studying the North American integration (see http://www.nafta-sec-alena.org/); indicators suggested by Asia Regional Integration Center (see http://aric.adb.org/) and Asian Development Bank (http://beta.adb.org/) applied to studying the integration processes taking place in Asia-Pacific region; Eurasian Development Bank uses its own methodology for studying the indicators of Eurasian integration (see http://www.eabr.org), which takes into account 3 types of indices: integration of a pair of countries (describes the intensity of ties between two post-Soviet countries), integration of the country as a part of a group of countries (characterizes the approximation of one post-soviet country to the biggest “regions” of the region), integration within the group of countries (characterizes the average level of interdependence between countries and dynamics of integration in time); Inter-American Development Bank uses a number of indicators to study the integration processes in the countries of Latin America. These indicators allow analyzing market shifts as well as public management indices, transparency development, democracy, etc. (see http://www.iadb.org).

Based on previous authors’ research (O. Bulatova et al., 2019) this article suggests combining existing indices of regional integration development into a respective system of integral evaluation, which will allow conducting complex comparative analysis of potential for integration process development.

Defining modern scientific and methodological basis for international regional integration, classification of certain statistical instruments to evaluate these processes provided in previous research allows drawing up the conclusion that integration process development is influenced by many factors, both internal and external.

For comprehensive accounting and evaluation of potential for integration processes development it is reasonable to apply integrated indices that provide comprehensive and quantitative description of regional integration processes taking place in modern economy under certain regional models at the present moment.

2. Building the integrated index of the potential for integration process development

As the analysis conducted shows, there is a need to develop certain methods of defining potential for further integration processes development. B. Balassa (1967) defined statistical and dynamical effects that appear in national economy as the result of deepening integration relations. Statistical effects are the result of redistribution of foreign trade flow of goods, factors of production, the result of introduction of the liberalization regime, expansion of markets of integrated countries may lead to positive trade creation effect and negative trade diversion effect. In the long run, structural transformations in economies of integrated countries result in dynamic effects related to the development of business competition inside integration association, establishment of joint infrastructure, lowering transaction costs, etc.

Integration processes development entails both quantitative and qualitative changes that influence structural shifts taking place in integrated countries. It is worth mentioning that it is still quite problematic to distinguish the influence of regional integration itself and deepening interaction between the countries due to the global integration development. A scientific inquiry by P. Lombaerde, G. Pietrangeli, C. Weeratunge (2008) that uses a number of indices that allow measuring the level of integration development at the level of regional groups, evaluating the role of every country in certain integration association based on its contribution, comparing processes of regional integration in different regions, etc., may be considered as a solution.

The evaluation of the development of regional economic integration requires various indices that allow analyzing the depth of integration relations between the countries that form integration association. B. Russet (2009) considers economic interaction to be one of the regional integration criteria which is manifested in intraregional trade that imposes direct effect for every member.

However, the most common in terms of assessing the level of integration interaction, determining the nature of the development of regional integration processes and their effectiveness, is the method of multidimensional evaluation, which involves the construction of appropriate comprehensive indicators. Scientific researches in this direction are aimed at developing an optimal integrated indicator, the structure of which most fully allows to provide a comprehensive assessment of the development of integration processes. In particular, De Lombaerde, Philippe & Dorrucci, Ettore & Genna, Gaspare & Mongelli, Francesco (2011), emphasizing the complex nature of the development of regional integration processes, the multi-vector nature of the changes, they are characterized, emphasize the need for continuous comprehensive monitoring to assess the system of indicators combined into appropriate integral estimates. In this respect, they propose a Composite Integration Index, which is a relative assessment of the level of development of the integration association and is based on 8 sectoral indices.

In a study by C.-Y. Park, R. Claveria (2018) it is proposed to apply the method of multidimensional evaluation to analyze the multifaceted measurement of regional integration processes based on the calculation of multidimensional regional integration index (MDRII), which includes 6 components (Trade and investment, Money and finance integration, Regional value chain, Infrastructure and connectivity, Movement of people, Institutional and social integration), combining 26 indicators (including integrated indices), which the authors tested to assess the development of integration processes at the level of individual countries.

A comprehensive indicator that reflects the stages of development of the integration process (acquis communautaire, Single Market integration, Economic and Monetary Union, economic convergence) and is proposed to be used to assess the integration aspirations of EU countries, was developed by J. König (2017). In a research by Mursalova, Kh.N. (2019), the methodological aspects of the application of complex indicators to assess the effectiveness of integration associations are analyzed, while the author does not specify what such an indicator should be, what its structure and the features of the calculation are.

In a study by Makkonen, Teemu (2016), in terms of forming the structure of a comprehensive index to assess the level of regional integration is determined by the lack of existing indices, and therefore expediency of consideration the component that would reflect the development of science, technology and innovation (Science, Technology, and Innovation indicators), which are drivers of economic growth in the processes of integration interaction in particular.

In the work of Gor, Seth. (2017) an analysis of the integrated integration index for the African region is presented (African Regional Integration Index), which includes 16 indicators combined in 5 areas (trade integration, productive integration, free movement of people, financial and macroeconomic integration and regional infrastructure). Michaela Stanickova & Lukáš Melecký (2018) offer a composite weighted index of regional resilience to assess integration processes in EU regions. According to the authors, the most important factors to be taken into account in such an index are as follows: community links, human capital and sociodemographic structure, labour market, economic performance and innovation, science and research.

Comparative analysis of approaches to the structuring of integrated indices allows us to reach conclusion about the predominance of a functional approach in the systematization of components, which allows taking into account the specific stages of development of integration cooperation and areas of interaction.

This study proposes an approach to assessing the development of integration processes, taking into account the internal and external components of the impact. In order to build the integrated index of the potential for integration process development it is necessary to introduce certain notations. Let us assume that there is a population of integration associations m and their level of development is characterized by the system of parameters (unique index). Let us set them as:

$Xi = (x_{i1}, x_{i2}, \ldots, x_{ij}, \ldots, x_{in})$, where

i – index of integrated association being analyzed ($i = \overline{1, m}$),

j – unique index which characterizes integration relations ($j = \overline{1, n}$).

Thus, x_{ij} represents the value of *j* index for integration association *i*.

When building integrated indices important methodical challenges are as follows: establishing the system of unique indices (parameters), which could provide suitable and comprehensive description of the stage of regional economic integration process development; choosing the form of integrated index itself, which will provide a generalized evaluation based on the system of unique indices built.

When addressing the first challenge, unique parameters may be seen as certain systematized indices that allow evaluating the depth of integration relations between the countries of the world. The systemization and classification of the system of unique indices prove that the level of integration processes development on the one hand is determined by intraregional factors that define scale, depth and specifics of integration relations development within existing regional integration associations characterized by the intraregional trade and its share in total external trade turnover, the share of high-tech export in total intraregional export, index of GDP per capita in integration association. On the other hand, it is determined by the influence of external factors that characterize the place and role of integration association in global processes (the share of integration association in the world trade turnover including high-tech export, investments, population, etc.). The choice of the above mentioned indices is based on the results of correlation-regression analysis. Its results are provided in Table 1.

Table 1. The results of correlation-regression analysis of intraregional trade development (X_{intra}) EU, NAFTA, ASEAN, APTA

EU

Factors	Regression coefficient	Elasticity coefficient
GDP per capita, millions of U.S. dollars	130.2647	1.305492
ICT export, millions of U.S. dollars	1.961292	0.152931
Foreign Direct Investments, millions of U.S. dollars	-0.14551	-0.277184
Population, thousands	29.04114	5.180026
$R^2 = 0.9743$		
$X_{intra} = 130,26*GDP + 1,96*X_{ICT} - 0,15*FDI + 29,04*Pop - 1483424,58$		

NAFTA

Factors	Regression coefficient	Elasticity coefficient
GDP per capita, millions of U.S. dollars	52.40168	2.119695
ICT export, millions of U.S. dollars	5.186308	0.560561
Foreign Direct Investments, millions of U.S. dollars	-0.03817	-0.16249
Population, thousands	-6.20109	-3.34038
$R^2 = 0.9725$		
$X_{intra} = 52,4*GDP + 5,19*X_{ICT} – 0,04*FDI – 6,2*Pop + 1494523,34$		

ASEAN

Factors	Regression coefficient	Elasticity coefficient
GDP per capita, millions of U.S. dollars	161.0476	1.82068
ICT export, millions of U.S. dollars	1.951287	0.492421
Foreign Direct Investments, millions of U.S. dollars	-0.24749	-0.76421
Population, thousands	0.035534	0.112483
$R^2 = 9941$		
$X_{intra} = 161,05*GDP + 1,95*X_{ICT} – 0,25*FDI + 0,04*Pop - 117405,82$		

APTA

Factors	Regression coefficient	Elasticity coefficient
GDP per capita, millions of U.S. dollars	79.12081	0.991164
ICT export, millions of U.S. dollars	3.253449	0.690553
Foreign Direct Investments, millions of U.S. dollars	-0.04201	-0.14778
Population, thousands	-0.29826	-5.20732
$R^2 = 0.9948$		
$X_{intra} = 79,12*GDP + 3,25*X_{ICT} – 0,04*FDI – 0,3*Pop + 709704,6$		

Calculations allowed setting the system of unique parameters that build the integrated index. To be able to compare all the unique parameters it is necessary to standardize them. This will allow combining them in the integrated index. Authors suggest calculating standardized evaluation using this formula:

$$P_{ij} = \frac{x_{ij} - x_{\min}}{x_{\max} - x_{\min}}$$

where:

x_{ij} – the value of the unique index *j* for integration association *i*,

x_{min}, x_{max} respectively, minimum and maximum value of the unique index j.

The calculation of standardized evaluation using the above mentioned formula is carried out when the increased value of certain parameter leads to the increase of the integrated index itself (*incentive index*).

On the contrary, if the increase in the unique parameter leads to the decrease in the final integrated value (*disincentive index*), then the following formula should be used:

$$P_{ij} = \frac{x_{\max} - x_{ij}}{x_{\max} - x_{\min}}$$

It becomes clear that certain unique indices describing different aspects of integration relations do not equally affect its overall state. As a result, when building an integrated index of evaluation of potential for integration processes development it is necessary to define the value of every unique index mentioned above, i.e. coefficient of significance for α_{ij}.

Table 2. Building the system of unique indices

Intraregional factors		External factors	
Intraregional trade (export) per capita, U.S. dollars	X_{Intra}	The share of regional trade of an integration association in the world trade, %	$X_{Intra}S$
The share of intraregional trade and its share in total external trade turnover, %	*ITS*	The share of investments of an integration association in world volume of investments, %	*FDIS*
The share of ICT in intraregional export, %	$X_{Intra\ \ ICT}$	The share of GDP of integration association in world GDP, %	*GDPS*
Index of investments per capita, U.S. dollars	*FDI*	The share of population of integration association in population of the world, %	*PopS*
GDP of integration association per capita, U.S. dollars	*GDP*	The share of ICT export of integration association in the world export, %	$X_{ICT}S$

If talking about the challenge of choosing the form of integrated index it appears that different types of weighted average are widely used in scientific research while building an overall index: arithmetical average, geometric average, square root average and some others. According to the analysis of practical application of different types of average it is advisable to consider the index of geometric weighted average as a form of an index while building integrated indices, when monotonous increase in certain parameter leads to the improvement of its state and overall index of its state requires maximization. This index may be written as:

$$I_i = \prod_{j=1}^{n} (P_{ij})^{\alpha_j} \quad \text{where in} \quad \alpha_j \geq 0 \text{ i } \sum_{j=1}^{n} \alpha_{ij} = 1 .$$

The advisability of using this type of average to calculate the integrated index of integration processes development is proved by the provisions of axiomatic approach (monotonicity axiom, positive linear homogeneity, multiplicative axiom and identity axiom) described in the index theory by I. Fisher. Taking into account all of the above, the overall integrated index of the potential for integration process development should be:

$$IPD_{\mathrm{i}} = \frac{I_{int_{\mathrm{i}}} + I_{ext_{\mathrm{i}}}}{2}$$

where:

- IPD_i – overall integrated index of the potential for integration processes development;
- I_{inti} – intraregional component of the potential for integration processes development;
- I_{exti} – external component of the potential for integration processes development.

The subindex that characterizes the influence of an intraregional component of overall integrated index of integration processes development should be calculated as follows:

$$I_{\mathrm{int}_i} = \sqrt[5]{P_{X\,\mathrm{int}\,ra_i} \cdot P_{ITS_i} \cdot P_{X\,\mathrm{int}\,ra\,ICT_i} \cdot P_{FDI_i} \cdot P_{GDP_i}}$$

The subindex that characterizes the influence of an external component on overall integrated index of integration processes development should be calculated as follows:

$$I_{ext_i} = \sqrt[5]{P_{X\,\mathrm{int}\,raS_i} \cdot P_{PopS_i} \cdot P_{X\,ICT\,S_i} \cdot P_{FDIS_i} \cdot P_{GDPS_i}}$$

The algorithm of calculating the integrated index of the potential for integration processes development is described in Figure 1. The suggested index ranges from 0 to 1, its proximity to 1 shows significant potential for integration processes development.

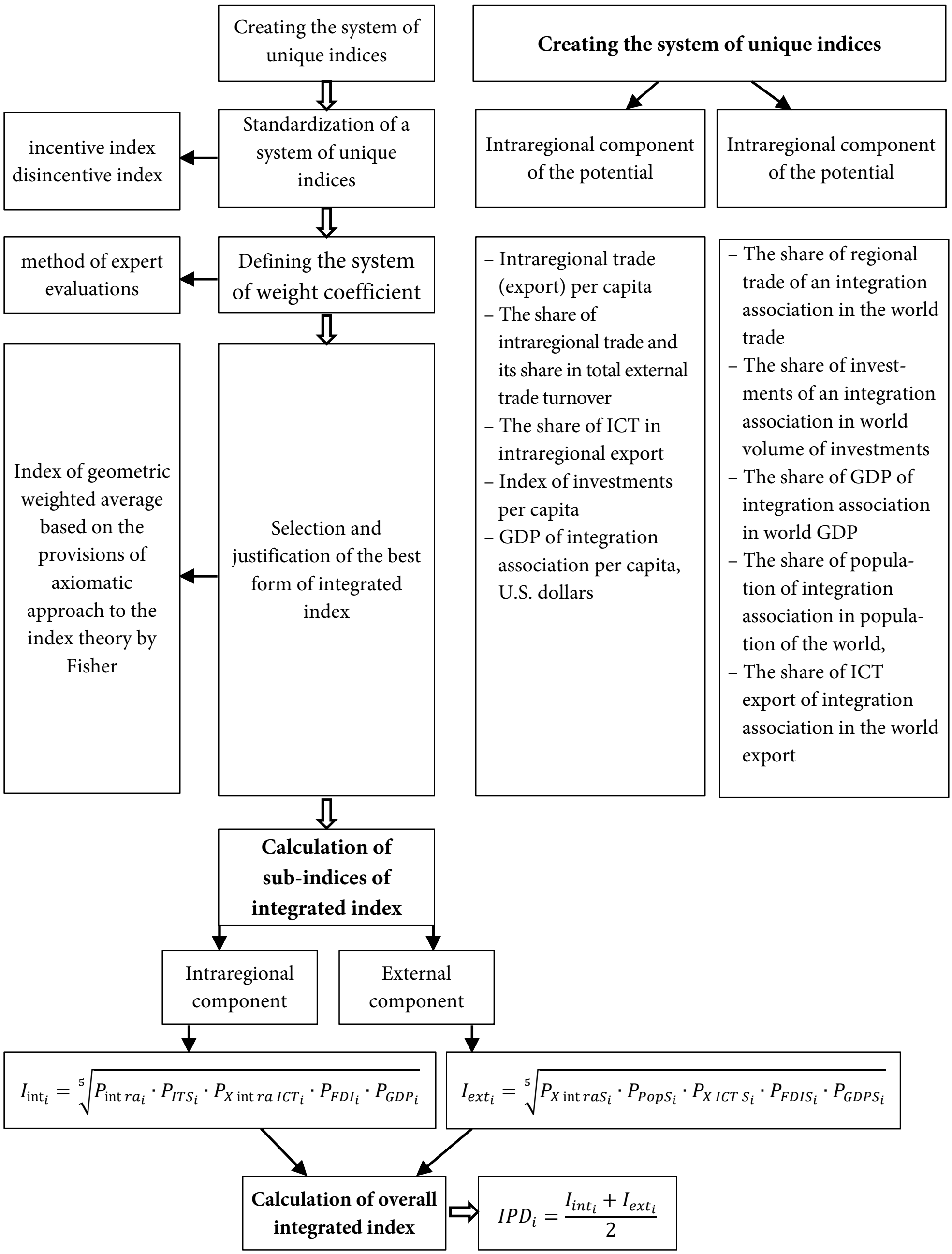

Figure1. The algorithm of building an integrated index of the potential for integration processes development

Using the integrated index for comprehensive comparative analysis of the potential for integration processes development allows taking into account the intensity of influence of many factors (both internal and external), which in its turn allows providing a comprehensive and quantitative description of economic integration processes that take place in the world economy under certain regional models.

3. Evaluation

Using the suggested method the authors calculate the integrated index of the potential for integration processes development for EU, NAFTA, ASEAN and APTA with regard to the intensity of the influence of factors that determine the intraregional component of the potential for development of integration processes as well as factors that allow evaluating the external component of the potential for integration processes development determined by the role of an integration association in the world economy.

UNCTAD database was used for calculations systematized in Tables 3-4. Integrated indices of the potential for integration processes development calculated are provided in Table 5 and in Figure 2.

Table 3. Dynamics of indices of intraregional component of the potential for integration processes

Years	Indices of intraregional component of the potential for integration processes																			
	Intraregional trade index (export) per capita, U.S. dollars				The share of intraregional trade in total external trade turnover , %				The share of ICT in intraregional export, %				Index of investments per capita, U.S. dollars				GDP of integration association per capita, U.S. dollars			
	APTA	ASEAN	EU	NAFTA	APTA	ASEAN	EU	NAFTA	APTA	ASEAN	EU	NAFTA	APTA	ASEAN	EU	NAFTA	APTA	ASEAN	EU	NAFTA
2000	15,0	186,5	3333,5	1647,2	8,0	22,9	67,7	55,8	5,1	23,5	67,4	44,5	101,6	490,7	4761,9	7783,4	915	1188	18274	28357
2001	15,6	161,3	3342,8	1509,6	8,5	22,3	67,3	55,1	6,1	22,6	70,2	44,8	110,4	486,4	4958,6	7239,1	948	1131	18446	28953
2002	19,3	170,2	3572,2	1464,9	9,2	22,7	67,5	56,1	7,9	22,8	71,7	45,0	120,4	528,7	5629,9	5842,3	1028	1251	20024	29586
2003	27,0	197,1	4294,2	1518,2	10,1	22,8	68,6	55,9	8,7	24,3	69,5	44,1	127,7	595,3	7218,4	7170,2	1153	1386	24271	30735
2004	37,6	243,8	5176,4	1711,5	10,7	23,7	68,5	56,1	9,3	23,4	69,1	46,7	143,2	674,3	8208,7	8020,0	1327	1542	27915	32613
2005	47,9	283,4	5533,3	1888,7	11,1	24,2	67,8	55,9	9,5	22,8	67,0	45,7	160,2	773,4	8789,0	8479,3	1535	1709	29069	34724
2006	57,3	325,4	6289,1	2042,8	10,9	24,0	68,1	53,9	9,0	21,8	72,0	45,3	180,8	978,3	11096,0	9686,1	1783	2021	30914	36678
2007	71,0	365,7	7289,8	2130,3	11,1	24,3	68,1	51,3	9,4	22,1	68,0	45,5	207,1	1284,7	14306,6	10966,7	2195	2383	35601	38221
2008	85,8	425,7	7921,9	2243,5	11,5	25,1	66,9	49,5	10,1	21,2	68,4	46,7	221,4	1339,3	12505,4	7504,8	2543	2733	38182	38704
2009	74,1	329,7	6043,4	1683,6	11,8	23,9	66,3	48,0	10,5	20,1	70,7	50,3	280,1	1513,4	13605,9	9210,2	2543	2733	38182	38704
2010	99,9	431,3	6606,0	2075,6	12,3	24,4	64,5	48,7	10,3	20,3	70,4	51,7	336,3	1920,2	13566,7	10404,3	3211	3318	33707	38542
2011	116,5	510,5	7634,2	2367,9	11,9	24,6	63,5	48,3	9,6	20,7	68,4	52,0	379,0	2105,2	13944,9	10193,9	3812	3806	36353	39993
2012	115,5	522,8	7051,0	2450,0	11,4	25,3	61,7	48,6	10,3	20,9	67,7	52,0	434,5	2437,3	15484,8	11339,3	4128	3981	34177	41134
2013	122,1	531,8	7351,6	2505,6	11,3	25,7	61,2	49,2	10,8	20,4	68,8	52,5	483,2	2603,7	16167,3	13541,8	4512	4062	35557	42201
2014	127,7	521,9	7532,6	2618,0	11,4	25,2	62,6	50,2	12,0	19,2	70,6	52,5	533,1	2822,2	16869,0	14518,1	4860	4049	36698	43383
2015	123,2	445,0	6558,6	2394,5	11,7	24,1	62,3	50,4	13,1	18,8	71,0	53,3	586,0	2875,9	17001,1	14597,6	5053	3891	32272	43660
2016	114,5	420,5	6633,4	2288,4	11,7	23,4	63,4	50,3	12,9	19,0	70,2	53,2	640,7	2980,0	16861,8	16509,7	5121	4036	32405	44079
2017	130,1	471,2	7241,5	2428,9	12,2	23,2	63,2	50,1	14,6	18,7	70,6	52,0	714,9	3501,7	19828,3	19204,8	5582	4297	33941	45793
2018	144,1	521,4	7965,9	2569,4	12,5	23,6	63,6	49,4	14,6	22,9	71,9	53,8	761,9	3640,6	19682,2	17921,7	6116	4543	36553	47856

Table 4. Dynamics of indices of external component of the potential for integration processes

Years	Indices of external component of the potential for integration processes																			
	The share of integration association in the world trade, %				The share of GDP of integration association in world GDP,%				The share of investments of an integration association in world volume of investments, %				The share of population of integration association in population of the world, %				The share of ICT export of integration association in the world export, %			
	APTA	ASEAN	EU	NAFTA	APTA	ASEAN	EU	NAFTA	APTA	ASEAN	EU	NAFTA	APTA	ASEAN	EU	NAFTA	APTA	ASEAN	EU	NAFTA
2000	7,4	6,7	38,0	19,0	6,9	1,9	26,5	35,0	3,5	3,5	31,5	43,8	41,5	8,5	7,9	6,8	10,4	17,0	27,3	21,2
2001	7,6	6,2	40,0	18,5	7,3	1,8	27,0	36,3	3,8	3,5	32,5	40,7	41,4	8,6	7,9	6,7	11,3	16,7	29,2	20,2
2002	8,4	6,3	40,7	17,0	7,7	1,9	28,2	36,1	4,3	3,9	37,7	33,8	41,4	8,6	7,8	6,7	14,8	16,8	27,8	17,3
2003	9,3	6,3	41,6	15,3	7,8	1,9	30,6	33,7	3,7	3,6	38,9	33,6	41,3	8,6	7,7	6,7	18,4	16,5	26,0	15,4
2004	10,2	6,2	40,9	14,3	8,0	1,9	31,4	32,1	3,6	3,5	38,5	32,9	41,3	8,6	7,7	6,7	21,4	16,0	25,8	14,1
2005	11,1	6,3	38,8	14,1	8,7	2,0	30,3	31,9	3,8	3,8	38,2	32,4	41,2	8,6	7,6	6,7	24,5	11,6	26,9	13,9
2006	11,9	6,3	37,9	13,7	9,4	2,2	29,9	31,4	3,5	3,9	39,3	30,4	41,1	8,6	7,5	6,7	25,7	11,3	27,3	13,5
2007	12,6	6,2	38,2	13,1	10,4	2,4	30,6	29,3	3,2	4,1	39,9	27,3	41,0	8,6	7,5	6,7	28,7	14,6	22,7	12,6
2008	12,8	6,1	36,8	12,6	11,1	2,5	30,0	27,4	4,1	5,2	41,8	22,5	40,9	8,6	7,4	6,6	29,8	13,6	22,4	12,7
2009	14,0	6,5	36,7	12,8	12,4	2,6	28,3	27,8	4,5	5,0	38,8	23,8	40,8	8,6	7,3	6,6	31,5	14,5	19,8	12,4
2010	15,0	6,9	33,9	12,8	13,7	3,0	25,7	26,8	4,8	5,8	34,7	24,3	40,7	8,6	7,3	6,6	32,7	14,6	18,6	11,9
2011	15,3	6,8	33,2	12,5	14,8	3,1	25,0	25,3	5,3	6,2	34,7	23,3	40,6	8,6	7,2	6,6	33,8	13,8	18,4	11,6
2012	15,9	6,8	31,4	12,8	15,8	3,2	23,1	25,7	5,5	6,5	34,5	23,4	40,4	8,6	7,1	6,6	35,1	14,0	15,9	11,4
2013	16,5	6,7	32,1	12,8	16,9	3,2	23,4	25,9	5,7	6,5	33,3	26,0	40,3	8,6	7,1	6,6	37,2	14,4	15,3	11,0
2014	17,3	6,8	32,4	13,1	18,0	3,2	23,6	26,2	6,0	6,8	33,2	26,8	40,2	8,6	7,0	6,6	36,4	14,6	15,3	10,9
2015	18,8	7,0	32,6	13,9	19,9	3,3	22,0	28,1	6,6	6,9	33,0	26,8	40,0	8,6	6,9	6,5	37,1	14,9	14,0	10,9
2016	18,2	7,1	33,5	13,8	20,1	3,4	21,8	28,2	6,8	6,8	30,6	28,4	39,9	8,6	6,9	6,5	35,2	15,1	14,4	10,9
2017	18,0	7,4	33,3	13,4	20,7	3,4	21,5	27,8	6,6	6,9	31,2	28,8	39,8	8,6	6,8	6,5	35,5	15,3	14,0	10,3
2018	17,9	7,4	33,3	13,2	21,6	3,5	21,9	27,6	7,1	7,4	31,3	27,4	39,6	8,6	6,7	6,5	38,0	12,2	14,5	10,0

Table 5. Integrated indices of the potential for integration processes development calculated

Years	2000	2001	2002	2003	2004	2005	2006	2007	2008	2009	2010	2011	2012	2013	2014	2015	2016	2017	2018
Intraregional component of the potential for development of integration processes																			
APTA	0,0641	0,0669	0,0878	0,1073	0,1177	0,1487	0,1487	0,1419	0,1487	0,1477	0,1487	0,1487	0,1496	0,1514	0,1526	0,1457	0,1446	0,1493	0,1569
ASEAN	0,2277	0,2224	0,2264	0,2215	0,2228	0,1977	0,1920	0,2265	0,2262	0,2287	0,2335	0,2268	0,2382	0,2449	0,2498	0,2439	0,2405	0,2388	0,2297
EU	0,6002	0,6130	0,6722	0,7013	0,7141	0,7104	0,7113	0,7283	0,7386	0,7386	0,7166	0,7232	0,7072	0,7097	0,7105	0,6881	0,6871	0,6883	0,6931
NAFTA	0,6687	0,6499	0,6373	0,6248	0,6144	0,6192	0,5883	0,5597	0,5300	0,5450	0,5712	0,5632	0,5772	0,5978	0,6022	0,6059	0,6197	0,6129	0,5898
External component of the potential for integration processes development																			
APTA	0,1874	0,1948	0,2026	0,2711	0,3434	0,3897	0,4092	0,4387	0,4494	0,4715	0,4986	0,5182	0,5357	0,5487	0,5607	0,5775	0,5681	0,5734	0,5797
ASEAN	0,0790	0,0644	0,0393	0,0305	0,0425	0,0108	0,0133	0,0400	0,0270	0,0356	0,0429	0,0365	0,0398	0,0428	0,0454	0,0441	0,0449	0,0528	0,0300
EU	0,6498	0,6626	0,7095	0,7338	0,7467	0,7410	0,7404	0,6791	0,6656	0,6243	0,6007	0,6022	0,5558	0,5491	0,5513	0,5129	0,5177	0,5183	0,5255
NAFTA	0,5566	0,5218	0,4260	0,3598	0,3535	0,3919	0,3748	0,3158	0,2800	0,3018	0,3217	0,3141	0,3214	0,3388	0,3448	0,3490	0,3677	0,3622	0,3504
Overall integral index of the potential for integration processes development																			
APTA	0,1257	0,1309	0,1452	0,1892	0,2306	0,2692	0,2790	0,2903	0,2990	0,3096	0,3237	0,3335	0,3427	0,3500	0,3567	0,3616	0,3563	0,3614	0,3683
ASEAN	0,1534	0,1434	0,1328	0,1260	0,1326	0,1043	0,1026	0,1333	0,1266	0,1322	0,1382	0,1317	0,1390	0,1439	0,1476	0,1440	0,1427	0,1458	0,1298
EU	0,6250	0,6378	0,6909	0,7175	0,7304	0,7257	0,7259	0,7037	0,7021	0,6815	0,6587	0,6627	0,6315	0,6294	0,6309	0,6005	0,6024	0,6033	0,6093
NAFTA	0,6126	0,5859	0,5316	0,4923	0,4839	0,5055	0,4816	0,4377	0,4050	0,4234	0,4465	0,4387	0,4493	0,4683	0,4735	0,4774	0,4937	0,4876	0,4701

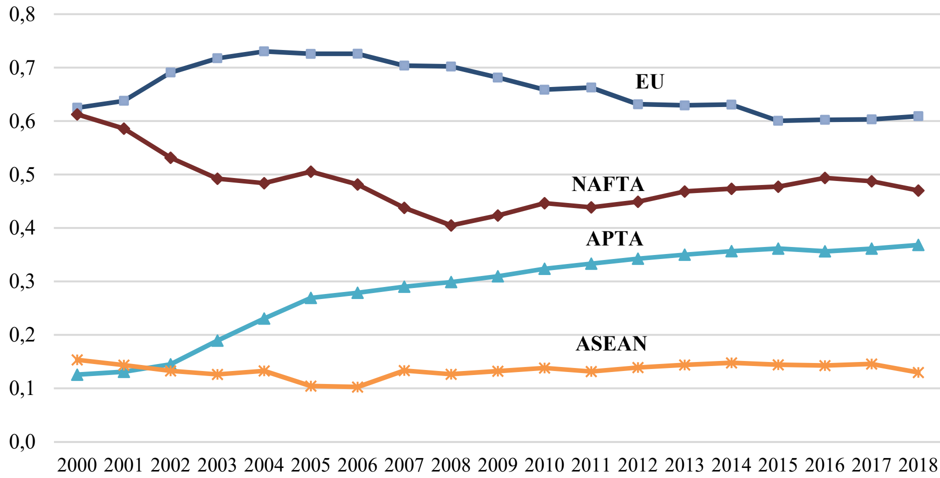

Figure 2. Dynamics of integrated index of potential for integration processes

Despite the obvious fact that the existing trend in integration cooperation development loses its ground, the EU, compared to other organizations, is still an association with the highest rate of the potential for development. Starting from 2007, the intraregional factors determined the nature of changes in integration cooperation inside the association. There is a tendency of decreasing sub-indices of both internal and external components, although the changes take place with a different rate. In particular, the decrease in the external component outpaces a similar decrease in the internal component. Thus, the index of the external component decreased by 29% (in 2018 compared to 2006 with top value of indices), while the internal one increased by 3.6%.

NAFTA is characterized by a relatively high level of potential for integration development, which, unlike the EU, is characterized by somewhat different laws of the change of the integral index. Thus, during the period of 2000-2008, there was a decrease in the level of development of integration processes, the integral indicator decreased by 34.7%, from 0.613 in 2000 to 0.405 in 2008, which is the lowest value of the potential of integration cooperation. In the period 2008-2018, there is an increase in the level of the integrated indicator, which increased by 16.1%, but is lower by 23.3% compared to the level of 2000.

So far, ASEAN is characterized by a low potential of the development of integration processes, which mostly consist of the indices of the intraregional component, whose value stays the same from 2000 until 2018 (0.23-0.25). At the same time, there are no significant changes in the dynamics of the integrated indicator. It is worth noting the decrease in the level of development potential of integration processes (for 2000-2018, a decrease of 15.3%).

APTA is characterized by the most intensive and accelerated growth rate of the potential for the integration processes development (the integrated index increased 2.93 times from 0.126 in 2000 to 0.368 in 2018). The development of integration processes APTA is associated with the most rapid growth of the external component of development, the integral value of which became 3.1 times higher, reaching the level of 0.58 in 2018, which is the maximum value of the external component of the development of the potential of integration processes in the analyzed associations.

4. Conclusion

In order to evaluate the potential for the integration processes development, it is advisable to apply integral indicators as they allow providing a comprehensive and quantitative description of processes of economic integration that take place in the world economy at a certain moment of time.

Today, there is no single model of a regional trade agreement, so in most cases each country or integration association applies an integrated approach to exploiting the possibilities of integration agreements regarding access to new markets, expansion of investment opportunities, reduction of transaction costs, establishment of unified technical norms and requirements, protection of intellectual property, establishment of a unified competitive policy, transparency of the mechanism of state regulation. Such an integrated approach as a whole contributes to the deepening of integration cooperation with partner countries and creation of predictable political conditions that influence the development of trade and economic cooperation with all regions within the framework of certain economic space between states, and thus the expansion of continental and transcontinental integration cooperation.

The greatest impact on the development of regionalization of the world economy with further building of the world's global space will be imposed by deepening and expansion of the most developed models of regional integration, which have already existed the world economy - European, North American and Asian-Pacific models. Involvement of other countries in this process, on a regional, continental or transcontinental basis, through the creation of free trade areas and other forms of "soft" integration will contribute to the deepening of the development of "new regionalism" and the emergence of totally new integrational entities that will no longer be of regional but of transcontinental nature, which corresponds to the level of international meta-regions in the world economy. Countries of the world choose their own strategy for participation in the processes of regional integration, depending on challenges determined by their level of socio-economic development, the existing potential, the nature of the development of external relations, etc. Considering the integration component of the strategy of further development of the countries of the world, it is necessary to pay attention to two aspects: first, to the features and lines of deepening the integration cooperation right inside the existing integration association; second is to define the lines of the integration policy towards other countries involved in the cooperation under continental and transcontinental models.

The study of further development of continental and transcontinental models of regional integration is a logical extension of the analysis of models of international economic integration development, which encompasses not only trade and economic, but also other spheres (first of all, an industrial one). At the same time, existing fair restrictions objectify further study of the development of transcontinental integration just in terms of the implementation of trade and economic cooperation, as deeper forms of integration will face institutional constraints.

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Type II Topp-Leone Frechet distribution: properties and applications

Rama Shanker[1], **Umme Habibah Rahman**[2]

ABSTRACT

The paper focuses on type II Topp-Leone Frechet distribution. Its properties including hazard rate function, reverse hazard rate function, Mills ratio, quantile function and order statistics have been studied. The maximum likelihood estimation used for estimating the parameters of the proposed distribution has been explained and expressions for the Fisher information matrix and confidence intervals have been provided. The paper discusses the applications of the distribution for modeling several datasets relating to temperature. Finally, the goodness of fit of the proposed distribution has been compared with that of the Frechet distribution.

Key words: Frechet distribution, Topp-Leone distribution, reliability properties, applications.

1. Introduction

Frechet distribution introduced by Mourice Rene Frechet (1927) is defined by its cumulative distribution function (cdf) and probability density function (pdf)

$$G(x;\alpha,\beta)=e^{-\alpha x^{-\beta}}\,;\,x>0,\alpha>0,\beta>0 \tag{1.1}$$

and $$g(x;\alpha,\beta)=\alpha\beta\,x^{-(\beta+1)}e^{-\alpha x^{-\beta}}\,;\,x>0,\alpha>0,\beta>0 \tag{1.2}$$

where $\alpha>0$ is a shape parameter $\beta>0$ is a scale parameter. It is an inverse of Weibull distribution introduced by Weibull (1951). Shanker and Shukla (2019) derived a generalization of Weibull distribution and discussed its statistical properties, estimation of parameter and applications. Frechet distribution is the type II extreme value distribution used for modeling extreme data from accelerate life testing, natural calamaties, rainfall, temperature, wind speed and so on. Nadarajah and Kotz (2003a, 2006) introduced exponentiated Frechet distribution and other exponentiated type

[1] Corresponding author. Department of Statistics, Assam University, Silchar. India.
E-mail: shankerrama2009@gmail.com. ORCID: https://orcid.org/0000-0002-5002-8904.
[2] Department of Statistics, Assam University, Silchar. India.
E-mail: umme.habibah.rahman17@gmail.com. ORCID: https://orcid.org/0000-0002-6168-6283.

distributions and discussed their statistical properties and parameter estimation. Mubarak (2012) discussed maximum likelihood and least squares estimates for the parameter of Frechet distribution based on progressive type II censoring. The transmuted Frechet distribution and Marshall-Olkin Frechet distribution were discussed by Mahmoud and Mandouh (2013) and Krishna and Jose (2013), respectively. The Bayesian estimation of the shape parameter of Frechet distribution using different prior distribution and various loss functions has been discussed by Nasir and Aslam (2015). The beta exponential Frechet distribution and Weibull- Frechet distribution have been discussed by Mead *et al* (2017) and Afify *et al* (2016), respectively.

The Topp-Leone distribution (TLD) proposed by Topp and Leone (1955) is one of the continuos distribution useful for generating new distribution. The most important characteristics of TLD is to provides closed forms for both the pdf and cdf. The TLD distribution received attention in statistics after the works of Nadarajah and Kotz (2003b) who studied some properties of TLD including moments, central moments and characteristic function. Ghitany *et.al.* (2005) discussed some reliability measures and stochastic orderings of TLD. The goodness of fit tests for the TLD has been studied by Al-Zahrani (2012). Reyad et al (2021) studied the properties, estimation and applications of Frechet Topp-Leone G-family of distributions.

The type-I TLD developed for empirical data with J-shaped histogram such as powered band functions and automatic calculating machine failure. Suppose a continuos random variable X following type-I TLD (TITLD) are given by

$$F_{TLG}(x) = [G(x)]^{\alpha}\,[2 - G(x)]^{\alpha} \tag{1.3}$$

and $$f_{TLG}(x) = 2\alpha\, g(x)[1 - G(x)][G(x)]^{\alpha-1}[2 - G(x)]^{\alpha-1} \tag{1.4}$$

where $g(x) = \dfrac{dG(x)}{dx}$ and $\alpha > 0$ is a shape parameter. It has been observed that the TL random variable with finite support has the same bounds as the cdf $G(x)$ of any other random variable.

By taking $G(x) = \left(1 - e^{-\lambda x}\right)^{\beta}$, where $\lambda > 0$ is a scale parameter and $\beta > 0$ is a shape parameter, as the cdf of generalized exponential distribution proposed by Gupta and Kundu (1999), Sangsanit and Bodhisuwan (2016) introduced the Topp-Leone generalized exponential distribution(TLGED) defined by its pdf and cdf,

$$f_{TLGE}(x;\alpha,\beta,\lambda) = 2\alpha\beta\lambda e^{-\lambda x}\left\{1 - \left(1 - e^{-\lambda x}\right)^{\beta}\right\}\left(1 - e^{-\lambda x}\right)^{\beta\alpha-1}\left\{2 - \left(1 - e^{-\lambda x}\right)^{\beta}\right\}^{\alpha-1} \tag{1.5}$$

and $$F_{TLGE}(x;\alpha,\beta,\lambda) = \left(1 - e^{-\lambda x}\right)^{\beta\alpha}\left\{2 - \left(1 - e^{-\lambda x}\right)^{\beta}\right\}^{\alpha} \tag{1.6}$$

Various statistical properties, estimation of parameters using maximum likelihood estimation and goodness of fit of TLGED have been studied by Sangsanit and Bodhisuwan (2016).

Recently, Elgarhy *et al* (2018) introduced Type-II Topp-Leone generalized family of distribution. Suppose, $g(x)$ and $G(x)$ are the pdf and cdf of the parent distribution. The pdf and cdf of a random variable X following Type-II Topp-Leone distribution (TIITLD) is defined by its cdf and pdf,

$$F(x;\beta)=1-\left[1-\{G(x)\}^2\right]^{\beta};x>0,\beta>0 \tag{1.7}$$

$$f(x;\beta)=2\beta\, g(x)G(x)\left[1-\{G(x)\}^2\right]^{\beta-1};x>0,\beta>0 \tag{1.8}$$

where $\beta>0$ is a shape parameter.

Since Frechet distribution has been extensively used in the modeling of data related to temperature ,it is hoped and expected that the proposed distribution which is an extension of Frechet distribution using Type II Topp-Leone distribution would provide a better fit for temperature data.

The main motivation of considering Type II Topp-Leone Frechet distribution is that Frechet distribution being two-parameter distribution is very much useful for modeling data relating to temperature and it is expected that the proposed distribution, being three-parameter distribution and based on the concept of Type II Topp-Leone distribution, would provide better fit over Frechet distribution. Some of the important properties of the proposed distribution including shapes of the pdf and cdf, asymptotic behaviour, hazard rate function, reverse hazard rate function, Mills ratio have been studied. Maximum likelihood estimation has been discussed for estimating parameters of the proposed distribuion. Finally, applications of the proposed distribution for modeling datasets relating to minimum temperature of Silchar, Assam have been discussed.

2. Type II Topp-Leone Frechet Distribution

Using the cdf and pdf of Frechet distribution in (1.7) and (1.8), the cdf and the pdf of type II Topp-Leone Frechet distribution (TIITLFD) can be expressed as

$$F(x;\alpha,\beta,\lambda)=1-\left\{1-\left(1-e^{-\alpha x^{-\beta}}\right)^2\right\}^{\lambda};x>0,\alpha>0,\beta>0,\lambda>0 \tag{2.1}$$

$$f(x;\alpha,\beta,\lambda)=2\alpha\beta\,\lambda\, x^{-(\beta+1)}\, e^{-2\alpha x^{-\beta}}\left\{1-\left(e^{-\alpha x^{-\beta}}\right)^2\right\}^{\lambda-1};x>0,\alpha>0,\beta>0,\lambda>0 \tag{2.2}$$

where α and λ are shape parameters and β is a scale parameter. Further,

$$\lim_{x\to-\infty}F(x;\alpha,\beta,\lambda)=0$$

$$\text{and }\lim_{x\to+\infty}F(x;\alpha,\beta,\lambda)=1$$

This shows that TIITLFD is a proper density function.

Graphs of the pdf and the cdf of TIITLFD are shown in fig.1 and fig.2 for varying values of the parameters α, β and λ .

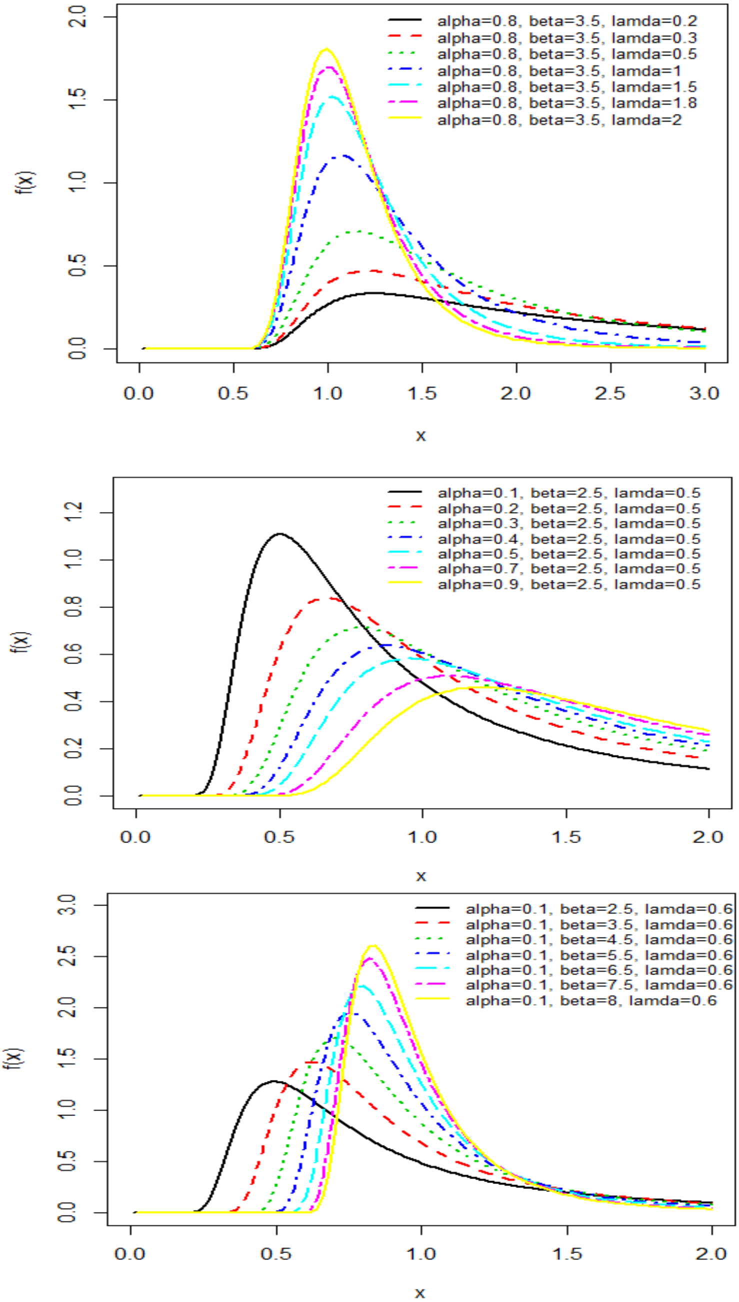

Figure1. Graphs of the pdf of TIITLFD for varying values of parameters

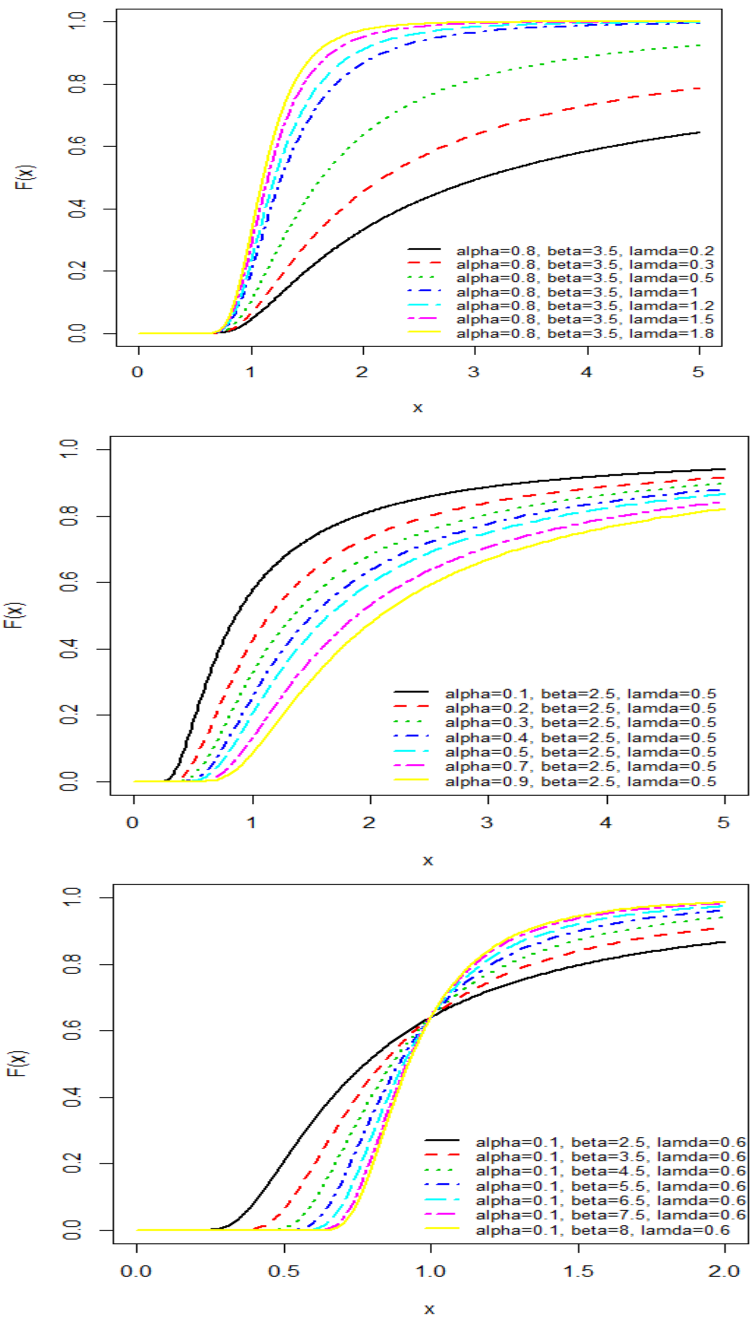

Figure2. Graphs of the cdf of TIITLFD for varying values of parameters

3. Statistical Properties

In this section, statistical properties including asymptotic behaviour, survival function, hazard function, reverse hazard rate and mills ratio of TLLTLFD has been studied.

3.1. Asymptotic behavior

The asymptotic behavior of TIITLFD for $x \to 0$ and $x \to \infty$ are

$$\lim_{x\to 0} f(x;\alpha,\beta,\lambda) = \lim_{x\to 0}\left[2\alpha\beta\lambda\, x^{-(\beta+1)} e^{-2\alpha x^{-\beta}} \left\{1-\left(e^{-\alpha x^{-\beta}}\right)^2\right\}^{\lambda-1}\right] = 0$$

and $$\lim_{x\to \infty} f(x;\alpha,\beta,\lambda) = \lim_{x\to \infty}\left[2\alpha\beta\lambda\, x^{-(\beta+1)} e^{-2\alpha x^{-\beta}} \left\{1-\left(e^{-\alpha x^{-\beta}}\right)^2\right\}^{\lambda-1}\right] = 0.$$

These results confirm that the proposed distribution has a mode.

3.2. Reliability properties

The survival function (or the reliability function) is the propability that a subject survives longer than the expected time. The survival function of the TIITLFD is given by

$$S(x;\alpha,\beta,\lambda) = 1 - F(x;\alpha,\beta,\lambda) = \left\{1-\left(e^{-\alpha x^{-\beta}}\right)^2\right\}^{\lambda}.$$

The hazard function (also known as the hazard rate, instantaneous failure rate or force of mortality) is the probability to measure the instant death rate of a subject. Suppose X be a continuous random variable with pdf $f(x)$ and cdf $F(x)$. The hazard rate function of X is defined as

$$h(x) = \lim_{\Delta x \to 0} \frac{P(X < x + \Delta x / X > x)}{\Delta x} = \frac{f(x)}{1-F(x)}$$

The corresponding $h(x)$ of TIITLFD can be obtained as

$$h(x;\alpha,\beta,\lambda) = \frac{2\alpha\beta\lambda x^{-(\beta+1)} e^{-2\alpha x^{-\beta}}}{1-\left(e^{-\alpha x^{-\beta}}\right)^2}$$

The reverse hazard rate is the ratio between the probability density function and its distribution function. The reverse hazard function of TIITLFD is given by

$$h_r(x) = \frac{2\alpha\beta\lambda x^{-(\beta+1)}\left\{1-\left(e^{-\alpha x^{-\beta}}\right)^2\right\}^{\lambda-1}}{1-\left\{1-\left(e^{-\alpha x^{-\beta}}\right)^2\right\}^{\lambda}}$$

The mills ratio is the ratio between the cdf and the pdf. The mills ratio of TIITLFD is

$$\frac{1}{h_r(x;\alpha,\beta,\lambda)} = \frac{1-\left\{1-\left(e^{-\alpha x^{-\beta}}\right)^2\right\}^{\lambda-1}}{2\alpha\beta\lambda x^{-(\beta+1)} e^{-2\alpha x^{-\beta}}\left\{1-\left(e^{-\alpha x^{-\beta}}\right)^2\right\}^{\lambda-1}}$$

Graphs of the survival function and the hazard function of TIITLFD are shown in fig.3 and fig.4 for varying values of the parameters α, β and λ .

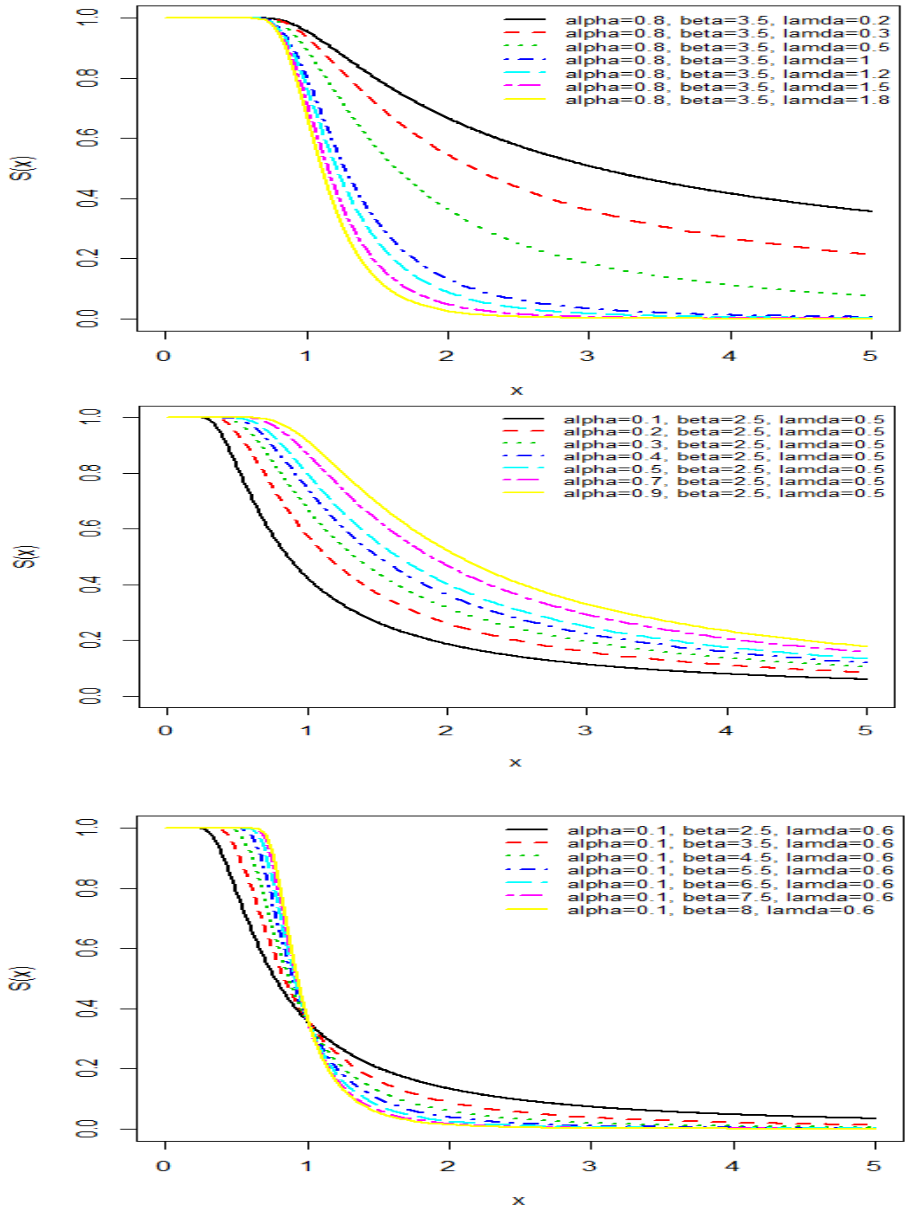

Figure 3. Graphs of survival function of TIITLFD for varying values of parameters

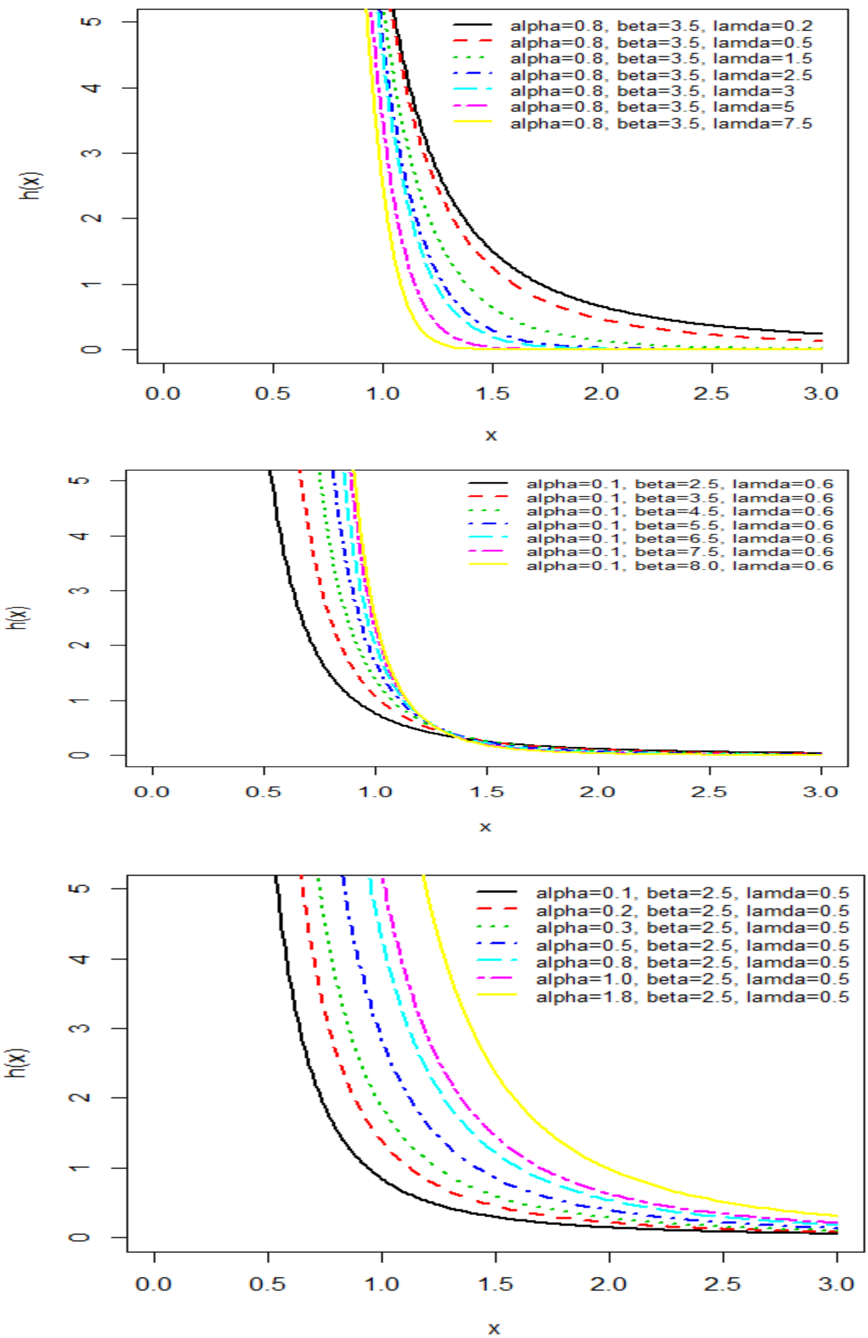

Figure 4. Graphs of hazard function of TIITLFD for varying values of parameters

3.3. Quantile function

The quantile function is defined as

$$Q(u) = F^{-1}(u)$$

Therefore, the corresponding quantile function for TIITLFD can be expressed as

$$Q(u) = F^{-1}(x) = \left[\frac{2\alpha}{-\ln\left\{1-(1-u)^{\frac{1}{\lambda}}\right\}} \right]^{\frac{1}{\beta}}$$

Let U has the uniform $U(0,1)$ distribution. Taking $u = 0.5$, the median of TIITLFD can be obtained as

$$Q(0.5) = F^{-1}(0.5) = \left[\frac{2\alpha}{-\ln\left\{1-(1-0.5)^{\frac{1}{\lambda}}\right\}} \right]^{\frac{1}{\beta}}$$

Thus, the formula for generating random samples from TIITLFD for simulating random variable X is given by

$$X = Q(u) = F^{-1}(u) = \left[\frac{2\alpha}{-\ln\left\{1-(1-u)^{\frac{1}{\lambda}}\right\}} \right]^{\frac{1}{\beta}}$$

4. Distribution of order statistics

Let $x_1, x_2, ..., x_n$ be the random samples from TIITLFD(α, β, λ). The pdf of i^{th} order statistics is given by

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} f_X(x)[F_X(x)]^{i-1}[1-F_X(x)]^{n-1}$$

The pdf of i^{th} order statistics $X_{(i)}$ of TIITLFD is given by

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-1)!} 2\alpha\beta\lambda\, x^{-(\beta+1)} e^{-2\alpha x^{-\beta}} \left\{1-\left(e^{-\alpha x^{-\beta}}\right)^2\right\}^{\lambda-1} \left[1-\left\{1-\left(e^{-\alpha x^{-\beta}}\right)^2\right\}^{\lambda}\right]^{i-1}$$
$$\times \left[1-1-\left\{1-\left(e^{-\alpha x^{-\beta}}\right)^2\right\}^{\lambda}\right]^{n-i}$$

Therefore, the pdf of the first order statistic $X_{(1)}$ can be expressed as

$$f_{1:n}(x) = n 2\alpha\beta\lambda\, x^{-(\beta+1)} e^{-2\alpha x^{-\beta}} \left\{1-\left(e^{-\alpha x^{-\beta}}\right)^2\right\}^{\lambda-1} \left[1-1-\left\{1-\left(e^{-\alpha x^{-\beta}}\right)^2\right\}^{\lambda}\right]^{n-1}$$

The pdf of the highest order statistic $X_{(n)}$ can be expressed as

$$f_{n:n}(x) = n2\alpha\beta\lambda x^{-(\beta+1)} e^{-2\alpha x^{-\beta}} \left[1 - \left\{1 - \left(e^{-\alpha x^{-\beta}}\right)^2\right\}^{\lambda}\right]^{n-1}$$

5. Maximum likelihood estimation

Let $x_1, x_2, ..., x_n$ be a random sample of size n from a TIITLFD(α, β, λ). The log-likelihood function can be expressed as

$$\log L = \sum_{i=1}^{n} \log f\left(x; \alpha, \beta, \lambda\right) = n\left(\log 2 + \log \alpha + \log \beta + \log \lambda\right) - \left(\beta + 1\right) \sum_{i=1}^{n} x_i$$
$$- 2\alpha \sum_{i=1}^{n} x_i^{-\beta} + \left(\lambda - 1\right) \sum_{i=1}^{n} \log\left\{1 - \left(e^{-\alpha x_i^{-\beta}}\right)^2\right\}$$

The maximum likelihood estimate (MLE) $\left(\hat{\alpha}, \hat{\beta}, \hat{\lambda}\right)$ of $\left(\alpha, \beta, \lambda\right)$ of TIITLFD are the solutions of the following log- likelihood equations

$$\frac{\partial \log L}{\partial \alpha} = \frac{n}{\alpha} - 2\sum_{i=1}^{n} x_i^{-\beta} + 2(\lambda - 1)\sum_{i=1}^{n} \frac{x^{-\beta}\left(e^{-\alpha x_i^{-\beta}}\right)^2}{1 - \left(e^{-\alpha x_i^{-\beta}}\right)^2} = 0$$

$$\frac{\partial \log L}{\partial \beta} = \frac{n}{\beta} - \sum_{i=1}^{n} \log x_i + 2\alpha\beta \sum_{i=1}^{n} x_i^{-\beta} \log x_i + 2\alpha(\lambda - 1)\sum_{i=1}^{n} \frac{x_i^{-\beta} \log x_i \left(e^{-\alpha x_i^{-\beta}}\right)^2}{1 - \left(e^{-\alpha x_i^{-\beta}}\right)^2} = 0$$

$$\frac{\partial \log L}{\partial \lambda} = \frac{n}{\beta} - \sum_{i=1}^{n} \log\left[1 - \left(e^{-\alpha x_i^{-\beta}}\right)^2\right] = 0$$

These log-likelihood equation can't be solved analytically and required statistical software with iterative numerical techniques. These equations can be solved using R-software.

The 3×3 observed information matrix of TIITLFD can be presented as,

$$\begin{pmatrix} \hat{\alpha} \\ \hat{\beta} \\ \hat{\lambda} \end{pmatrix} \sim \left[\begin{pmatrix} \alpha \\ \beta \\ \lambda \end{pmatrix}, \begin{pmatrix} \frac{\partial^2 \log L}{\partial \alpha^2} & \frac{\partial^2 \log L}{\partial \alpha\, \partial \beta} & \frac{\partial^2 \log L}{\partial \alpha\, \partial \lambda} \\ \frac{\partial^2 \log L}{\partial \beta\, \partial \alpha} & \frac{\partial^2 \log L}{\partial \beta^2} & \frac{\partial^2 \log L}{\partial \beta\, \partial \lambda} \\ \frac{\partial^2 \log L}{\partial \lambda\, \partial \alpha} & \frac{\partial^2 \log L}{\partial \lambda \partial \beta} & \frac{\partial^2 \log L}{\partial \lambda^2} \end{pmatrix} \right]$$

The inverse of the information matrix results in the well-known variance-covariance matrix. The 3×3 approximate Fishar information matrix corresponding to the above observed information matrix is given by

$$I^{-1} = -E\begin{bmatrix} \frac{\partial^2 \log L}{\partial\alpha^2} & \frac{\partial^2 \log L}{\partial\alpha\,\partial\beta} & \frac{\partial^2 \log L}{\partial\alpha\,\partial\lambda} \\ \frac{\partial^2 \log L}{\partial\beta\,\partial\alpha} & \frac{\partial^2 \log L}{\partial\beta^2} & \frac{\partial^2 \log L}{\partial\beta\,\partial\lambda} \\ \frac{\partial^2 \log L}{\partial\lambda\,\partial\alpha} & \frac{\partial^2 \log L}{\partial\lambda\partial\beta} & \frac{\partial^2 \log L}{\partial\lambda^2} \end{bmatrix}$$

The solution of the Fishar information matrix will yield asymptotic variance and covariance of the ML estimators for $\left(\hat{\alpha},\hat{\beta},\hat{\lambda}\right)$. The approximate 100(1-α)% confidence intervals for (α,β,λ) respectively are $\hat{\alpha} \pm Z_{\frac{\alpha}{2}}\frac{\sigma_{\alpha\alpha}}{n}$, $\hat{\beta} \pm Z_{\frac{\alpha}{2}}\frac{\sigma_{\beta\beta}}{n}$ and $\hat{\lambda} \pm Z_{\frac{\alpha}{2}}\frac{\sigma_{\lambda\lambda}}{n}$, where Z_{α} is the upper $100\alpha^{th}$ percentile of the standard normal distribution.

6. Applications

In this study, monthly mean temperature series of Silchar city, Assam, India from January 1988-July 2018 (30 years) which is collected by India Meteorological Department, Pune, India has been analyzed. For the application purpose , the datasets from January to July has been considered. The data sets are given in Table 1.

Table 1. Minimum temperature data for the district Silchar (Assam, India) from 1988-2018

Months	Temperature Data (Minimum Temperature)
JANUARY	11.7, 9.7, 11.6, 11.5, 13.5, 12.2, 12.5, 11.5, 12.4, 11.9, 13.3, 12.3, 12.3, 11.3, 13.5, 12.5, 13.9, 12.5, 11.9, 11.4, 12.8, 12.9, 11.1, 11.7, 12.3, 10.3, 12.1, 14.1, 12.6, 12.3, 11.9
FEBRUARY	12.2, 12.8, 13.8, 16.0, 14.0, 14.9, 13.1, 13.8, 14.8, 13.6, 15.4, 15.6, 13.9, 14.2, 14.6, 15.0, 14.1, 16.4, 16.9, 14.6, 12.6, 13.6, 10.8, 13.4, 12.7, 14.3, 12.8, 13.5, 16.3, 14.6, 14.4
MARCH	15.8, 16.1, 15.1, 19.5, 19.6, 17.3, 18.4, 16.7, 19.6, 19.0, 17.2, 18.3, 18.2, 17.9, 17.6, 17.6, 19.8, 19.7, 18.8, 16.5, 18.4, 17.9, 18.9, 17.5, 18.8, 17.6, 16.4, 17.6, 19.4, 17.3, 16.9
APRIL	19.2, 19.7, 17.5, 20.9, 22.0, 20.3, 21.1, 21.2, 21.8, 20.1, 21.5, 23.6, 21.6, 21.8, 20.8, 22.1, 21.0, 22.0, 21.5, 20.9, 22.1, 21.9, 21.7, 21.2, 20.4, 20.9, 20.8, 21.2, 21.7, 20.6, 19.9
MAY	20.0, 22.7, 21.3, 22.2, 22.5, 22.2, 24.0, 24.2, 23.5, 22.9, 24.2, 23.5, 23.5, 23.7, 22.9, 23.7, 23.8, 22.2, 23.3, 24.1, 23.7, 23.4, 23.4, 23.1, 23.0, 22.1, 23.2, 23.1, 23.2, 23.3, 21.4
JUNE	22.6, 23.3, 22.4, 24.8, 25.0, 24.5, 25.3, 25.3, 24.6, 24.6, 25.3, 25.3, 25.2, 25.4, 25.0, 24.9, 25.3, 25.6, 24.5, 25.1, 25.0, 25.1, 24.7, 25.3, 24.3, 25.1, 25.4, 25.1, 25.4, 24.6, 23.6
JULY	24.3, 23.2, 21.9, 25.9, 25.3, 25.2, 25.7, 25.6, 25.3, 25.7, 25.6, 25.4, 25.6, 25.8, 25.4, 25.9, 25.0, 25.4, 26.0, 25.7, 25.5, 25.9, 25.8, 25.5, 25.7, 24.6, 25.8, 25.7, 25.6, 25.3, 24.6

In order to compare the TIITLFD with Frechet distribution (FD), we consider the criteria like Bayesian information criterion *(BIC),* Akaike Information Criterion *(AIC),* Akaike Information Criterion Corrected *(AICC)* and $-2\log L$. The better distribution corresponds to lesser values of *AIC, BIC, AICC* and $-2\log L$. The formulae for calculating *AIC, BIC* and *AICC* are as follows:

$$AIC = 2K - 2\log L,\ BIC = k\log n - 2\log L,\ AICC = AIC + \frac{2k(k+1)}{(n-k-1)},$$

where k is the number of parameters, n is the sample size and *-2 logL* is the maximized value of log likelihood function. The ML estimates of the parameters of the considered distributions along with values of $-2\log L, AIC, AICC$ and BIC for the datasets in table 1 are presented in table 2.

Table 2. ML estimates of the parameters of the considered distributions along with values of $-2\log L$,AIC, AICC and BIC

Month	Distribution	ML Estimates of Parameters			$-2\log L$	*AIC*	*AICC*	*BIC*
		α	β	λ				
JANUARY	TIITLFD	855.7040	2.2127	580.9820	83.65	89.65	90.54	93.95
	FD	902968.012	6.4906	-----	109.77	113.77	114.66	116.64
FEBRUARY	TIITLFD	610.0494	1.9935	329.3614	104.71	110.71	115.60	115.01
	FD	237844.5	5.6279	-----	128.31	132.31	133.20	135.18
MARCH	TIITLFD	3225.3562	2.3447	1082.718	99.44	105.44	106.33	109.74
	FD	716761.3	5.5199	-----	139.62	143.62	144.51	146.49
APRIL	TIITLFD	8331.643	2.4482	9238.186	90.04	96.04	96.93	100.34
	FD	437995.8	5.0533	-----	152.98	156.99	157.88	159.85
MAY	TIITLFD	12063.11	3.1993	2436.307	76.37	82.37	83.26	86.67
	FD	312889.7	4.7984	-----	160.60	164.60	165.49	167.47
JUNE	TIITLFD	25189.16	3.3265	7596.683	78.99	68.99	69.89	73.30
	FD	905187.7	5.0160	-----	161.98	165.98	166.86	168.84
JULY	TIITLFD	25713.03	3.3140	5699.862	64.38	70.38	71.27	74.69
	FD	462271.8	4.7767	-----	166.41	170.41	171.30	173.28

It is obvious from above table 2 that TIITLFD provides much better fit than Frechet distribution for data relating to minimum temperature and hence the proposed distribution can be considered an improtant distribution for modeling minimum temperaure data.

6. Concluding remarks

In this paper Type II Topp-Leone Frechet distribution (TIITLFD) has been proposed. Its statistical properties including behaviour of pdf, cdf and hazard rate function have been discussed. The distribution of the order statistics has been given. The maximum likelihood estimation for estimating parameters of the proposed distribution has been discussed. The applications of the proposed distribution for modeling data relating to temperature has been explained and the goodness of fit of the TIITLFD and Frechet distribution has been presented for ready comparison.

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Record data from Kies distribution and related statistical inferences

Nesreen M. Al-Olaimat[1], Husam A. Bayoud[2], Mohammad Z. Raqab[3]

ABSTRACT

The Kies probability model was proposed as an alternative to the extended Weibull models as it provides a more efficient fit to some real-life data sets in comparison to the aforementioned models. The paper proposes classical and Bayesian inferences for the Kies distribution based on records. Maximum likelihood estimates are studied jointly with asymptotic and bootstrap confidence intervals. Moreover, Bayes estimates, along with credible intervals are discussed assuming squared and LINEX loss functions. The proposed estimation methods have been investigated and compared via simulation studies. A real data set has been analysed for illustrative purposes.

Key words: Bayesian estimates, Kies distribution, maximum likelihood estimation, records.

1. Introduction

For its importance in many practical fields, the Weibull distribution has received the attention of several authors in the literature. Moreover, many modified versions of the Weibull distribution were developed in the literature. One of the modified versions of the Weibull distribution is known as Kies Distribution and was firstly proposed by Kies (1958). Recently, Kies distribution has received the attention of different authors, including Kumar and Dharmaja (2014), who studied some of its important statistical aspects and showed that it possess increasing, decreasing and bathtub hazard rate functions that would make it a good alternative for some versions of the extended Weibull distributions, namely the generalized Weibull (GW) distribution, modified Weibull (MW) distribution, beta Weibull (BW) distribution and beta generalized Weibull (BGW) distribution. In 2013, Kumar and Dharmaja studied the one-parameter Kies distribution as a special case, called the reduced Kies (RK) distribution, which is shown to possess certain special properties that are analogous to those of the Weibull distribution. In 2017, they proposed a generalized version of the extended reduced Kies distribution, called a modified Kies (MK) distribution, see Kumar and Dharmaja (2017a). In addition, Kumar and Dharmaja (2017b) introduced and studied an exponentiated reduced Kies distribution with two parameters.

[1]Department of Mathematics, The University of Jordan, Amman 11942, Jordan.
E-mail: nesrreenolaimat@gmail.com. ORCID: https://orcid.org/0000-0003-2862-3739.
[2]College of Sciences and Humanities, Fahad Bin Sultan University, Tabuk, Saudi Arabia.
E-mail: hbayoud@fbsu.edu.sa; husam.awni@yahoo.com. ORCID: https://orcid.org/0000-0002-0066-7206.
[3]Department of Mathematics, The University of Jordan, Amman 11942, Jordan.
E-mail: mraqab@ju.edu.jo; mraqab4@gmail.com. ORCID: https://orcid.org/0000-0002-3047-1466.

The cumulative distribution function (CDF), probability density function (PDF), hazard rate and cumulative hazard rate functions of the two-parameter Kies distribution K(λ, β) are given by:

$$F(x;\lambda,\beta) = 1 - e^{-\lambda\left(\frac{x}{1-x}\right)^{\beta}}, \tag{1}$$

$$f(x;\lambda,\beta) = \frac{\beta\lambda x^{\beta-1}}{(1-x)^{\beta+1}} e^{-\lambda\left(\frac{x}{1-x}\right)^{\beta}}, \tag{2}$$

$$h(x;\lambda,\beta) = \frac{\beta\lambda x^{\beta-1}}{(1-x)^{\beta+1}}, \tag{3}$$

and

$$H(x;\lambda,\beta) = \lambda\left(\frac{x}{1-x}\right)^{\beta}, \tag{4}$$

respectively, where $0 < x < 1$, $\lambda > 0$ and $\beta > 0$.

The Kies distribution has a bounded range, which makes it appropriate model for fitting real data sets with a bounded range. However, there are many situations in which the observations can take values only in a limited range, like proportions, percentages or fractions. Papke and Wooldridge (1996) pointed out that variables in many economic applications such as the fraction of total weekly hours spent on working, the proportion of income spent on non-durable consumption, industry market shares, and a fraction of land area allocated to agriculture are all bounded between zero and one. Moreover, Genc (2013) indicated that when the reliability is measured as a percentage or ratio, it is important to have models defined on the unit interval in order to have reasonable results.

This paper studies classical and Bayesian inferences for the parameters of the Kies distribution based on records. Records play an important role in several fields of statistics which date back to Chandler (1952), who firstly defined and provided groundwork for mathematical theory of records. However, record statistics arise in many practical fields including hydrology, meteorology, sporting and athletic events wherein only records are usually considered, for more details and applications on records, readers may refer to Arnold et al. (1998), Ahsanullah (2004), Ahsanullah and Raqab (2006) and Ahsanullah and Nevzorov (2015).

Let $\{X_j, j \geq 1\}$ be a sequence of independent and identically distributed (iid) continuous random variables (r.v.'s) with CDF $F(x)$ and PDF $f(x)$. An observation X_j is defined to be an upper record if $X_j > X_i$ for every $j > i$, and an analogous definition can be given for lower records (with the inequality being reversed). By convention, the first record X_1 is called the trivial record because it is an upper and a lower record value simultaneously.

The set of the upper record values is given by the r.v.'s $X_{U(k)}$ for $k \geq 1$ where
$U(1) = 1$, $U(k) = min\{j : j > U(k-1), X_j > X_{U(k-1)}\}$.

Suppose we have a random sample (not ordered) of size n, say $\{X_1, X_2, ..., X_n\}$, the set

$$\{X_{U(1)} = X_1, X_{U(2)}, ..., X_{U(m)}\},$$

presents a set of upper record values with size $1 \leq m \leq n$ that is obtained from the random sample. The sequence $U(k)$, $k \geq 1$ is called the sequence of upper record times. For simplicity, we denote the sequence of upper record values $\{X_{U(j)}\}_{j=1}^{m}$ by $\{Y_j\}_{j=1}^{m}$.

In this paper, we will need the following lower and upper incomplete gamma functions

$$\int_0^z t^{\alpha-1}e^{-\mu t}dt = \mu^{-\alpha}\gamma(\alpha,\mu z), \tag{5}$$

$$\int_z^\infty t^{\alpha-1}e^{-\mu t}dt = \mu^{-\alpha}\Gamma(\alpha,\mu z), \tag{6}$$

respectively. Additionally,

$$\int_z^\infty t^{-\alpha}e^{-t}dt = z^{\frac{-\alpha}{2}}e^{(\frac{-\alpha}{2})}W_{-\frac{\alpha}{2},(\frac{1-\alpha}{2})}(z), \tag{7}$$

where $W_{c_1,c_2}(g)$ is the Whittaker function, which is defined, for $|\arg(-g)| < \frac{3\pi}{2}$, as

$$W_{c_1,c_2}(g) = \frac{\Gamma(-2c_2)}{\Gamma(\frac{1}{2}-c_2-c_1)}M_{c_1,c_2}(g) + \frac{\Gamma(2c_2)}{\Gamma(\frac{1}{2}+c_2-c_1)}M_{c_1,-c_2}(g), \tag{8}$$

in which

$$M_{c_1,c_2}(g) = e^{-\frac{g}{2}}g^{c_2+\frac{1}{2}}\sum_{k=0}^{\infty}\left\{\frac{(\frac{1}{2}-c_1+c_2)_k}{(1+2c_2)_k}\frac{g^k}{k!}\right\}, \tag{9}$$

the series given in Eq. (9) converges for all finite values of g. Also, the pochhammer symbol is defined as follows:

$$(a)_k = a(a+1)(a+2)...(a+k-1) = \frac{\Gamma(a+k)}{\Gamma(a)} = \prod_{i=1}^{k}(a+i-1), \tag{10}$$

where $(a)_0 = 1$ and $(1)_k = k!$.

The rest of this paper is organized as follows: forms of the single moment and some properties for records from K(λ, β) are derived in Section 2. In Section **??**, classical estimation methods are proposed for the parameters of the Kies distribution based on upper records. In Section 4, the Bayes estimators based on the squared error and linear exponential loss functions are computed using gamma priors for the two unknown parameters. Further in Section 5, we consider a real data set for illustrative purposes. In Section 6, simulation studies are carried out in order to study the performance of the proposed estimation methods. Finally, the paper is concluded in Section 7.

2. Distributional properties of records from Kies distribution

The aim of this section is to present some properties and derive the form of the kth moment of the m^{th} record from K(λ, β). The PDF of the m^{th} record value and the joint PDF of the m^{th} and s^{th} records are given, respectively, Arnold et al. (1998) by

$$f_m(y) = \frac{[H(y)]^{m-1}}{\Gamma(m)}f(y), \tag{11}$$

and

$$f_{m,s}(y,z) = \frac{[H(y)]^{m-1}}{\Gamma(m)}h(y)\frac{[H(z)-H(y)]^{s-m-1}}{\Gamma(s-m)}f(z), \tag{12}$$

where $-\infty < y < z < \infty$, $H(.)$ and $h(.)$ are the cumulative hazard and the hazard rate functions, respectively.

Result 2.1. *By using Eqs. (2), (3) and (4), the PDF of the m^{th} record and the joint PDF of the m^{th} and s^{th} records from K(λ, β) given in Eqs. (11) and (12),respectively, become*

$$f_m(y) = \frac{\beta\lambda^m}{\Gamma(m)}(\frac{y}{1-y})^{m\beta}\frac{1}{y(1-y)}e^{-\lambda(\frac{y}{1-y})^\beta}, \tag{13}$$

$$f_{m,s}(y,z) = \frac{\lambda^s\beta^2}{\Gamma(m)}(\frac{y}{1-y})^{m\beta}\frac{1}{y(1-y)}\frac{[(\frac{z}{1-z})^\beta-(\frac{y}{1-y})^\beta]^{s-m-1}}{\Gamma(s-m)}\frac{z^{\beta-1}}{(1-z)^{\beta+1}}e^{-\lambda(\frac{z}{1-z})^\beta}, \tag{14}$$

where $0 < y < z < 1$ and λ, $\beta > 0$.

Using Eqs. (13) and (5), the CDF F_m of the m^{th} record value from the Kies distribution is given by

$$F_m(y) = \frac{\gamma(m,\lambda(\frac{y}{1-y})^\beta)}{\Gamma(m)}, m \geq 1, \tag{15}$$

where $0 < y < 1$ and λ, $\beta \geq 0$.

Result 2.2. *Suppose that the random variable X follows a Kies distribution. Then, one can prove that*

$$X \stackrel{D}{=} \frac{(\frac{1}{\lambda}X^*)^{\frac{1}{\beta}}}{1+(\frac{1}{\lambda}X^*)^{\frac{1}{\beta}}},$$

where D means converges in distribution and $X^* = -\log(1-U)$ where U is Uniform(0, 1). It is obvious that X^* follows a standard exponential distribution. Consequently, using the result, A.4.10, Page(174) of Houchens (1984), the corresponding sequence of records can be described by

$$Y_m \stackrel{D}{=} \frac{(\frac{1}{\lambda}\sum_{i=1}^m X_i^*)^{\frac{1}{\beta}}}{1+(\frac{1}{\lambda}\sum_{i=1}^m X_i^*)^{\frac{1}{\beta}}}, \tag{16}$$

where $\{X_i^*\}_{i=1}^m$ is a sequence of $i.i.d.$ $Exp(1)$ random variables

Result 2.3. *If the random variable X has a Kies distribution, then k^{th} moment $\mu_m^{(k)} = E(Y_m^k)$ for the m^{th} record from the Kies distribution is given by*

$$\begin{aligned}\mu_m^{(k)} = \Psi(m,\lambda,\beta,k) =& \frac{1}{\Gamma(m)}\sum_{j=0}^{\infty}(-1)^j\frac{(k)_j}{j!}\lambda^{-(\frac{k+j}{\beta})}\gamma(m+\frac{k+j}{\beta},\lambda) \\ &+\frac{1}{\Gamma(m)}\sum_{j=0}^{\beta(m-1)}(-1)^j\frac{(k)_j}{j!}\lambda^{\frac{j}{\beta}}\Gamma(m-\frac{j}{\beta},\lambda) \\ &+\frac{1}{\Gamma(m)}[\sum_{j=\beta(m-1)+1}^{\infty}(-1)^j\frac{(k)_j}{j!}\lambda^{\frac{m}{2}+\frac{j}{2\beta}-\frac{1}{2}}e^{\frac{m}{2}-\frac{j}{2\beta}-\frac{1}{2}} \\ &\times W_{\frac{m}{2}-\frac{j}{2\beta}-\frac{1}{2},(\frac{m}{2}-\frac{j}{2\beta})}(\lambda)].\end{aligned} \tag{17}$$

Proof. By using Eq. (13), the k^{th} moment for the m^{th} record from the Kies distribution is

$$E(Y_m^k) = \int_0^1 \frac{\beta\lambda^m}{\Gamma(m)}(\frac{y_m}{1-y_m})^{m\beta}\frac{y_m^k}{y_m(1-y_m)}e^{-\lambda(\frac{y_m}{1-y_m})^\beta}dy_m. \tag{18}$$

On substituting $(\frac{y_m}{1-y_m})^\beta = t$ in Eq(18), we get

$$E(Y_m^k) = \frac{\lambda^m}{\Gamma(m)}\int_0^\infty (\frac{t^{\frac{1}{\beta}}}{1+t^{\frac{1}{\beta}}})^k t^{m-1}e^{-\lambda t}dt.$$

On splitting the integral and expanding $(1+t^{\frac{1}{\beta}})^{-k}$ using Newton's Generalization of the binomial theorem, we get the following

$$E(Y_m^k) = \frac{\lambda^m}{\Gamma(m)}\int_0^1 \frac{t^{m+\frac{k}{\beta}-1}}{(1+t^{\frac{1}{\beta}})^k}e^{-\lambda t}dt + \frac{\lambda^m}{\Gamma(m)}\int_1^\infty \frac{t^{m+\frac{k}{\beta}-1}}{t^{\frac{k}{\beta}}(t^{\frac{-1}{\beta}}+1)^k}e^{-\lambda t}dt, \tag{19}$$

$$\begin{aligned} E(Y_m^k) &= \frac{\lambda^m}{\Gamma(m)}\sum_{j=0}^\infty \frac{(-1)^j(k)_j}{j!}\int_0^1 (t^{\frac{k+j+m\beta}{\beta}-1})e^{-\lambda t}dt \\ &+ \frac{\lambda^m}{\Gamma(m)}\sum_{j=0}^\infty \frac{(-1)^j(k)_j}{j!}\int_1^\infty (t^{\frac{m\beta-j}{\beta}-1})e^{-\lambda t}dt, \end{aligned} \tag{20}$$

where $(.)_j$ is the Pochhammer symbol given by (10), if we put $u=\lambda t$ we get

$$\begin{aligned} E(Y_m^k) &= \frac{1}{\Gamma(m)}\sum_{j=0}^\infty \frac{(-1)^j(k)_j}{j!}\lambda^{-\frac{k+j}{\beta}}\int_0^\lambda (u^{\frac{k+j+m\beta}{\beta}-1})e^{-u}du \\ &+ \frac{1}{\Gamma(m)}\sum_{j=0}^\infty \frac{(-1)^j(k)_j}{j!}\lambda^{\frac{j}{\beta}}\int_\lambda^\infty (u^{\frac{m\beta-j}{\beta}-1})e^{-u}du, \end{aligned} \tag{21}$$

since the exponent $m-\frac{j}{\beta}$ in the second integral carries positive and negative values, therefore, on splitting the second summation we get the following:

$$\begin{aligned} E(Y_m^k) &= \frac{1}{\Gamma(m)}\sum_{j=0}^\infty \frac{(-1)^j(k)_j}{j!}\lambda^{-\frac{k+j}{\beta}}\int_0^\lambda (u^{\frac{k+j+m\beta}{\beta}-1})e^{-u}du \\ &+ \frac{1}{\Gamma(m)}\sum_{j=0}^{\beta(m-1)} \frac{(-1)^j(k)_j}{j!}\lambda^{\frac{j}{\beta}}\int_\lambda^\infty u^{m-\frac{j}{\beta}-1}e^{-u}du \\ &+ \frac{1}{\Gamma(m)}\sum_{j=\beta(m-1)+1}^\infty \frac{(-1)^j(k)_j}{j!}\lambda^{\frac{j}{\beta}}\int_\lambda^\infty u^{-(1+\frac{j}{\beta}-m)}e^{-u}du, \end{aligned} \tag{22}$$

which leads to (17) in the light of (5), (6) and (7). □

The expected value of the m^{th} record value $[E(Y_m)]$ is the first moment, which is given by:

$$\mu_m^{(1)} = \Psi(m,\lambda,\beta,1).$$

In addition, the variance of the m^{th} record value is

$$var(Y_m) = \Psi(m,\lambda,\beta,2) - [\Psi(m,\lambda,\beta,1)]^2.$$

For illustrative purposes, $E(Y_m)$ and variance of some records of the Kies distribution, namely 3^{rd}, 5^{th}, 7^{th} and 10^{th}, are computed and summarized in Tables (1) and (2) assuming different values of λ and β. It can be observed from these tables that $E(Y_m)$(Variance) increases(decreases) with m, which is expected.

Table 1: Expected values and variances of records from K(λ, β) with λ= 0.75 and 1

m	$\lambda = 0.75$				$\lambda = 1$			
	$\beta = 0.75$		$\beta = 2$		$\beta = 0.75$		$\beta = 2$	
	$E(Y_m)$	Variance	$E(Y_m)$	Variance	$E(Y_m)$	Variance	$E(Y_m)$	Variance
3	0.80600	0.01810	0.64400	0.00519	0.74800	0.02350	0.61100	0.00549
5	0.90300	0.00387	0.70800	0.00242	0.86600	0.00625	0.67800	0.00267
7	0.94000	0.00111	0.74500	0.00142	0.95500	0.00201	0.71700	0.00161
10	0.96400	0.00027	0.77900	0.00080	0.97800	0.00021	0.75400	0.00092

Table 2: Expected values and variances of records from K(λ, β) with λ= 2 and 3

m	$\lambda = 2$				$\lambda = 3$			
	$\beta = 0.75$		$\beta = 2$		$\beta = 0.75$		$\beta = 2$	
	$E(Y_m)$	Variance	$E(Y_m)$	Variance	$E(Y_m)$	Variance	$E(Y_m)$	Variance
3	0.57000	0.03240	0.52800	0.00585	0.45300	0.03110	0.47900	0.00577
5	0.73200	0.01470	0.59900	0.00317	0.62400	0.01920	0.55000	0.00332
7	0.81600	0.00659	0.64200	0.00203	0.72600	0.01060	0.59500	0.00222
10	0.88200	0.00228	0.68500	0.00124	0.81500	0.00450	0.63900	0.00140

3. Classical estimation

3.1. Maximum likelihood estimation

Let $data = \{y_1, y_2, \ldots, y_m\}$ be the first m upper record values arising from a sequence of iid K(λ, β) with CDF, PDF and hazard rate being defined in Eqs. (1), (2) and (3), respectively. The likelihood function of the $data$ is given by (see Arnold et al. (1998).

$$\begin{aligned} L(data;\lambda,\beta) &= f(y_m;\lambda,\beta)\prod_{i=1}^{m-1} h(y_i;\lambda,\beta) \\ &= \beta^m \lambda^m e^{-\lambda(\frac{y_m}{1-y_m})^\beta} \prod_{i=1}^{m} \frac{y_i^{\beta-1}}{(1-y_i)^{\beta+1}}. \end{aligned} \tag{23}$$

Thus, the log-likelihood function $l(data|\lambda,\beta) = \log L(data;\lambda,\beta)$ can be written as

$$l(data|\lambda,\beta) = m\log\lambda + m\log\beta - \lambda(\frac{y_m}{1-y_m})^\beta + (\beta-1)\sum_{i=1}^{m}(\log y_i) - (\beta+1)\sum_{i=1}^{m}\log(1-y_i), \tag{24}$$

where $0 < y_1 < y_2 < ... < y_m < 1$, $\beta > 0$ and $\lambda > 0$. The following proposition shows the existence and uniqueness of the MLEs of λ and β.

Proposition 3.1. *The log-likelihood function $l(data|\lambda,\beta)$ is unimodal function of λ and β.*

Proof. Note that $l(data|\lambda,\beta)$ is a continuous function in λ and β, and is strictly concave as the Hessian matrix is negative definite. Thus, $l(data|\lambda,\beta)$ is unimodal of λ and β.
This shows the existence and uniqueness of the MLEs of the unknown parameters λ and β. □

Substituting $R_i = \frac{y_i}{1-y_i}, i = 1,2,...,m$ and solving the following system of equations (equations 25 and 26)

$$0 = \frac{\partial l(data|\lambda,\beta)}{\partial \lambda} = \frac{m}{\lambda} - R_m^{\beta}, \tag{25}$$

$$0 = \frac{\partial l(data|\lambda,\beta)}{\partial \beta} = \frac{m}{\beta} - \lambda R_m^{\beta} \log R_m + \sum_{i=1}^{m} \log(R_i), \tag{26}$$

we immediately obtain the MLEs of β and λ as

$$\hat{\beta} = \frac{m}{\sum_{i=1}^{m-1} \log(\frac{R_m}{R_i})}, \tag{27}$$

and

$$\hat{\lambda} = \frac{m}{R_m^{\hat{\beta}}}. \tag{28}$$

3.2. Asymptotic confidence interval

Since it is not easy to derive the exact distribution of the MLEs, we cannot obtain the exact confidence intervals (CIs) for the parameters λ and β. Consequently, the asymptotic CIs (ACIs) of the parameters are derived using the asymptotic distribution of the MLEs. To this end, we need to find the variance-covariance matrix of the MLEs. The observed information matrix of λ and β is given by

$$I(\lambda,\beta) = -\begin{pmatrix} \frac{\partial^2 l(data|\lambda,\beta)}{\partial^2 \lambda} & \frac{\partial^2 l(data|\lambda,\beta)}{\partial \lambda \partial \beta} \\ \frac{\partial^2 l(data|\lambda,\beta)}{\partial \beta \partial \lambda} & \frac{\partial^2 l(data|\lambda,\beta)}{\partial^2 \beta} \end{pmatrix},$$

where

$$\begin{aligned} \frac{\partial^2 l(data|\lambda,\beta)}{\partial^2 \lambda} &= -\frac{m}{\lambda^2}, \\ \frac{\partial^2 l(data|\lambda,\beta)}{\partial \lambda \partial \beta} &= \frac{\partial^2 l(data|\lambda,\beta)}{\partial \beta \partial \lambda} = -R_m^{\beta} \log R_m, \\ \frac{\partial^2 l(data|\lambda,\beta)}{\partial^2 \beta} &= -(\frac{m + \lambda \beta^2 R_m^{\beta} \log^2 R_m}{\beta^2}). \end{aligned}$$

Therefore, the approximate variance−covariance matrix for the MLE of $\theta = (\lambda, \beta)$ is given by

$$V = -\begin{pmatrix} \frac{\partial^2 l(data|\lambda,\beta)}{\partial^2\lambda} & \frac{\partial^2 l(data|\lambda,\beta)}{\partial\lambda\partial\beta} \\ \frac{\partial^2 l(data|\lambda,\beta)}{\partial\beta\partial\lambda} & \frac{\partial^2 l(data|\lambda,\beta)}{\partial^2\beta} \end{pmatrix}^{-1}_{(\lambda,\beta)=(\hat{\lambda},\hat{\beta})} = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix}, \tag{29}$$

where

$$\begin{aligned} V_{11} &= \frac{m+\lambda\beta^2 R_m^{\beta}\log^2(R_m)}{\frac{m}{\lambda^2}(m+\lambda\beta^2 R_m^{\beta}\log^2(R_m)) - \beta^2(R_m^{\beta}\log(R_m))^2} \\ V_{12} &= V_{21} = \frac{-R_m^{\beta}\log(R_m)}{\frac{m}{\lambda^2\beta^2}(m+\lambda\beta^2 R_m^{\beta}\log^2(R_m)) - (R_m^{\beta}\log(R_m))^2} \\ V_{22} &= \frac{1}{\frac{m}{\beta^2}+\lambda R_m^{\beta}\log^2(R_m) - \frac{(\lambda R_m^{\beta}\log(R_m))^2}{m}}. \end{aligned}$$

The asymptotic joint distribution of the MLEs $\hat{\lambda}$ and $\hat{\beta}$ is approximated by bivariate normal, and is given by:

$$\begin{pmatrix} \hat{\lambda} \\ \hat{\beta} \end{pmatrix} \overset{N}{\sim} \left[\begin{pmatrix} \lambda \\ \beta \end{pmatrix}, \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix} \right]. \tag{30}$$

Hence, by replacing λ and β by their MLEs, we get an estimate of V as follows:

$$\hat{V} = \begin{pmatrix} \frac{m}{(R_m^{\hat{\beta}})^2}(1+\hat{\beta}^2\log^2 R_m) & \frac{-\hat{\beta}^2\log R_m}{R_m^{\hat{\beta}}} \\ \frac{-\hat{\beta}^2\log R_m}{R_m^{\hat{\beta}}} & \frac{\hat{\beta}^2}{m}. \end{pmatrix}$$

Consequently, asymptotic $100(1-\alpha)\%$ CIs for the parameters λ and β are, respectively, given by:

$$(L_\lambda, U_\lambda) = (\hat{\lambda} - z_{1-\frac{\alpha}{2}}\sqrt{\hat{V}_{11}}), \hat{\lambda} + z_{1-\frac{\alpha}{2}}\sqrt{\hat{V}_{11}}), \tag{31}$$

and

$$(L_\beta, U_\beta) = (\hat{\beta} - z_{1-\frac{\tau}{2}}\sqrt{\hat{V}_{22}}, \hat{\beta} + z_{1-\frac{\tau}{2}}\sqrt{\hat{V}_{22}}), \tag{32}$$

where z_α is $100\alpha^{th}$ percentile of the standard normal distribution. However, some cases provide negative lower bounds of the asymptotic CI while the parameters λ and β are positive. In order to avoid such a case, we propose using a log−transformation for parameters in order to construct a modified asymptotic confidence intervals for λ and β following the lines of Ren and Gui (2020). Since, for a parameter, η, $g(\eta) = \log(\eta)$ is differentiable with $g\prime(\eta) \neq 0$, hence $Var[g(\hat{\eta})] = \frac{Var(\hat{\eta})}{\hat{\eta}^2}$. Therefore, modified asymptotic $(1-\alpha)100\%(0<\alpha<1)$ CIs for λ and β can be easily obtained, respectively, as follows:

$$\left(\frac{\hat{\lambda}}{e^{\frac{z_{1-\frac{\tau}{2}}}{\hat{\lambda}}\sqrt{V_{11}}}}, \hat{\lambda}e^{\frac{z_{1-\frac{\tau}{2}}}{\hat{\lambda}}\sqrt{V11}} \right) \text{ and } \left(\frac{\hat{\beta}}{e^{\frac{z_{1-\frac{\tau}{2}}}{\hat{\beta}}\sqrt{V_{22}}}}, \hat{\beta}e^{\frac{z_{1-\frac{\tau}{2}}}{\hat{\beta}}\sqrt{V_{22}}} \right). \tag{33}$$

3.3. Bootstrap method

Since the asymptotic CIs results do not perform quite well for a small sample size, the percentile Bootstrap method, which is denoted by Boot-p, is presented in this section to construct approximate CIs for λ and β using the following algorithm, see for example, Ahmed (2014):

Step 1) From the records $y_1, y_2, ..., y_m$, compute the MLEs $\hat{\lambda}_{ML}$ and $\hat{\beta}_{ML}$.

Step 2) Using $\hat{\lambda}_{ML}$ and $\hat{\beta}_{ML}$ that are obtained in Step 1, generate a random sample of records from K(λ, β), called a bootstrap sample.

Step 3) Based on the Bootstrap sample that is obtained in Step 2, compute the corresponding MLEs $\hat{\lambda}^*$ and $\hat{\beta}^*$ of λ and β, respectively.

Step 4) Repeat Steps (2) and (3) B-times to obtain $\{\hat{\lambda}_1^*, \hat{\lambda}_2^*, \ldots, \hat{\lambda}_B^*\}$ and $\{\hat{\beta}_1^*, \hat{\beta}_2^*, \ldots, \hat{\beta}_B^*\}$.

Step 5) Arrange $\{\hat{\lambda}_1^*, \hat{\lambda}_2^*, ..., \hat{\lambda}_B^*\}$ and $\{\hat{\beta}_1^*, \hat{\beta}_2^*, \ldots, \hat{\beta}_B^*\}$ in ascending order and obtain $\{\hat{\lambda}_{(1)}^*, \hat{\lambda}_{(2)}^*, \ldots, \hat{\lambda}_{(B)}^*\}$ and $\{\hat{\beta}_{(1)}^*, \hat{\beta}_{(2)}^*, ..., \hat{\beta}_{(B)}^*\}$.

Step 6) The approximate $100(1-\alpha)\%$ Boot-p CIs for λ and β are given by $\left(\hat{\lambda}_{(B\frac{\alpha}{2})}^*, \hat{\lambda}_{(B(1-\frac{\alpha}{2}))}^*\right)$ and $\left(\hat{\beta}_{(B\frac{\alpha}{2})}^*, \hat{\beta}_{(B(1-\frac{\alpha}{2}))}^*\right)$, respectively.

4. Bayesian estimation

In this section, we derive the posterior densities of the parameters β and λ based on the upper record values, then obtain the corresponding Bayes estimates of these parameters under different loss functions. Symmetric and asymmetric loss functions are considered in our study, which are squared error (SE) and linear exponential (LINEX) loss functions. The SE loss function of the parameter η and an estimate $\hat{\eta}$ is given by:

$$L_{SE}(\hat{\eta}, \eta) = (\hat{\eta} - \eta)^2 . \tag{34}$$

As the SE loss function leads to identical penalization for underestimation and overestimation, an asymmetric loss function, known as LINEX loss function, was proposed by Zellner (1986). The LINEX loss function of the parameter η and an estimate $\hat{\eta}$ is given by:

$$L_{LINEX}(\hat{\eta}, \eta) = b[e^{\nu(\hat{\eta}-\eta)} - \nu(\hat{\eta} - \eta) - 1], \tag{35}$$

where $b > 0$ is the scale of the loss function. In our study, we assume $b = 1$. The parameter $\nu \neq 0$ indicates the shape parameter of the loss function. The LINEX loss function is affected by ν, the sign of ν indicates the direction of the asymmetry, and the magnitude of ν indicates the degree of the asymmetry. It is known that assuming $\nu > 0$ means that overestimation is considered to be more costly than underestimation, while assuming $\nu < 0$ means the reverse situation, and when ν is close to zero, the LINEX loss function is almost symmetric and is approximately equal to the SE loss function. Thus, for small values of ν, estimation results obtained by both LINEX and SE are close, for more details about the LINEX loss function readers may refer to Zellner (1986).

A natural choice of the priors of λ and β would be to assume that the two quantities are independent with gamma distributions; namely $Gamma(a_1,b_1)$ and $Gamma(a_2,b_2)$, respectively, where the hyper-parameters a_1,a_2,b_1 and b_2 are nonnegative numbers chosen to reflect prior knowledge about the parameters λ and β.

The joint prior distribution of λ and β is obtained as follows:

$$g(\lambda,\beta) \propto \lambda^{a_1-1}e^{-b_1\lambda}\beta^{a_2-1}e^{-b_2\beta}. \tag{36}$$

In light of the upper record $data=\{y_1,y_2,...,y_m\}$, the joint posterior distribution of λ and β is obtained as follows:

$$\pi(\lambda,\beta|data) \propto L(data|\lambda,\beta)g(\lambda,\beta), \tag{37}$$

where $L(data|\lambda,\beta)$ is the likelihood function given in Eq. (23) and $g(\lambda,\beta)$ is the joint prior density that is given in Eq. (36). By substituting Eqs. (36) and (23) in Eq. (37), the joint posterior density of λ and β is immediately given by:

$$\pi(\lambda,\beta|data) \propto \lambda^{m+a_1-1}\beta^{m+a_2-1}e^{-\beta b_2}e^{-\lambda(b_1+(R_m)^\beta)}\prod_{i=1}^{m}R_i^\beta. \tag{38}$$

It can be seen that the joint posterior distribution in Eq. (38) can be represented as follows:

$$\pi(\lambda,\beta|data) \propto \pi_1(\beta|data)\pi_2(\lambda|\beta,data), \tag{39}$$

where

$$\pi_1(\beta|data) \propto \frac{\beta^{m+a_2-1}e^{-\beta b_2}\prod_{i=1}^{m}R_i^\beta}{\left(b_1+R_m^\beta\right)^{m+a_1}}, \tag{40}$$

and $\pi_2(\lambda|\beta,data)$ is a gamma density with shape and scale parameters equal to $m+a_1$ and $\left[b_1+R_m^\beta\right]^{-1}$, respectively.

Subsequently, the Bayes estimate of any function of λ and β, say $\eta(\lambda,\beta)$, under SE and LINEX loss functions separately are respectively given by:

$$\hat{\theta}_{BS} = \frac{\int_0^\infty\int_0^\infty \eta(\lambda,\beta)\pi_1(\beta|data)\pi_2(\lambda|\beta,data)d\beta d\lambda}{\int_0^\infty\int_0^\infty \pi_1(\beta|data)\pi_2(\lambda|\beta,data)d\beta d\lambda}, \tag{41}$$

and

$$\hat{\theta}_{BL} = -\frac{1}{\nu}\log\left(\frac{\int_0^\infty\int_0^\infty e^{-\nu\eta(\lambda,\beta)}\pi_1(\beta|data)\pi_2(\lambda|\beta,data)d\beta d\lambda}{\int_0^\infty\int_0^\infty \pi_1(\beta|data)\pi_2(\lambda|\beta,data)d\beta d\lambda}\right). \tag{42}$$

Unfortunately, the Bayes estimates in Eqs. (41) and (42) cannot be derived in explicit forms. Therefore, we propose to approximate the Bayes estimates and the corresponding credible intervals by using an importance sampling technique as suggested by Chen and Shao (1999). Similar procedure was used, for example, by Chen et al. (2000), Kundu and Pradhan (2009), Pradhan and Kundu (2009), Pradhan and Kundu (2011) and Bayoud (2016).

It can be easily seen that the marginal posterior of β in Eq. (40) can be rewritten as follows:

$$\pi_1(\beta|data) \propto g_1(\beta|data)g_2(\beta), \tag{43}$$

where $g_1(\beta|data)$ is a gamma density with shape and scale parameters equal to $(m+a_2)$

and $\frac{1}{b_2}$, respectively, and

$$g_2(\beta) = \frac{\prod_{i=1}^{m} R_i^{\beta}}{\left(b_1 + R_m^{\beta}\right)^{m+a_1}}. \tag{44}$$

Now, we propose the following algorithm, along the line of Kundu and Pradhan (2009), to compute the approximate Bayes estimates and to construct the associated credible intervals for the parameters β and λ.

Let $data = \{y_1, y_2, \ldots, y_m\}$ be a set of m upper records and let a_i and $b_i, (i = 1,2)$ be pre-assumed hyper-parameters chosen based on prior information about the underlying parameters β and λ.

Step 1) Generate a random sample of size M from the gamma density function $g_1(\beta|data)$, say $\{\beta_1, \beta_2, ..., \beta_M\}$;

Step 2) For each β_j, generate λ_j from the gamma density function $\pi_2(\lambda|\beta_j, data)$, say $\{\lambda_1, \lambda_2, ..., \lambda_M\}$;

Step 3) Compute $g_2(\beta_i)$, for $j = 1,2,...,M$;

Step 4) Under the SEL function, a simulation consistent estimate of $\eta(\lambda,\beta)$ can be obtained using the importance sampling technique as:

$$\hat{\eta}_{BS}(\lambda,\beta) = \frac{\sum_{j=1}^{M} \eta(\lambda_j,\beta_j)\, g_2(\beta_j)}{\sum_{j=1}^{M} g_2(\beta_j)}.$$

Hence, $\hat{\beta}_{BS} = \frac{\sum_{j=1}^{M} \beta_j\, g_2(\beta_j)}{\sum_{j=1}^{M} g_2(\beta_j)}$ and $\hat{\lambda}_{BS} = \frac{\sum_{j=1}^{M} \lambda_j\, g_2(\beta_j)}{\sum_{j=1}^{M} g_2(\beta_j)}$.

Step 5) Under the LINEX function, a simulation consistent estimate of $\eta(\lambda,\beta)$ can be obtained using the importance sampling technique as:

$$\hat{\theta}_{BL} = \hat{\eta}_{BL}(\lambda,\beta) = -\frac{1}{v}\log \frac{\sum_{j=1}^{M} e^{-v\eta(\lambda_j,\beta_j)} g_2(\beta_j)}{\sum_{j=1}^{M} g_2(\beta_j)}.$$

Hence, $\hat{\beta}_{BL} = -\frac{1}{v}\log \frac{\sum_{j=1}^{M} e^{-v\beta_j} g_2(\beta_j)}{\sum_{j=1}^{M} g_2(\beta_j)}$ and $\hat{\lambda}_{BL} = -\frac{1}{v}\log \frac{\sum_{j=1}^{M} e^{-v\lambda_j} g_2(\beta_j)}{\sum_{j=1}^{M} g_2(\beta_j)}$.

Step 6) Compute

$$w_j = \frac{g_2(\beta_j)}{\sum_{j=1}^{M} g_2(\beta_j)} for j = 1,2,\ldots,M;$$

Step 7) Arrange the set $\{(\beta_1,w_1),(\beta_2,w_2),...,(\beta_M,w_M)\}$ as $\left\{(\beta_{(1)},w_{[1]}),(\beta_{(2)},w_{[2]}),\ldots,(\beta_{(M)},w_{[M]})\right\}$, where $\beta_{(1)} \leq \beta_{(2)},\ldots,\leq \beta_{(M)}$ are order statistics of β_j from the sample of size M obtained in Step (1) with $w[k]$ being the value of w_i's associated with k^{th} order statistic of β_i's, say $\beta_{(k)}$.
Similarly, we obtain $\left\{(\lambda_{(1)},w_{[1]}),(\lambda_{(2)},w_{[2]}),\ldots,(\lambda_{(M)},w_{[M]})\right\}$, which are order statistics of λ_j from the sample of size M obtained in Step (2) and $w[k]$ as defined above.

Step 8) The $100(1-\alpha)\%$ credible interval (CrI) for η is given by $(\hat{\eta}_{(100\frac{\alpha}{2})}, \hat{\eta}_{(100(1-\frac{\alpha}{2}))})$, where $\hat{\eta}_{(100p)}$ is a simulation consistent Bayes estimate for η, which is given by $\eta_{(M_p)}$ such that M_p is the integer satisfying:

$$\sum_{j=1}^{M_p} w_{[j]} \leq p < \sum_{j=1}^{M_p+1} w_{[j]}.$$

Remark 4.1. *Since $\hat{\beta}_{BS}$ and $\hat{\lambda}_{BS}$ are unique Bayes estimates for β and λ, respectively, then they are admissible based on Theorem 2.4 of Lehmann and Casella (1998).*

Remark 4.2. *Since $\hat{\beta}_{BL}$ and $\hat{\lambda}_{BL}$ are unique Bayes estimates for β and λ, respectively, then they are admissible based on Theorem 2.4 of Lehmann and Casella (1998).*

5. Data analysis

In this section, record statistics from a real data set obtained from K(λ, β) are analyzed in order to illustrate the proposed estimation methods. All the computations are performed using Mathematica codes.

5.1. Real data: total annual rainfall

In this example, we analyze the total annual rainfall (in inches) during 25 years from 1984-2008 recorded at Los Angeles Civic Center. This data is given below, see http : // www.laalmanac.com/weather/we08aa.php:

12.82	17.86	7.66	2.48	8.08	7.35	11.99	21.00	7.36
8.11	24.35	12.44	12.40	31.01	9.09	11.57	17.94	4.42
16.42	9.25	37.96	13.19	3.21	13.53	9.08		

This data set was studied by Tarvirdizade and Ahmadpour (2016). Firstly, all observations have been divided over 100 in order to transform them to be in $(0,1)$, the support of K(λ, β). Then, the well-known Kolmogorov-Smirnov (K-S) goodness of fit test is used to test whether the Kies distribution adequately fits this data set or not. The MLEs of λ and β have been computed based on the complete sample numerically using Newton Raphson method to be 11.1410 and 1.4171, respectively. The corresponding K-S test statistic and the associated P-value are equal to 0.1674 and 0.4851, respectively. Accordingly, one cannot reject the hypothesis that the data set comes from K(λ, β).
It can be easily seen that the upper records obtained from this data set are: 0.1282, 0.1786, 0.2100, 0.2435, 0.3101, 0.3796.
Based on these records, the MLEs, 95% ACIs, Bayes estimates and the corresponding 95% credible intervals are computed for the underlying parameters λ and β. To study how sensitive are the Bayes estimates for the choice of the hyper-parameters, the following priors are considered: *Prior* $0: a_1 = b_1 = a_2 = b_2 = 0$, *Prior* $1: a_1 = 24, b_1 = 2, a_2 = 7, b_2 = 5$, and *Prior* $2: a_1 = 12, b_1 = 1, a_2 = 12, b_2 = 9$.
Tables (3) and (4) summarize the results of point and interval estimates, respectively, based on both the classical and the Bayesian approaches.

Table 3: Estimates for λ and β based on the real data set

Parameter	MLE.	Bayes Estimates										
		Prior 0			*Prior* 1				*Prior* 2			
		LINEX			SE.	LINEX			SE.	LINEX		
		$v=-0.01$	$v=0.5$	$v=2$		$v=-0.01$	$v=0.5$	$v=2$		$v=-0.01$	$v=0.5$	$v=2$
λ	12.0148	12.1468	8.3301	4.8332	12.0110	12.0330	11.0390	9.2047	12.0510	12.0940	10.2930	7.8200
β	1.4135	1.4152	1.3363	1.1582	1.4370	1.4376	1.4076	1.3304	1.3695	1.3700	1.3462	1.2823

Table 4: 95% ACIs and CrIs of λ and β based on the real data set

Parameter	ACI	CrI		
		Prior 0	*Prior* 1	*Prior* 2
λ	(4.5358, 31.8260)	(4.4128, 23.3846)	(8.5085, 16.3880)	(6.7847, 18.7840)
β	(0.6350, 3.1463)	(0.5187, 2.7489)	(0.9224, 2.2084)	(0.9052, 2.0203)

6. Simulation study

In this section, a simulation study is conducted to evaluate the performance of the proposed estimation methods based on Kies record data. Simulations are performed using three sets of parameter values $(\lambda = 1, \beta = 2)$, $(\lambda = 2, \beta = 1)$ and $(\lambda = \beta = 2)$, mainly to compare the MLEs with the Bayes estimators and also to explore their effects on different parameter values. A given number m of upper records are generated from K(λ, β) using Eq. (16). The MLEs and the approximate Bayes estimates are computed using the importance sampling procedure. Bayes estimates are computed under the SE and LINEX loss functions assuming the following priors, which are assumed based on the considered cases:
Prior 0: $a_1 = 0$, $b_1 = 0$, $a_2 = 0$, $b_2 = 0$.
For $\lambda = 1$, $\beta = 2$:
Prior 1: $a_1 = 2$, $b_1 = 2$, $a_2 = 16$, $b_2 = 8$ and *Prior* 2: $a_1 = 4$, $b_1 = 4$, $a_2 = 8$, $b_2 = 4$.
For $\lambda = 2$, $\beta = 1$:
Prior 3: $a_1 = 4$, $b_1 = 2$, $a_2 = 8$, $b_2 = 8$ and *Prior* 4: $a_1 = 8$, $b_1 = 4$, $a_2 = 16$, $b_2 = 16$.
For $\lambda = 2$, $\beta = 2$:
Prior 5: $a_1 = 8$, $b_1 = 4$, $a_2 = 8$, $b_2 = 4$ and *Prior* 6: $a_1 = 10$, $b_1 = 5$, $a_2 = 10$, $b_2 = 5$.

These priors are proposed so as λ has the same mean but different variances, similarly for β. The main purpose of this is to reflect the sensitivity of our inferences to the choice of the hyper-parameters. The shape parameter of LINEX loss function v is assumed to equal -0.01, 0.5 and 2, separately.

Simulation studies are performed with $M = 1000$ iterations using Mathematica codes. The mean squared error (MSE) of the proposed MLEs and Bayes estimates is computed. The point estimation results are reported in Tables (5), (6) and (7) assuming the true parameters are $(\lambda = 1, \beta = 2)$, $(\lambda = 2, \beta = 1)$ and $(\lambda = \beta = 2)$, respectively, assuming $m = 5$, 6, 7 and 8. Further, the performance of the proposed classical CIs and Bayes CrIs are studied in terms of the average length (AL) and the coverage probability (CP). Tables (8), (9) and (10) present the ALs and CPs of the 95% ACIs, Boot-p CIs and CrIs for λ and β assuming $m = 5$, 6, 7 and 8.

Tables (5), (6) and (7) show that the performance of the Bayes estimates is better than that of the MLEs for both parameters in terms of MSEs. It can be also seen that the informative Bayes estimates under LINEX loss function with positive ν outperform the other estimates in most considered cases. However, non-informative Bayes estimates and the MLEs perform almost the same in most considered cases, but for positive ν the non-informative Bayes estimates under LINEX loss function outperform, in terms of the MSE, the MLEs. As expected, the Bayes estimates under some prior assumptions compete the corresponding Bayes estimates under other priors. For example, the MSEs of the Bayes estimates under *Prior* 4 are getting smaller than their counterparts under *Prior* 3. It is evident that all Bayes estimates under the informative priors behave better than the MLEs and the non-informative Bayes estimates. Clearly, the MSE of the proposed estimates decreases as m increases for both λ and β.

In view of interval estimation, Tables (8), (9) and (10) summarize the ALs and CPs of ACIs, Boot-p CIs and CrIs of λ and β when $(\lambda,\beta) = (1,2),(2,1)$ and $(2,2)$, respectively. The informative Bayes credible intervals are superior to the ACIs and the Boot-p CIs in the sense of coverage probability optimality criterion. It is noteworthy that the coverage probabilities of the Bayes credible intervals are generally well matched to their nominal levels. However, non informative Bayes credible intervals and ACIs are superior to the Boot-p CIs as they produce higher coverage probability with less average lengths. In general, there is a clear evidence that the informative credible intervals is the most valid method as it gives the highest simulated coverage probabilities comparing the intervals established by the classical approach.

Table 5: Average and MSE Values of the MLEs and Bayes estimates when $\lambda = 1$ and $\beta = 2$

m	Parameter	Criterion	MLE.	Bayes Estimates										
				Prior 0				*Prior* 1				*Prior* 2		
				LINEX			SE.	LINEX			SE.	LINEX		
				$\nu=-0.01$	$\nu=0.5$	$\nu=2$		$\nu=-0.01$	$\nu=0.5$	$\nu=2$		$\nu=-0.01$	$\nu=0.5$	$\nu=2$
m=5	β	Average	4.285	3.730	3.005	2.813	2.143	2.144	2.097	1.971	2.199	2.201	2.129	1.952
		MSE	12.258	9.9641	3.0226	0.6083	0.0504	0.0507	0.0370	0.0236	0.1420	0.1431	0.1062	0.0650
	λ	Average	0.550	0.491	0.672	0.808	0.941	0.961	0.913	0.804	1.015	1.038	0.987	0.866
		MSE	0.670	0.834	0.598	0.412	0.1192	0.1099	0.1005	0.1019	0.0905	0.0911	0.0785	0.0746
m=6	β	Average	2.776	2.995	2.671	2.324	2.092	2.093	2.052	1.941	2.160	2.162	2.101	1.951
		MSE	4.9816	5.1280	2.0570	0.5392	0.0350	0.0352	0.0271	0.0221	0.1177	0.1183	0.0938	0.0642
	λ	Average	0.939	0.936	0.950	0.982	1.048	1.053	1.004	0.893	1.015	1.038	0.987	0.866
		MSE	0.6109	0.6132	0.5151	0.3858	0.1100	0.1049	0.0887	0.0749	0.0905	0.0911	0.0785	0.0746
m=7	β	Average	2.492	2.501	2.237	2.101	1.984	1.984	1.949	1.857	2.155	2.157	2.1050	1.979
		MSE	2.4238	2.4667	1.2455	0.4004	0.0304	0.0305	0.0228	0.0210	0.093	0.094	0.0737	0.0503
	λ	Average	1.158	1.205	1.121	1.108	1.048	1.073	1.023	0.899	0.955	0.998	0.949	0.833
		MSE	0.5639	0.5658	0.4832	0.3647	0.0802	0.0841	0.0694	0.0549	0.0828	0.0831	0.0751	0.0727
m=8	β	Average	2.376	2.421	2.181	2.115	2.0922	2.0828	2.051	1.968	2.114	2.115	2.074	1.977
		MSE	0.7529	0.8120	0.6131	0.3737	0.0153	0.0154	0.0103	0.0072	0.0501	0.0504	0.0383	0.0218
	λ	Average	1.100	1.112	1.081	1.072	1.051	1.086	1.037	0.921	0.993	1.050	1.000	0.880
		MSE	0.5416	0.5430	0.4717	0.3597	0.0787	0.0745	0.0624	0.0541	0.0791	0.0606	0.0688	0.0683

Table 6: Average and MSE Values of the MLEs and Bayes estimates when $\lambda = 2$ and $\beta = 1$

m	Parameter	Criterion	MLE.	Bayes Estimates										
				Prior 0			*Prior* 3				*Prior* 4			
				LINEX			SE.	LINEX			SE.	LINEX		
				$v=-0.01$	$v=0.5$	$v=2$		$v=-0.01$	$v=0.5$	$v=2$		$v=-0.01$	$v=0.5$	$v=2$
m=5	β	Average	1.626	1.703	1.405	1.357	1.068	1.069	1.049	0.998	1.046	1.046	1.034	1.002
		MSE	1.5740	1.5929	0.9497	0.3181	0.0285	0.0286	0.0246	0.0181	0.0123	0.0123	0.0109	0.0087
	λ	Average	1.784	1.762	1.708	1.798	2.003	2.014	1.888	1.617	2.039	2.044	1.960	1.760
		MSE	1.7627	1.7681	1.5471	1.4310	0.2154	0.2184	0.1907	0.2612	0.1028	0.1164	0.1006	0.1278
m=6	β	Average	1.444	1.398	1.208	1.198	1.0785	1.079	1.062	1.017	1.026	1.026	1.016	0.988
		MSE	0.5791	0.5814	0.3471	0.2047	0.0250	0.0251	0.0241	0.0148	0.0105	0.0105	0.0097	0.0085
	λ	Average	1.944	1.925	1.901	1.899	2.083	2.106	1.980	1.702	2.019	2.021	1.941	1.747
		MSE	1.7591	1.752	1.4231	1.390	0.2113	0.2168	0.1888	0.2477	0.0892	0.0971	0.0893	0.1277
m=7	β	Average	1.315	1.297	1.2577	1.189	1.038	1.038	1.024	0.986	1.040	1.041	1.031	1.006
		MSE	0.2623	0.2541	0.1914	0.1310	0.0172	0.0173	0.0154	0.0124	0.0094	0.0094	0.0085	0.0068
	λ	Average	1.536	1.621	1.781	1.812	1.914	1.977	1.866	1.618	1.988	1.985	1.910	1.729
		MSE	1.5867	1.5973	1.2577	1.0010	0.1879	0.1387	0.1543	0.2381	0.0881	0.0969	0.0853	0.1197
m=8	β	Average	1.174	1.179	1.128	1.118	1.044	1.044	1.033	1.003	1.032	1.0321	1.024	1.001
		MSE	0.1703	0.1715	0.1128	0.0891	0.0131	0.0132	0.0116	0.0087	0.0093	0.0093	0.0082	0.0066
	λ	Average	1.771	1.684	1.702	1.779	1.898	1.899	1.818	1.604	2.027	2.046	1.969	1.780
		MSE	1.0664	1.0981	0.9087	0.8727	0.16697	0.1272	0.1394	0.2344	0.0879	0.0884	0.0810	0.1136

Table 7: Average and MSE Values of the MLEs and Bayes estimates when $\lambda = 2$ and $\beta = 2$

m	Parameter	Criterion	MLE.	Bayes Estimates										
				Prior 0			*Prior* 5				*Prior* 6			
				LINEX			SE.	LINEX			SE.	LINEX		
				$v=-0.01$	$v=0.5$	$v=2$		$v=-0.01$	$v=0.5$	$v=2$		$v=-0.01$	$v=0.5$	$v=2$
m=5	β	Average	3.517	3.621	2.971	2.607	2.181	2.182	2.111	1.932	1.997	1.998	1.942	1.801
		MSE	6.8171	7.0197	2.5659	0.5555	0.1649	0.1658	0.1264	0.1123	0.1077	0.1080	0.0990	0.1084
	λ	Average	3.145	3.208	2.748	2.510	1.966	1.953	1.869	1.672	2.050	2.057	1.989	1.815
		MSE	4.3416	4.5390	1.9262	1.1788	0.1269	0.1373	0.1335	0.1872	0.0992	0.1061	0.0931	0.1076
m=6	β	Average	3.054	2.841	2.641	2.4871	2.090	2.091	2.027	1.862	2.039	2.040	1.989	1.863
		MSE	3.5818	3.6457	1.7527	0.4956	0.1401	0.1407	0.1170	0.1073	0.1070	0.1076	0.0978	0.0988
	λ	Average	1.695	1.642	1.773	1.893	1.983	1.991	1.909	1.710	2.066	2.055	1.988	1.820
		MSE	2.2436	2.2749	1.4881	1.0301	0.0876	0.0837	0.0814	0.1401	0.0806	0.0772	0.0674	0.0856
m=7	β	Average	2.509	2.612	2.331	2.210	2.110	2.111	2.062	1.942	2.166	2.167	2.118	1.996
		MSE	2.1891	2.2171	1.2571	0.3701	0.1061	0.1066	0.0869	0.0730	0.0920	0.0929	0.0895	0.0515
	λ	Average	1.704	1.698	1.710	1.724	2.062	2.071	1.996	1.804	2.081	2.062	1.989	1.807
		MSE	0.8909	0.8931	0.6138	0.4003	0.0700	0.0830	0.0662	0.0832	0.0621	0.0667	0.0649	0.0761
m=8	β	Average	2.962	2.979	2.651	2.341	2.139	2.140	2.089	1.957	2.112	2.112	2.078	1.990
		MSE	2.1486	2.2129	1.1741	0.3540	0.09485	0.0952	0.0799	0.0680	0.0562	0.0565	0.0451	0288
	λ	Average	1.7852	1.704	1.803	1.814	1.944	1.890	1.825	0.0636	0.0798	0.0801	0.0677	0.0408
		MSE	0.8812	0.8921	0.6013	0.3124	0.0697	0.0821	0.0651	0.0831	0.0608	0.0652	0.0611	0.0753

Table 8: ALs and CPs of 95% CIs of $\lambda = 1$ and $\beta = 2$

Cases		ACI		Boot-p		CrIs					
						Prior 0		*Prior* 1		*Prior* 2	
m		β	λ	β	λ	β	λ	β	λ	β	λ
m=5	CP	0.80	0.93	0.70	0.81	0.93	0.83	0.97	0.96	0.99	0.95
	AL	3.8612	6.0170	7.5359	12.7930	4.8621	2.3738	4.8741	2.1439	6.9520	1.8891
m=6	CP	0.88	0.94	0.76	0.85	0.94	0.78	0.98	0.96	0.99	0.95
	AL	2.6151	5.8688	4.3292	7.0043	4.2388	2.1614	5.0343	2.0812	7.3202	1.8049
m=7	CP	0.88	0.95	0.82	0.87	0.95	0.77	0.99	0.97	0.99	0.96
	AL	2.2432	5.6800	2.9700	4.9216	3.8525	2.0112	5.2122	1.9840	7.7010	1.7304
m=8	CP	0.88	0.95	0.83	0.94	0.96	0.72	0.99	0.99	0.99	0.96
	AL	2.0205	5.4132	2.5839	4.9142	3.6011	1.8501	5.3762	1.9293	8.0700	1.5527

Table 9: ALs and CPs of 95% CIs of $\lambda = 2$ and $\beta = 1$

Cases		ACI		Boot-p		CrIs					
						Prior 0		*Prior* 3		*Prior* 4	
m		β	λ	β	λ	β	λ	β	λ	β	λ
m=5	CP	0.84	0.93	0.71	0.77	0.89	0.81	0.96	0.97	0.99	0.93
	AL	3.3471	6.4309	6.2641	28.2670	2.4086	4.1827	3.4144	3.8074	2.4416	3.3268
m=6	CP	0.86	0.93	0.73	0.83	0.91	0.79	0.97	0.98	0.99	0.94
	AL	2.6966	5.8499	4.7969	7.5577	2.4037	3.5744	3.6622	3.6126	2.4941	3.2894
m=7	CP	0.86	0.95	0.75	0.83	0.92	0.79	0.98	0.98	0.99	0.95
	AL	2.3408	5.6125	4.0902	6.7176	2.0447	3.2721	3.8511	3.3761	2.6494	3.2165
m=8	CP	0.90	0.97	0.79	0.86	0.93	0.78	0.99	0.98	0.99	0.99
	AL	1.9024	5.5002	2.7860	4.6248	1.8546	2.9352	3.9861	3.0051	2.6921	3.0488

Table 10: ALs and CPs of 95% CIs of $\lambda = 2$ and $\beta = 2$

Cases		ACI		Boot-p		CrIs					
						Prior 0		*Prior* 5		*Prior* 6	
m		β	λ	β	λ	β	λ	β	λ	β	λ
m=5	CP	0.85	0.91	0.72	0.82	0.91	0.91	0.96	0.97	0.99	0.95
	AL	6.7013	6.4842	13.3060	12.9420	5.6959	4.4634	7.0471	3.2719	6.1916	3.1703
m=6	CP	0.89	0.91	0.79	0.83	0.91	0.90	0.97	0.98	0.99	0.95
	AL	5.2326	5.8095	9.5716	6.3452	4.6384	4.0721	7.4476	3.1824	6.4401	3.0779
m=7	CP	0.89	0.96	0.80	0.83	0.94	0.88	0.99	0.97	0.99	0.96
	AL	4.5255	5.6413	7.4424	5.5048	4.1055	3.6944	7.7343	3.0254	6.7930	3.0302
m=8	CP	0.93	0.98	0.82	0.88	0.94	0.86	0.99	0.98	0.99	0.98
	AL	3.9058	5.6261	5.9419	4.9526	3.5840	3.3658	8.1000	2.9724	7.0417	2.8790

7. Conclusion

In this paper, classical and Bayesian inferences were proposed for the two-parameter Kies distribution based on upper records. Some distributional properties of the Kies distribution based on records were studied. Uniqueness and existence of the MLEs were discussed. Asymptotic and bootstrap confidence intervals were constructed. In the context of Bayesian estimation, the Bayes estimates of the parameters cannot be obtained in explicit forms. So, approximate Bayes estimates along with their associated credible intervals were obtained by employing importance sampling technique under SE and LINEX loss functions assuming non-informative and informative priors for both parameters. The performance of the different estimation methods was assessed via Monte Carlo simulations. Generally, from the simulation study, it was concluded that the proposed informative Bayes estimates outperform the classical estimates in all considered cases. However, non-informative Bayesian and the classical estimation methods perform almost the same under SE and LINEX under small ν, while better results of the Bayesian methods are obtained under LINEX assuming other positive values of ν. Classical confidence intervals (asymptotic and Boot-P) and Bayes credible intervals were also constructed for the unknown parameters. It is clearly evident that the Bayes credible intervals compete the classical confidence intervals in terms of the coverage probability in all cases. It was also noticed that the Asymptotic CI outperforms the Boot-p CI in all cases. Finally, a real data set was analyzed for illustrative purposes.

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Estimation of the density and cumulative distribution functions of the exponentiated Burr XII distribution

Amal S. Hassan[1], Salwa M. Assar[2], Kareem A. Ali[3], Heba F. Nagy[4,]

ABSTRACT

The exponentiated Burr Type XII (EBXII) distribution has wide applications in reliability and economic studies. In this article, the estimation of the probability density function and the cumulative distribution function of EBXII distribution is considered. We examine the maximum likelihood estimator, the uniformly minimum variance unbiased estimator, the least squares estimator, the weighted least squares estimator, the maximum product spacing estimator, the Cramér–von-Mises estimator, and the Anderson–Darling estimator. We derive analytical forms for the bias and mean square error. A simulation study is performed to investigate the consistency of the suggested methods of estimation. Data relating to the wind speed and service times of aircraft windshields are used with the studied methods. The simulation studies and real data applications have revealed that the maximum likelihood estimator performs more efficiently than its remaining counterparts.

Key words: exponentiated Burr Type XII model, least squares estimator, maximum likelihood estimator, uniform minimum variance unbiased estimator, weighted least squares estimator.

Mathematical Subject Classification: 62F10.

1. Introduction

The Burr Type XII (BXII) distribution has gained special attention in physics, actuarial studies, reliability and applied statistics. Characteristics of the BXII distribution are near to several distributions like exponential, normal, lognormal, etc. Extra properties about the BXII distribution can be found in Headrick et al. (2010).

[1] Department of Mathematical Statistics, Cairo University, Faculty of Graduate Studies for Statistical Research, Egypt. E-mail: dr.amalelmoslamy@gmail.com. ORCID: https://orcid.org/0000-0003-4442-8458.

[2] Department of Mathematical Statistics, Cairo University, Faculty of Graduate Studies for Statistical Research, Egypt. E-mail: salwaassar@yahoo.com. ORCID: https://orcid.org/0000-0001-7450-7486.

[3] Department of Mathematical Statistics, Cairo University, Faculty of Graduate Studies for Statistical Research, Egypt. E-mail: ali.kareem64@yahoo.com. ORCID: https://orcid.org/0000-0002-2522-7733.

[4] Corresponding Author, Department of Mathematical Statistics, Cairo University, Faculty of Graduate Studies for Statistical Research. Egypt, E-mail: heba_nagy_84@yahoo.com. ORCID: https://orcid.org/0000-0003-0262-205X.

The exponentiated Burr Type XII distribution is a generalization to the BXII distribution through adding a new shape parameter. The cumulative distribution function (CDF) of the EBXII distribution is defined as follows:

$$G(x)=\left(1-\left(1+x^{c}\right)^{-k}\right)^{\beta}; \; x, k, c, \beta>0, \tag{1}$$

where, k, c and β are shape parameters. The probability density function (PDF) of the EBXII corresponding to (1) is specified by

$$g(x)=ck\beta x^{c-1}\left(1+x^{c}\right)^{-k-1}\left(1-\left(1+x^{c}\right)^{-k}\right)^{\beta-1}; \; x, k, c, \beta>0. \tag{2}$$

Statistical developments on the EBXII model have been studied by several authors. Among them, AL-Hussaini and Hussein (2011) studied maximum likelihood (ML) and Bayesian estimation to the parameters of the EBXII distribution under Type II censored data. Kumar et al. (2017) established several explicit expressions and recurrence relations for single and product moments of r-th lower record values from the EBXII distribution.

Statistical inference is one of the most popular topics in research and scientific studies whether from the theoretical or applied aspects. Most traditional studies have been focused on inferring the parameter(s) involved in the distribution. The importance of statistical distributions is not limited to the characterization of statistical phenomena, but rather to the calculation of many population metrics such as moments, probability weighted moments, failure rate function, etc. However, it would be more useful to study the efficient estimation of the PDF and CDF. The estimation of the PDF and the CDF is important for many reasons. For instance, the best estimators for the PDF can be used to estimate functionals of the PDF such as estimation of Kullback-Leibler divergence, as provided by Hurvich et al. (1990), estimation of Fisher information (see Mielniczuk and Wojtyś (2010), the estimation of the differential entropy (see Nilsson and Kleijn (2007), and estimation of the Rényi entropy. Similarly, the best estimators for the CDF can be used to estimate functionals of the CDF like estimation of quantiles (see Saleh et al. (1988) and estimation of the Lorenz curve (see Woo and Yoon (2001)).

Some studies on the estimation of PDF and CDF have appeared in recent literature for some continuous distributions, for instance, Pareto distribution by Asrabadi (1990) and (Dixit and Nooghabi (2010), Dixit and Nooghabi (2011)), exponentiated Pareto distribution by Jabbari (2010), generalized Rayleigh distribution by Alizadeh et al. (2013), generalized exponential Poisson distribution by Bagheri et al. (2014), exponentiated Weibull by Alizadeh et al. (2015a), generalized exponential distribution by Alizadeh et al. (2015b), exponentiated Gumbel distribution by Bagheri et al. (2016a), Weibull extension distribution by Bagheri et al. (2016b), Lindley distribution by Maiti and Mukherjee (2018), generalized logistic distribution by Tripathi et al. (2017), Frechet distribution by Maleki and Deiri (2017) and Topp-Leone distribution by Benkhelifa (2017), exponentiated gamma distribution (Rasekhi (2018)), and Gompertz distribution (Dey et al. (2018)).

Our objective here is to investigate the efficient estimation of the PDF and the CDF of the EBXII model due to its wide statistical applications and developments. Different parametric methods of estimation, namely ML, uniformly minimum variance unbiased (UMVU), least squares (LS), weighted least squares (WLS), Cramér-von-Mises (CvM), Anderson–Darling (AD) and maximum product spacing (MPS) are considered. This paper is organized as follows. Sections (2) and (3) provide ML and UMVU estimators of the PDF and CDF with their mean square errors (MSEs). Section (4) includes other parametric methods of estimation. Section (5) comprises a simulation study in order to compare different suggested estimators. A real data set is analyzed for illustrative purpose in Section (6). The article ends with concluding remarks in Section (7).

2. Maximum likelihood estimators

In this section we obtain the ML estimators of the PDF and the CDF of the EBXII distribution. Let $X_1, X_2, \ldots, X_n$ be a random sample with size n from the EBXII distribution with known parameters k and c. The log likelihood function of the EBXII distribution is given by

$$L\left(\beta|\underline{x},c,k\right)=nln\left(\beta\right)+nln\left(k\right)+nln\left(c\right)+\left(c-1\right)\sum_{i=1}^{n}\ln\left(x_i\right)-\left(k+1\right)\sum_{i=1}^{n}\ln\left(1+x_i^c\right)$$
$$+\left(\beta-1\right)\sum_{i=1}^{n}\ln\left(1-\left(1+x_i^c\right)^{-k}\right). \tag{3}$$

The ML estimator of β, say $\hat{\beta}$, is given as

$$\hat{\beta}=\frac{n}{-\sum_{i=1}^{n}\ln\left(1-\left(1+x_i^c\right)^{-k}\right)}=\frac{n}{T}, \qquad T=-\sum_{i=1}^{n}\ln\left(1-\left(1+x_i^c\right)^{-k}\right).$$

We can rewrite CDF(1) as follows:

$$-\ln G(x) = \beta V, \quad V = -\ln(1-(1+x^c)^{-k}).$$

It can be seen that V has an exponential distribution with scale parameter β. Then T has a gamma (n,β), random variable with density function given by

$$f\left(t\right)=\frac{\beta^n}{\Gamma(n)}t^{n-1}e^{-\beta t}, \; t>0. \tag{4}$$

Therefore, $\hat{\beta}=\frac{n}{T}=S$, has an inverse gamma $(n,n\beta)$ distribution with PDF given by

$$f\left(s\right)=\frac{\left(n\beta\right)^{n}}{\Gamma(n)}s^{-n-1}e^{-\frac{n\beta}{s}},\quad s>0\,. \tag{5}$$

Applying the invariance property of the ML method, the required PDF and CDF estimators are obtained as follows:

$$\hat{g}\left(x\right)=ck\,\hat{\beta}x^{c-1}\left(1+x^{c}\right)^{-k-1}\left(1-\left(1+x^{c}\right)^{-k}\right)^{\hat{\beta}-1},$$

$$\text{and}\quad \hat{G}\left(x\right)=\left(1-\left(1+x^{c}\right)^{-k}\right)^{\hat{\beta}}. \tag{6}$$

Now, we show that $\hat{g}\left(x\right)$ and $\hat{G}\left(x\right)$ are biased estimators of $g\left(x\right)$ and $G\left(x\right)$ respectively. Further, the MSEs are obtained. Theorem (1) calculates $E(\hat{g}\left(x\right)^{r})$ and $E(\hat{G}\left(x\right)^{r})$.

Theorem 1: We have

$$E(\hat{g}\left(x\right)^{r})=\frac{2c^{r}k^{r}b^{r}d^{r\left(\frac{1}{k}+1\right)}\left(1-d\right)^{-r}\left(n\beta\right)^{n}}{\Gamma(n)}\left(\frac{n\beta}{-rln\left(1-d\right)}\right)^{\frac{r-n}{2}}\mathrm{K}_{r-n}(2\sqrt{-n\beta rln\left(1-d\right)}),$$

and

$$E(\hat{G}\left(x\right)^{r})=2\frac{\left(n\beta\right)^{n}}{\Gamma(n)}\left(\frac{n\beta}{-rln\left(1-d\right)}\right)^{\frac{-n}{2}}\mathrm{K}_{-n}(2\sqrt{-n\beta rln\left(1-d\right)}),$$

where $b=x^{c-1}, d=(1+x^{c})^{-k}$.

Proof: First by using $b=x^{c-1}, d=(1+x^{c})^{-k}$, then $\hat{g}\left(x\right)$ can be rewritten as follows:

$\hat{g}\left(x\right)=cksbd^{\frac{1}{k}+1}\left(1-d\right)^{s-1}$, and $\hat{G}\left(x\right)=\left(1-d\right)^{s}$.

Thus,

$$E\left(\hat{g}\left(x\right)^{r}\right)=\int_{0}^{\infty}c^{r}k^{r}s^{r}b^{r}d^{r\left(\frac{1}{k}+1\right)}\left(1-d\right)^{rs-r}\frac{\left(n\beta\right)^{n}}{\Gamma(n)}s^{-n-1}e^{\frac{-n\beta}{s}}ds$$

$$=c^{r}k^{r}b^{r}d^{r\left(\frac{1}{k}+1\right)}\left(1-d\right)^{-r}\frac{\left(n\beta\right)^{n}}{\Gamma(n)}\int_{0}^{\infty}s^{r-n-1}e^{-s\left(-rln\left(1-d\right)\right)}e^{\frac{-n\beta}{s}}ds$$

$$=\frac{2c^r k^r b^r d^{r\left(\frac{1}{k}+1\right)}(1-d)^{-r}(n\beta)^n}{\Gamma(n)}\left(\frac{n\beta}{-rln(1-d)}\right)^{\frac{r-n}{2}}\mathrm{K}_{r-n}(2\sqrt{-n\beta rln(1-d)}).$$

Here, $\mathrm{K}_\nu(.)$ denotes the modified Bessel's function of the second kind of order ν (see equation (3.471.9) in (Gradshteyn and Ryzhik (2000)). Similarly, $E(\hat{G}(x)^r)$ takes the following form:

$$E(\hat{G}(x)^r)=2\frac{(n\beta)^n}{\Gamma(n)}\left(\frac{n\beta}{-rln(1-d)}\right)^{\frac{-n}{2}}\mathrm{K}_{-n}(2\sqrt{-n\beta rln(1-d)}).$$

Theorem 2: The MSEs for $\hat{g}(x)$ and $\hat{G}(x)$ respectively are given by

$$MSE(\hat{g}(x))=c^2k^2b^2d^{2\left(\frac{1}{k}+1\right)}(1-d)^{-2}(2\frac{(n\beta)^n}{\Gamma(n)}\left(\frac{n\beta}{-2ln(1-d)}\right)^{\frac{2-n}{2}}\mathrm{K}_{2-n}(2\sqrt{-2n\beta ln(1-d)}$$

$$-4\beta(1-d)^{-\beta}\frac{(n\beta)^n}{\Gamma(n)}\left(\frac{n\beta}{-ln(1-d)}\right)^{\frac{1-n}{2}}\mathrm{K}_{1-n}(2\sqrt{-n\beta ln(1-d)})+\beta^2(1-d)^{-2\beta}).$$

and

$$MSE(\hat{G}(x))=2\frac{(n\beta)^n}{\Gamma(n)}\left(\frac{n\beta}{-2ln(1-d)}\right)^{\frac{-n}{2}}\mathrm{K}_{-n}(2\sqrt{-2n\beta ln(1-d)}-4(1-d)^{\beta}\frac{(n\beta)^n}{\Gamma(n)}\left(\frac{n\beta}{-ln(1-d)}\right)^{\frac{-n}{2}}\mathrm{K}_{-n}(2\sqrt{-n\beta ln(1-d)}+(1-d)^{2\beta}.$$

Proof: Since

$$MSE(\hat{g}(x))=E(\hat{g}(x)^2)-2g(x)E(\hat{g}(x))+g(x)^2.$$

Hence, $E(\hat{g}(x)^2)$ and $E(\hat{g}(x))$ can be obtained by setting r =1 and r =2 in Theorem (1), hence the $MSE(\hat{g}(x))$ is easily calculated. The proof of $MSE(\hat{G}(x))$ is similar.

3. Uniformly minimum variance unbiased estimators

In this section, UMVU estimators of the PDF and CDF of the EBXII distribution are considered. In addition, the rth moment and the MSE of these estimators are derived.

Let $X_1,\ldots,X_n$ be a random sample of size n from the EBXII distribution. Then, $T=-\sum_{i=1}^{n}\ln\left(1-\left(1+x_i^c\right)^{-k}\right)$ is complete sufficient statistic for the parameter β

(assumed k and c are known parameters). Recall that T has a gamma (n, β) distribution with density function (4). According to the Lehmann-Scheffe theorem, if $g(x_1|t) = g^*(t)$ is the conditional PDF of $X_1|T$, we have

$$E\left(g^*(T)\right) = \int g(x_1|t) f(t) dt = \int g(x_1, t) dt = g(x_1),$$

where $g(x_1, t)$ is the joint PDF of X_1 and T. Therefore, $g^*(t)$ is the UMVU estimator of $g(x)$.

Lemma 1: The conditional distribution of V given $T = t$ is obtained as

$$g_{V|T}(v|t) = \frac{(n-1)(t-v)^{n-2}}{t^{n-1}}, v < t < \infty, \qquad V = -\ln\left(1-\left(1+x_1^c\right)^{-k}\right).$$

Proof: We have

$$g_{V|T}(v|t) = \frac{g(v, t-v)}{f(t)} = \frac{\beta^n (t-v)^{n-2} e^{-\beta t}\Gamma(n)}{\Gamma(n-1)\beta^n t^{n-1} e^{-\beta t}} = \frac{(n-1)(t-v)^{n-2}}{t^{n-1}}, v < t < \infty.$$

In the following theorem the UMVU estimators for $g(x)$ and $G(x)$ are obtained.

Theorem 3: The uniformly minimum variance unbiased estimators for $g(x)$ and $G(x)$ are given by

$$\tilde{g}(x) = g^*(t) = \frac{(n-1)\left(t+\ln\left(1-\left(1+x^c\right)^{-k}\right)\right)^{n-2}}{t^{n-1}} \times \frac{kcx^{c-1}\left(1+x^c\right)^{-k-1}}{1-\left(1+x^c\right)^{-k}}, \quad -\ln\left(1-\left(1+x^c\right)^{-k}\right) < t < \infty,$$

and

$$\tilde{G}(x) = \left\{\frac{t+\ln\left(1-\left(1+x^c\right)^{-k}\right)}{t}\right\}^{n-1}.$$

Proof: The estimator $\tilde{g}(x)$ is the UMVU estimator for $g(x)$ can be proved by the Lehmann-Scheffe theorem and Lemma (1). In addition, $\tilde{G}(x)$ is the UMVU estimator of $G(x)$ from the fact that

$$\frac{d}{dx}\tilde{G}(x) = \frac{d}{dx}\left(\frac{t+\ln(1-(1+x^c)^{-k})}{t}\right)^{n-1} = \tilde{g}(x).$$

Further, we compute the MSEs for the two UMVU estimators of $\tilde{g}(x)$ and $\tilde{G}(x)$, suppose that

$$M = \frac{(n-1)kcx^{c-1}\left(1+x^{c}\right)^{-k-1}}{1-\left(1+x^{c}\right)^{-k}} \text{ and } p(x) = -\ln\left(1-\left(1+x^{c}\right)^{-k}\right).$$

Then, we obtain the following expectation:

$$E\left(\tilde{g}(x)^{r}\right) = \int_{p(x)}^{\infty} \tilde{g}(x)^{r} f(t)dt = M^{r}\frac{\beta^{n}}{\Gamma(n)}\int_{p(x)}^{\infty}\frac{\left(t-p(x)\right)^{nr-2r}}{t^{nr-r}}t^{n-1}e^{-\beta t}dt.$$

After some simplification, we obtain

$$E\left(\tilde{g}(x)^{r}\right) = \frac{M^{r}}{\Gamma(n)}\sum_{i=0}^{nr-2r}\binom{nr-2r}{i}(-1)^{i} p(x)^{i}\beta^{i+r}\int_{p(x)\beta}^{\infty} u^{n-i-r-1}e^{-u}du,$$

which $\int_{p(x)\beta}^{\infty} u^{n-i-r-1}e^{-u}du$ is the upper incomplete gamma function, so $E\left(\tilde{g}(x)^{r}\right)$ can be formulated as follows:

$$E\left(\tilde{g}(x)^{r}\right) = \sum_{i=0}^{nr-2r}\frac{\beta^{i+r}}{\Gamma(n)}\binom{nr-2r}{i}(-1)^{i} M^{r} p(x)^{i}\,\Gamma(n-i-r, p(x)\beta). \tag{7}$$

Similarly, we can prove that

$$E\left(\tilde{G}(x)^{r}\right) = \sum_{i=0}^{nr-r}\frac{\beta^{i}}{\Gamma(n)}\binom{nr-r}{i}(-1)^{i} p(x)^{i}\,\Gamma(n-i, p(x)\beta). \tag{8}$$

Theorem 4. The mean square errors for $\tilde{g}(x)$ and $\tilde{G}(x)$, respectively, are given by

$$MSE\left(\tilde{g}(x)\right) = \sum_{i=0}^{2n-4}\frac{M^{2}}{\Gamma(n)}\binom{2n-4}{i}(-1)^{i}\beta^{i+2}p(x)^{i}\,\Gamma(n-i-2, p(x)\beta) - (\mathrm{g}(x))^{2},$$

and

$$MSE\left(\tilde{G}(x)\right) = \sum_{i=0}^{2n-2}\frac{\beta^{i}}{\Gamma(n)}\binom{2n-2}{i}(-1)^{i} p(x)^{i}\,\Gamma(n-i, p(x)\beta) - (\mathrm{G}(x))^{2}.$$

Proof: Since

$$MSE\left(\tilde{g}(x)\right) = E\left(\tilde{g}(x)^{2}\right) - g(x)^{2},$$

where $E\left(\tilde{g}(x)^{2}\right)$ can be obtained by setting r =2 in (7), hence we can calculate $MSE\left(\tilde{g}(x)\right)$. The proof of $MSE\left(\tilde{G}(x)\right)$ is similar.

4. Other parametric methods of estimation

In this section, several methods of estimation such as LS, WLS, MPS, CvM and AD are considered. All these methods are based on the CDF. Let $X_{i:n}, i=1,\ldots,n,$ be the order statistics of a random sample from the EBXII distribution and assumed k and c are known parameters. Then, the LS, WLS, MPS, CvM and AD estimators of the PDF and the CDF of the EBXII distribution are derived in the following subsections.

4.1. Least squares and weighted least squares estimators

The ordinary least squares and the weighted least squares (Swain et al. (1988)) are well-known methods used for estimating the unknown parameters. The LS estimator of β, say, β' and the WLS estimator of β, say, $\bar{\beta}$, are given by minimizing the following quantities with respect to β

$$\sum_{i=1}^{n}\left(G(x_{i:n})-\frac{i}{n+1}\right)^2, \tag{9}$$

$$\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i\,(n-i+1)}\left(G(x_{i:n})-\frac{i}{n+1}\right)^2. \tag{10}$$

There is no closed form solution for β, in minimizing Equations (9) and (10), so the numerical technique is applied to find β' and $\bar{\beta}$. Hence, the LS and WLS estimators of the CDF and PDF for the EBXII distribution are obtained, respectively, as follows:

$$G'(x)=\left(1-\left(1+x^c\right)^{-k}\right)^{\beta'}, \quad g'(x)=ck\,\beta' x^{c-1}\left(1+x^c\right)^{-k-1}\left(1-\left(1+x^c\right)^{-k}\right)^{\beta'-1},$$

and

$$\bar{G}(x)=\left(1-\left(1+x^c\right)^{-k}\right)^{\bar{\beta}}, \quad \bar{g}(x)=ck\,\bar{\beta} x^{c-1}\left(1+x^c\right)^{-k-1}\left(1-\left(1+x^c\right)^{-k}\right)^{\bar{\beta}-1}.$$

4.2. Maximum product of spacing estimators

The MPS has been proposed by Cheng and Amin (1979) as an alternative method for the ML for the estimation parameters of continuous univariate distribution. Let a sample of size n be available from EBXII, we define the corresponding uniform spacings as follows:

$$D_i=G(x_{i:n})-G(x_{i-1:n}), \qquad i=1,2,\ldots,n \text{ where } G(x_{0:n})=0,\ G(x_{n+1:n})=1,$$

$$\sum_{i=1}^{n+1}D_i=1.$$

The MPS estimator of β, say $\breve{\beta}$, can be obtained by maximizing the geometric mean of the spacings,

$$D(\beta)=\left(\prod_{i=1}^{n+1}D_i\right)^{\frac{1}{n+1}}=\prod_{i=1}^{n+1}\left(G(x_{i:n})-G(x_{i-1:n})\right)^{\frac{1}{n+1}},$$

with respect to β. Equivalently, the following expression

$$D(\beta)=\frac{1}{n+1}\sum_{i=1}^{n+1}\ln\left(G(x_{i:n})-G(x_{i-1:n})\right), \tag{11}$$

can be maximized to obtain the desired estimator of β. It can be shown that $\breve{\beta}$ satisfies

$$\frac{1}{n+1}\sum_{i=1}^{n+1}\frac{1}{\left(G(x_{i:n})-G(x_{i-1,n})\right)}\left[D_0(x_{i:n})-D_0(x_{i-1:n})\right]=0, \tag{12}$$

$D_0(x_{i:n})=\left(1-\left(1+x_{i:n}{}^{c}\right)^{-k}\right)^{\beta}\ln\left(1-\left(1+x_{i:n}{}^{c}\right)^{-k}\right)$. There is no closed form solution for β in (12), so the numerical technique is applied to find $\breve{\beta}$. Now, the MPS estimators of the CDF and PDF are obtained as follows:

$$\breve{G}(x)=\left(1-\left(1+x^{c}\right)^{-k}\right)^{\breve{\beta}},\quad \breve{g}(x)=ck\breve{\beta}x^{c-1}\left(1+x^{c}\right)^{-k-1}\left(1-\left(1+x^{c}\right)^{-k}\right)^{\breve{\beta}-1}.$$

4.3. Cramér-von-Mises and Anderson Darling estimators

Cramér-von-Mises is a type of minimum distance estimators (also called maximum goodness of fit estimators), which is based on the difference between the estimate of the CDF and its empirical distribution function. The CvM estimator of β, say $\ddot{\beta}$, is obtained by minimizing the following function with respect to β

$$C(\beta)=\frac{1}{12n}+\sum_{i=1}^{n}\left(G(x_{i:n})-\frac{2i-1}{2n}\right)^{2}. \tag{13}$$

Equivalently, we solve the equation

$$\sum_{i=1}^{n}\left(G(x_{i:n})-\frac{2i-1}{2n}\right)D_0(x_{i:n})=0,$$

to obtain β in the previous equation. So, the CvM estimators of the CDF and PDF are obtained as follows:

$$\ddot{G}(x)=\left(1-\left(1+x^{c}\right)^{-k}\right)^{\ddot{\beta}},\quad \ddot{g}(x)=ck\ddot{\beta}x^{c-1}\left(1+x^{c}\right)^{-k-1}\left(1-\left(1+x^{c}\right)^{-k}\right)^{\ddot{\beta}-1}.$$

Additionally, the Anderson-Darling method was initially discussed by Anderson and Darling (1952) as another type of minimum distance estimators. The AD estimator of β, say β°, is obtained by minimizing, with respect to β the function

$$A(\beta) = -n + \sum_{i=1}^{n} (2i-1)\left[\log(G(x_{i:n}) + \log(1-G(x_{n+1-i:n})\right]. \tag{14}$$

Analogously, these estimators can also be obtained by solving the following non-linear equation

$$\sum_{i=1}^{n} (2i-1)\left[\frac{D_0(x_{i:n})}{G(x_{i:n})} + \frac{D_0(x_{n+1-i:n})}{1-G(x_{n+1-i:n})}\right] = 0,$$

with respect to β. Therefore, the AD estimators of the CDF and PDF for the EBXII distribution are obtained as follows:

$$G^{\circ}(x) = \left(1-\left(1+x^{c}\right)^{-k}\right)^{\beta^{\circ}}, \quad g^{\circ}(x) = ck\,\beta^{\circ} x^{c-1}\left(1+x^{c}\right)^{-k-1}\left(1-\left(1+x^{c}\right)^{-k}\right)^{\beta^{\circ}-1}.$$

5. Simulation study

A simulation study is carried out in this section to determine the efficient estimate of the PDF and CDF between the following estimates: ML, UMVU, LS, WLS, MPS, CvM and AD. The MSE is used to compare between these estimators. One thousand random samples with sizes n=10,25,50,75 and 100 are generated from the EBXII distribution with different sets of parameters (β, k, c) = (0.5,2,2.5), (1,2,2.5), (2,2,2.5), (1.5,2,1.5), (2,2,1.5) and (4,2,1.5). The MSEs of each estimate are displayed in Tables 1 and 2. It can be detected from these tables that ML and UMVU estimates are more efficient than the other corresponding estimates.

In addition, Figures 1 to 6 display the MSEs values for the PDF and CDF of the EBXII distribution by proposed methods. The left-hand side graph in each figure is related to the PDF estimates and the corresponding right-hand side graph is the CDF estimates. Generally, the efficiency of all estimates improves as sample size increases.

Figure 7 and Figure 8 show the performance of a different set of the parameters. We detect from these figures that the ML and UMVU estimates of the set of parameters (β, k, c)= (4, 2, 1.5) have better statistical properties than the other corresponding studied sets.

Table 1. The MSEs of the PDF and CDF estimates of the EBXII distribution for (β, k, c)= (0.5,2,2.5), (1,2,2.5) and (2,2,2.5)

Sample size	Methods	$\beta=0.5$		$\beta=1$		$\beta=2$	
		PDF	CDF	PDF	CDF	PDF	CDF
n=10	ML	0.11	7.917e-3	0.026	8.763e-3	0.021	7.979e-3
	UMVU	0.115	8.007e-3	0.031	8.753e-3	0.026	7.957e-3
	MPS	0.201	8.227e-3	0.03	8.849e-3	0.023	8.047e-3
	CvM	0.107	0.012	0.04	0.012	0.037	0.013
	AD	0.111	8.739e-3	0.028	9.135e-3	0.024	8.788e-3
	LS	0.111	0.013	0.042	0.013	0.039	0.014
	WLS	0.116	0.012	0.039	0.012	0.037	0.013
n=25	ML	0.039	2.952e-3	9.415e-3	2.884e-3	8.208e-3	2.962e-3
	UMVU	0.037	2.915e-3	0.01	2.871e-3	8.865e-3	2.951e-3
	MPS	0.056	2.965e-3	0.01	2.971e-3	8.741e-3	3.058e-3
	CvM	0.032	3.902e-3	0.013	3.824e-3	0.011	3.917e-3
	AD	0.045	3.257e-3	0.011	3.333e-3	9.48e-3	3.368e-3
	LS	0.033	3.906e-3	0.013	4.017e-3	0.011	3.939e-3
	WLS	0.032	3.626e-3	0.012	3.482e-3	9.925e-3	3.495e-3
n=50	ML	0.012	1.483e-3	4.635e-3	1.393e-3	4.181e-3	1.46e-3
	UMVU	0.012	1.482e-3	4.844e-3	1.398e-3	4.365e-3	1.464e-3
	MPS	0.016	1.535e-3	5.113e-3	1.47e-3	4.501e-3	1.534e-3
	CvM	0.015	1.969e-3	6.104e-3	1.832e-3	5.476e-3	1.914e-3
	AD	0.014	1.752e-3	5.435e-3	1.629e-3	4.875e-3	1.709e-3
	LS	0.015	1.972e-3	6.126e-3	1.836e-3	5.485e-3	1.917e-3
	WLS	0.014	1.791e-3	5.635e-3	1.686e-3	5.048e-3	1.765e-3
n=75	ML	8.442e-3	9.864e-4	3.014e-3	9.171e-4	2.685e-3	9.409e-4
	UMVU	8.49e-3	9.893e-4	3.07e-3	9.121e-4	2.729e-3	9.351e-4
	MPS	0.01	1.032e-3	3.142e-3	9.3e-4	2.74e-3	9.485e-4
	CvM	0.011	1.319e-3	4.13e-3	1.235e-3	3.733e-3	1.291e-3
	AD	9.722e-3	1.175e-3	3.616e-3	1.086e-3	3.247e-3	1.13e-3
	LS	0.011	1.323e-3	4.134e-3	1.236e-3	3.732e-3	1.291e-3
	WLS	9.86e-3	1.186e-3	3.716e-3	1.111e-3	3.319e-3	1.151e-3
n=100	ML	5.985e-3	7.63e-4	2.508e-3	7.553e-4	2.135e-3	7.462e-4
	UMVU	6.005e-3	7.619e-4	2.562e-3	7.567e-4	2.183e-3	7.481e-4
	MPS	6.877e-3	7.802e-4	2.672e-3	7.855e-4	2.257e-3	7.785e-4
	CvM	7.305e-3	9.491e-4	3.329e-3	1.005e-3	2.808e-3	9.835e-4
	AD	6.731e-3	8.642e-4	2.959e-3	8.928e-4	2.5e-3	8.764e-4
	LS	7.351e-3	9.495e-4	3.339e-3	1.007e-3	2.814e-3	9.858e-4
	WLS	6.819e-3	8.725e-4	2.978e-3	8.966e-4	2.515e-3	8.792e-4

Table 2. The MSEs of the PDF and CDF estimates of the EBXII distribution for the set of parameters (β, k, c) = (1.5,2,1.5), (2,2,1.5) and (4,2,1.5)

Sample size	Methods	$\beta=1.5$		$\beta=2$		$\beta=4$	
		PDF	CDF	PDF	CDF	PDF	CDF
n=10	ML	0.022	7.947e-3	0.013	7.405e-3	7.499e-3	7.8e-3
	UMVU	0.028	7.907e-3	0.017	7.287e-3	9.843e-3	7.826e-3
	MPS	0.029	7.971e-3	0.017	7.291e-3	9.129e-3	7.948e-3
	CvM	0.03	0.012	0.019	0.011	0.012	0.011
		0.026	8.779e-3	0.016	8.207e-3	9.04e-3	8.78e-3
	AD	0.032	0.014	0.02	0.011	0.013	0.011
	LS	0.03	0.012	0.019	0.01	0.012	0.011
	WLS						
n=25	ML	8.38e-3	3.083e-3	6.48e-3	3.386e-3	3.285e-3	3.042e-3
	UMVU	9.201e-3	3.076e-3	7.156e-3	3.398e-3	3.574e-3	2.997e-3
	MPS	0.01	3.187e-3	7.699e-3	3.544e-3	3.601e-3	3.008e-3
	CvM	0.011	3.973e-3	8.25e-3	4.274e-3	4.578e-3	4.11e-3
	AD	9.843e-3	3.544e-3	7.366e-3	3.782e-3	3.99e-3	3.596e-3
	LS	0.011	3.985e-3	8.338e-3	4.285e-3	4.616e-3	4.119e-3
	WLS	0.01	3.709e-3	7.932e-3	4.043e-3	4.29e-3	3.823e-3
n=50	ML	3.934e-3	1.504e-3	3.015e-3	1.522e-3	1.631e-3	1.377e-3
	UMVU	4.112e-3	1.499e-3	3.178e-3	1.529e-3	1.714e-3	1.377e-3
	MPS	4.453e-3	1.544e-3	3.43e-3	1.614e-3	1.801e-3	1.437e-3
	CvM	5.304e-3	1.961e-3	3.972e-3	1.935e-3	2.246e-3	1.884e-3
	AD	4.687e-3	1.732e-3	3.507e-3	1.716e-3	1.979e-3	1.659e-3
	LS	5.34e-3	1.964e-3	4.004e-3	1.944e-3	2.258e-3	1.889e-3
	WLS	4.858e-3	1.788e-3	3.634e-3	1.761e-3	2.052e-3	1.712e-3
n=75	ML	2.699e-3	1.01e-3	1.972e-3	9.846e-4	1.121e-3	9.636e-4
	UMVU	2.768e-3	1.004e-3	2.048e-3	9.896e-4	1.153e-3	9.608e-4
	MPS	2.934e-3	1.023e-3	2.203e-3	1.043e-3	1.201e-3	9.888e-4
	CvM	3.451e-3	1.252e-3	2.615e-3	1.299e-3	1.435e-3	1.219e-3
	AD	3.109e-3	1.129e-3	2.339e-3	1.158e-3	1.277e-3	1.086e-3
	LS	3.465e-3	1.253e-3	2.63e-3	1.303e-3	1.439e-3	1.22e-3
	WLS	3.169e-3	1.148e-3	2.361e-3	1.165e-3	1.297e-3	1.102e-3
n=100	ML	1.951e-3	7.399e-4	1.354e-3	7.04e-4	8.727e-4	7.468e-4
	UMVU	1.987e-3	7.365e-4	1.38e-3	7.005e-4	8.922e-4	7.453e-4
	MPS	2.087e-3	7.477e-4	1.434e-3	7.1e-4	9.219e-4	7.614e-4
	CvM	2.619e-3	9.819e-4	1.815e-3	9.273e-4	1.189e-3	1.007e-3
	AD	2.318e-3	8.687e-4	1.61e-3	8.245e-4	1.052e-3	8.908e-4
	LS	2.625e-3	9.817e-4	1.819e-3	9.276e-4	1.192e-3	1.007e-3
	WLS	2.361e-3	8.829e-4	1.634e-3	8.345e-4	1.065e-3	9.022e-4

Note: in the tables, e denoted to base 10.

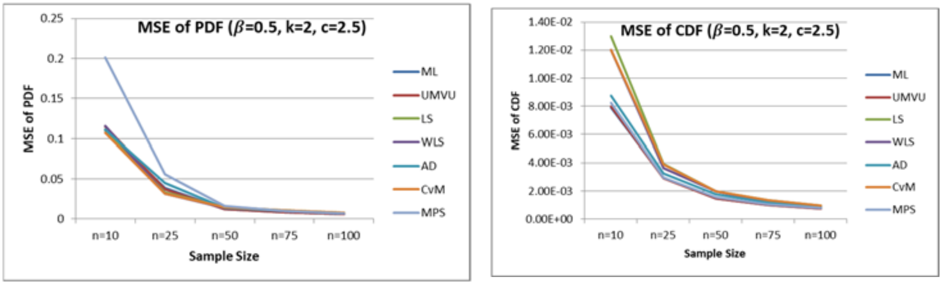

Figure1. The MSEs of the PDF and CDF for the parameter set $(\beta, k, c) = (0.5,2,2.5)$

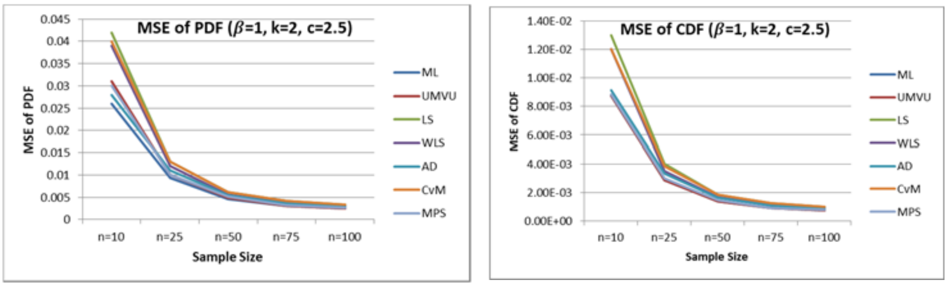

Figure 2. The MSEs of the PDF and CDF for the parameter set $(\beta, k, c) = (1,2,2.5)$

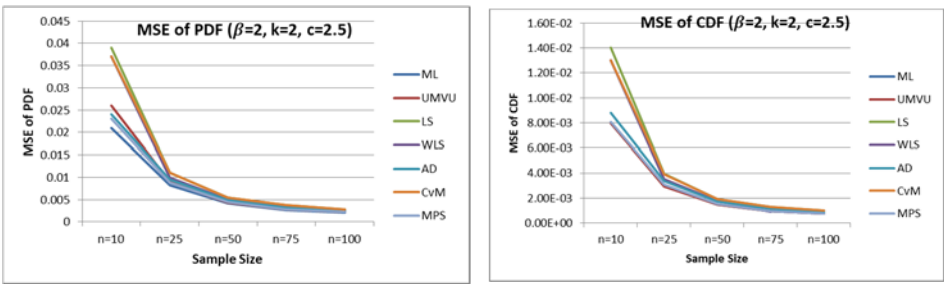

Figure 3. The MSEs of the PDF and CDF for the parameter set $(\beta, k, c) = (2,2,2.5)$

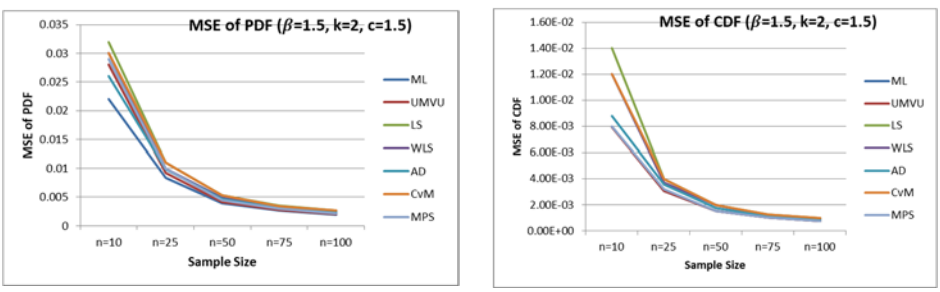

Figure 4. The MSEs of the PDF and CDF for the parameter set $(\beta, k, c) = (1.5,2,1.5)$

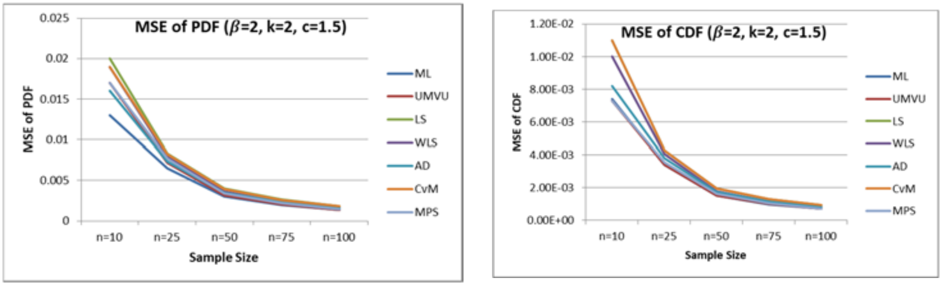

Figure 5. The MSEs of the PDF and CDF for the parameter set $(\beta, k, c) = (2,2,1.5)$

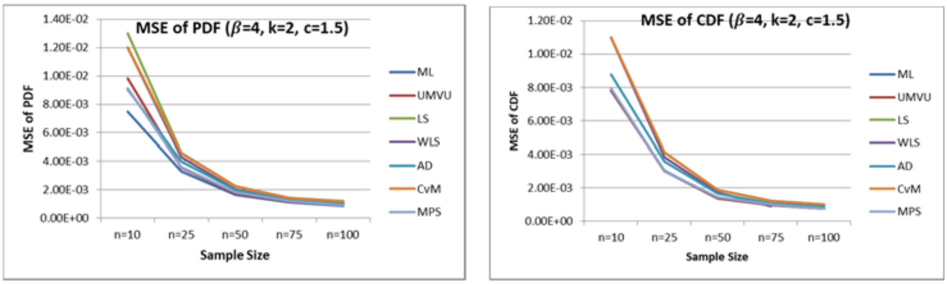

Figure 6. The MSEs of the PDF and CDF for the parameter set $(\beta, k, c) = (4,2,1.5)$

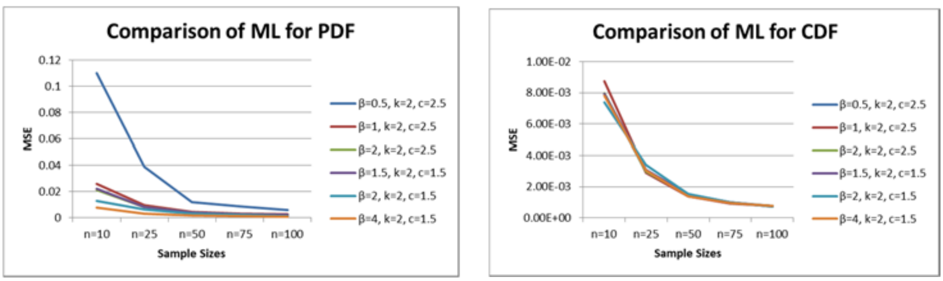

Figure 7. The MSEs of the PDF and CDF of the ML estimators for all the parameter sets

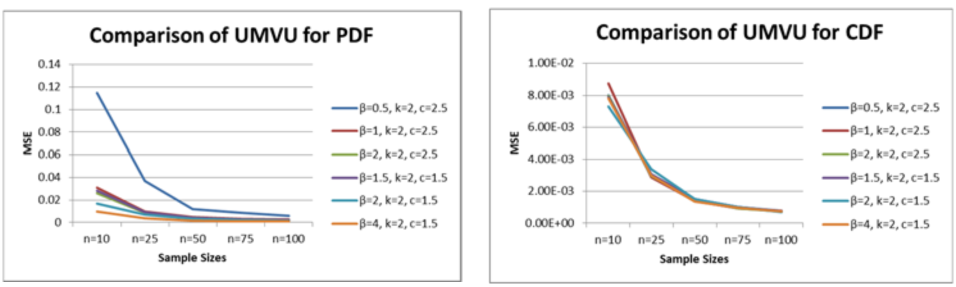

Figure 8. The MSEs of the PDF and CDF of the UMVU estimators for all parameter sets

6. Application to real data

Real data sets are considered to compare between ML, LS, WLS, CvM, AD and MPS methods. The first data consist of 31 observations that represent the Average Monthly Wind Speed (m/s) at Kolkata (from 1st March, 2009 to 31st March, 2009); these data were introduced by Bhattacharya and Bhattacharjee (2010). The second data set represents the data on service times of 63 aircraft windshield given by Murthy et al. (2004). For both data sets, all the three parameters are considered as unknown parameters. The parameters are estimated by ML, MPS, LS, WLS, CvM and AD methods. ML, MPS estimators are obtained by maximizing Equations (3) and (11), respectively, with respect to β, k and c. LS, WLS, CvM and AD estimators can be obtained by minimizing Equations (9), (10), (13) and (14), respectively, with respect to β, k and c. We compared the estimation methods by means of model selection criteria. The criteria like Akaike information criterion (AIC), Bayesian information criterion (BIC), and corrected Akaike information criterion (AICc) are considered. The model with the minimum AIC, BIC and AICc is chosen as the best model to fit the data. In addition, the PDF plot (estimated PDFs versus the empirical histogram for the data) and the CDF plot (estimated CDFs versus the empirical CDF for the data) are used in the model selection. Tables 3 and 4 give the parameter estimates and the values of the model selection for different methods.

Table 3. Estimates of the parameters and the corresponding AIC, BIC and AICc for first data

Methods	c Estimate	k Estimate	β Estimate	AIC	BIC	AICc
ML	2.139	1.631	1.333	56.304	60.605	56.946
LS	3.295	0.686	0.599	57.841	62.143	58.484
WLS	3.113	0.856	0.701	56.921	61.223	57.564
CvM	3.278	0.757	0.647	57.237	61.539	57.88
AD	3.461	0.701	0.585	57.317	61.619	57.96
MPS	1.198	1.040	0.989	79.264	83.566	79.687

Table 4. Estimates of the parameters and the corresponding AIC, BIC and AICc for second data

Methods	c Estimate	k Estimate	β Estimate	AIC	BIC	AICc
ML	1.378	1.206	1.988	235.331	241.76	235.527
LS	1.148	2.103	6.789	270.693	277.123	270.89
WLS	1.174	2.302	8.321	291.356	297.786	291.553
CvM	1.154	2.142	7.156	275.685	282.115	275.882
AD	1.381	1.242	2.456	237.674	244.103	237.87
MPS	1.402	1.1	1.734	235.739	242.168	235.939

As seen from Tables 3 and 4, the ML estimates give the smallest values compared with the other estimates. Figures 9 and 10 represent plots of the CDFs and PDFs of the EBXII distribution based on the fitted ML, LS, WLS, CvM, AD and MPS methods to the data, the figures indicate the superiority of the ML method over the other methods.

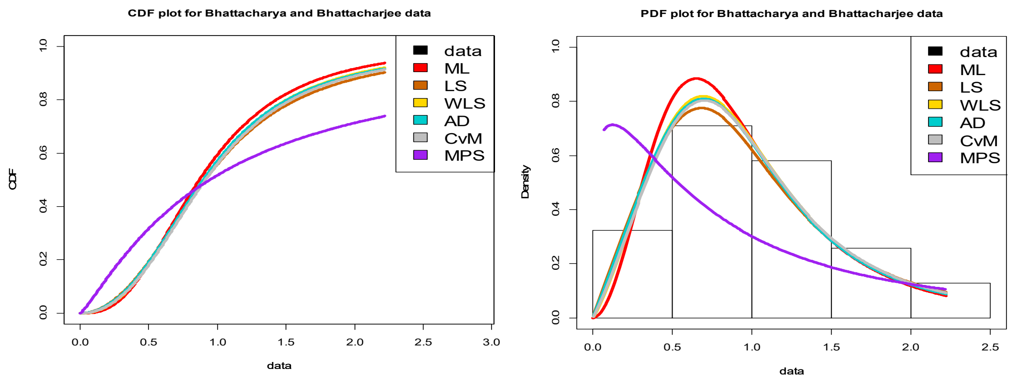

Figure 9. CDF and PDF plots for Wind Speed (m/s) data fitted by different methods of estimation

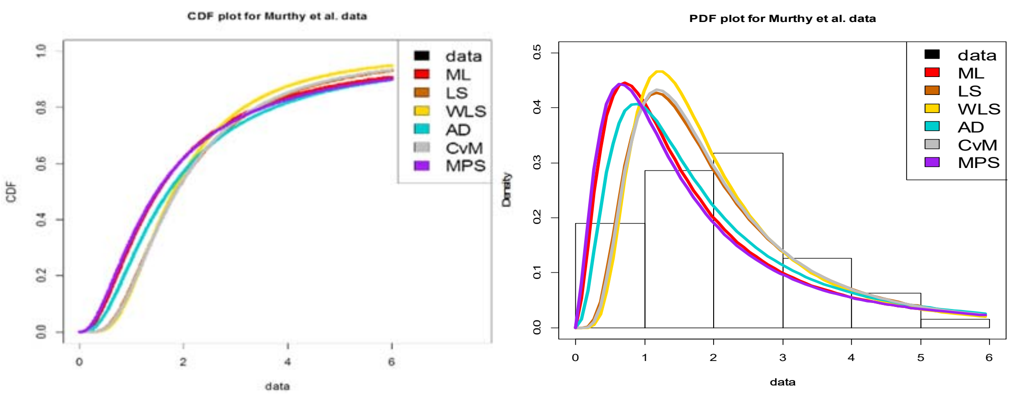

Figure 10. CDF and PDF plots for service times of 63 aircraft windshield fitted by different methods of estimation

7. Conclusion

In this paper, we consider seven different estimators of the PDF and CDF of the EBXII distribution when the shape parameters k and c are assumed to be known. Maximum likelihood estimator, uniformly minimum variance unbiased estimator, least squares estimator, weighted least squares estimator, maximum product spacing estimator, Cramér-von-Mises estimator and Anderson-Darling estimator are obtained. The MSEs of the maximum likelihood and uniformly minimum variance unbiased estimators are given in explicit forms. A simulation study is performed to compare the behaviours of the proposed estimates. A real data set is considered for illustrative purposes. The results show that the maximum likelihood and uniformly minimum variance unbiased estimates perform better than the other estimators.

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Relationships for moments of the progressively Type-II right censored order statistics from the power Lomax distribution and the associated inference

Jagdish Saran[1], Narinder Pushkarna[2], Shikha Sehgal[3]

ABSTRACT

In this paper, we establish several recurrence relations between single and product moments of progressively Type-II right censored order statistics from the power Lomax distribution. The relations enable the computation of all the single and product moments of progressively Type-II right censored order statistics for all sample sizes n and all censoring schemes $(R_1, R_2, \ldots, R_m), m \le n$, in a simple recursive manner. The maximum likelihood approach is used for the estimation of the parameters and the reliability characteristic. A Monte Carlo simulation study has been conducted to compare the performance of the estimates for different censoring schemes.

Key words: progressively Type-II right censored order statistics, single moments, product moments, recurrence relations, power Lomax distribution, maximum likelihood estimation.

Mathematics Subject Classification: 62G30; 62G05

1. Introduction

The Lomax distribution, proposed by Lomax (1954) was introduced originally for modelling business data and has been widely applied in a variety of contexts. In lifetime models, it is considered as an important model and belongs to the family of decreasing failure rate. Bryson (1974) found that this distribution can be used as heavy tailed alternative to the exponential, Weibull and gamma distributions.

Many authors constructed generalizations of the Lomax distribution. For example, Ghitany et al. (2007) introduced the Marshall-Olkin extended Lomax distribution, Abdul-Moniem and Abdel-Hameed (2012) introduced the exponentiated Lomax distribution, Tahir et al. (2015) introduced the Weibull Lomax distribution, Al-Zahrani and Sagor (2014) studied the Poisson Lomax distribution. Recently, Tahir et al. (2016)

[1] Department of Statistics, University of Delhi, India. E-mail: jagdish_saran52@yahoo.co.in.
[2] Department of Statistics, University of Delhi, India. E-mail: narinderpushkarna@ramjas.du.ac.in.
[3] Corresponding author, Department of Statistics, University of Delhi, India. E-mail: shikhastats@gmail.com. ORCID: https://orcid.org/0000-0002-2333-9264.

and Afify et al. (2016) introduced the Gumbel-Lomax distribution and the Transmuted Weibull Lomax distribution, respectively, and studied their mathematical and statistical properties.

A new extension of the Lomax distribution was proposed by Rady et al. (2016) as three parameter power Lomax distribution, by considering the power transformation $X = T^{1/\beta}$, where the random variable $(r.v.)$ T follows the Lomax distribution with parameters α and λ . Then the distribution of $r.v.$ X with three parameters α, β and λ is referred to as "power Lomax distribution", where α and β are the shape parameters and λ is the scale parameter of the distribution.

The probability density function (p.d.f.) of $r.v.$ X following the power Lomax distribution is given as

$$f(x) = \frac{\alpha\beta}{\lambda} x^{\beta-1} \left(1 + \frac{x^\beta}{\lambda}\right)^{-(\alpha+1)}, \quad x > 0, \alpha, \beta, \lambda > 0. \tag{1.1}$$

The corresponding cumulative distribution function (c.d.f.) is given by

$$F(x) = 1 - \left(1 + \frac{x^\beta}{\lambda}\right)^{-\alpha}, \quad x > 0, \alpha, \beta, \lambda > 0. \tag{1.2}$$

The reliability (survival) function $R(x)$ of the power Lomax distribution is given as

$$R(x) = \left(1 + \frac{x^\beta}{\lambda}\right)^{-\alpha}, \quad x > 0, \alpha, \beta, \lambda > 0, \tag{1.3}$$

and the failure rate function (hazard function) of the power Lomax distribution is given by

$$h(x) = \frac{f(x)}{R(x)} = \frac{\alpha\beta x^{\beta-1}}{\lambda + x^\beta}, \quad x > 0, \alpha, \beta, \lambda > 0. \tag{1.4}$$

From Eqs. (1.1) and (1.2), one can observe that the characterizing differential equation for the power Lomax distribution is given as

$$\alpha\beta(1 - F(x)) = (x + \lambda x^{1-\beta}) f(x). \tag{1.5}$$

Note: For $\beta = 1$ in Eq. (1.1), the p.d.f. reduces to that of the Lomax distribution.

2. Progressively Type-II right censored order statistics

The progressive Type-II right censoring scheme is quite useful in reliability and life-testing experiments because it allows the experimenter for items to be removed before the termination of the experiment to save time and cost. The progressive censoring scheme and associated inferential procedures have been discussed by several authors including Aggarwala and Balakrishnan (1996, 1998), Balakrishnan and Aggarwala (2000), Cohen (1963, 1976, 1991), Cohen and Whitten (1988), Balakrishnan and Sandhu (1995), Athar et al. (2014), Saran and Pushkarna (2001, 2014), Saran and Pande (2012), Pushkarna et al. (2015) and Saran et al. (2018). The progressive censoring scheme can be described as follows:

Let $X_1, X_2, \dots, X_n$ be a sequence of independent and identically distributed (i.i.d.) random variables representing failure times of n identical units placed on a life-test. Under the progressively Type-II right censoring scheme, at the time of i^{th} failure ($i = 1,2,\dots,m,$ where $m \leq n$), R_i surviving items are removed at random from the experiment, where $R_1, R_2, \dots, R_m$ are fixed integers. In other words, if a censoring scheme $(R_1, R_2, \dots, R_m)$ is fixed such that immediately following the first failure, R_1 surviving items are removed from the experiment at random; immediately following the first failure after that point; i.e. after second observed failure, R_2 surviving items are removed from the experiment at random; this process continues until, at the m^{th} observed failure, R_m items are removed from the experiment.

Thus, in this type of sampling, m failures are observed and

$\sum_{i=1}^{m} R_i$ items are progressively censored so that $n = m + \sum_{i=1}^{m} R_i$. The withdrawal of items may be seen as a model describing drop-outs of units due to failures, which have causes other than the specific one under study. In this sense, progressive censoring schemes are applied in clinical trials as well. The drop-outs of patients may be caused, e.g. by personal or ethical decisions, and they are regarded as random withdrawals.

Let $X_{1:m:n}^{(R_1,R_2,\dots,R_m)} < X_{2:m:n}^{(R_1,R_2,\dots,R_m)} < \dots < X_{m:m:n}^{(R_1,R_2,\dots,R_m)}$, be the m ordered observed failure times in a sample of size n from the Power Lomax distribution as defined by (1.1), under the progressively Type-II right censoring scheme $(R_1, R_2, \dots, R_m)$, $m \leq n$.

Then, the joint p.d.f. of $X_{1:m:n}^{(R_1,R_2,\dots,R_m)}, X_{2:m:n}^{(R_1,R_2,\dots,R_m)}, \dots, X_{m:m:n}^{(R_1,R_2,\dots,R_m)}$ is given by (Balakrishnan and Sandhu (1995))

$$f_{1,2,\dots,m:m:n}(x_1, x_2, \dots, x_m) = A(n, m-1) \prod_{i=1}^{m} f(x_i)[1 - F(x_i)]^{R_i},$$

$$0 < x_1 < x_2 < \dots < x_m < \infty, \tag{2.1}$$

where

$$A(n, m-1) = n(n - R_1 - 1)(n - R_1 - R_2 - 2) \dots (n - R_1 - R_2 - \dots - R_{m-1} - m + 1), \tag{2.2}$$

$f(x)$ and $F(x)$ are given by (1.1) and (1.2), respectively. Here, note that all the factors in $A(n,m-1)$ are positive integers. Also, it may be observed that the different factors in
$A(n,m-1)$ represent the number of units still on test immediately preceding the $1^{st}, 2^{nd}, \dots, m^{th}$ observed failures, respectively.

Similarly, for convenience in notation, let us define for $q = 0,1,\dots,(p-1)$,

$$A(p,q) = p(p-R_1-1)(p-R_1-R_2-2)\dots\left(p-R_1-R_2-\dots-R_q-q\right),$$

with all the factors being positive integers.

We shall denote the k^{th} single moment of the i^{th} progressively Type-II right censored order statistics, from (2.1), as

$$\mu_{i:m:n}^{(R_1,R_2,\dots,R_m)(k)} = E\left[X_{i:m:n}^{(R_1,R_2,\dots,R_m)}\right]^k$$

$$= A(n,m-1)\underset{0<x_1<\dots<x_m<\infty}{\int\int\dots\int} x_i^k \prod_{t=1}^{m} f(x_t)[1-F(x_t)]^{R_t}\,dx_t\,,\ 1\le i\le m\le n\,, k\ge 0, \tag{2.3}$$

and the $(r,s)^{th}$ product moment of the $i^{th}$and j^{th} progressively Type-II right censored order statistics from (2.1), as

$$\mu_{i,j:m:n}^{(R_1,R_2,\dots,R_m)(r,s)} = E\left[\left\{X_{i:m:n}^{(R_1,R_2,\dots,R_m)}\right\}^r\left\{X_{j:m:n}^{(R_1,R_2,\dots,R_m)}\right\}^s\right]$$

$$= A(n,m-1)\underset{0<x_1<\dots<x_m<\infty}{\int\int\dots\int} x_i^r x_j^s \prod_{t=1}^{m} f(x_t)[1-F(x_t)]^{R_t}\,dx_t;$$

$$1\le i<j\le m\le n, r,s\ \ge 0, \tag{2.4}$$

where $A(n,m-1)$ is defined before.

In Sections 3 and 4, utilizing the characterizing differential Eq. (1.5), we have derived recurrence relations for the single and the product moments of progressively Type-II right censored order statistics from the power Lomax distribution. These relations along with the recursive algorithm presented in Section 5 would enable one to compute all the single and product moments of progressively Type-II right censored order statistics for all sample sizes n and all censoring schemes $(R_1, R_2, \dots, R_m)$, $m \le n$, in a simple recursive manner. In Section 6, for the estimation of the parameters and the reliability characteristics, maximum likelihood approach is used. In Section 7, Monte Carlo simulation study is conducted to compare the performance of the estimates for different censoring schemes.

3. Recurrence relations for single moments

In this section, we shall exploit the relation (1.5) to establish recurrence relations for the single moments of progressively Type-II right censored order statistics from the power Lomax distribution. The results are presented in the form of the following theorems.

Theorem 3.1: For $2 \leq m \leq n$ and for $k \geq 0$,

$$\mu_{1:m:n}^{(R_1,R_2,\dots,R_m)(k+\beta)}\left[1-\frac{\alpha\beta}{(k+\beta)}(R_1+1)\right] = \frac{\alpha\beta}{(k+\beta)}(n-R_1-1)\mu_{1:m-1:n}^{(R_1+R_2+1,R_3,\dots,R_m)(k+\beta)} - \lambda\mu_{1:m:n}^{(R_1,R_2,\dots,R_m)(k)}, \quad (3.1)$$

and for $m=1, n=1,2,\dots.$ and $k \geq 0$,

$$\mu_{1:1:n}^{(n-1)(k+\beta)}\left(\frac{n\alpha\beta}{k+\beta}-1\right) = \left[\lambda\mu_{1:1:n}^{(n-1)(k)}\right]. \quad (3.2)$$

Proof: From (2.3), we have

$$\mu_{1:m:n}^{(R_1,R_2,\dots,R_m)(k+\beta)} + \lambda\,\mu_{1:m:n}^{(R_1,R_2,\dots,R_m)(k)}$$

$$= A(n,m-1)\underset{0<x_2<x_3\dots<x_m<\infty}{\int\int\dots\int}\left\{\int_0^{x_2}\left(x_1^{k+\beta}+\lambda x_1^k\right)f(x_1)[1-F(x_1)]^{R_1}dx_1\right\}$$

$$\times f(x_2)[1-F(x_2)]^{R_2}\dots f(x_m)[1-F(x_m)]^{R_m}dx_2\,dx_3\dots dx_m$$

$$= A(n,m-1)\underset{0<x_2<x_3<\dots<x_m<\infty}{\int\int\dots\int} I(x_2)\prod_{t=2}^{m} f(x_t)[1-F(x_t)]^{R_t}\,dx_t\,, \quad (3.3)$$

where

$$I(x_2) = \int_0^{x_2}\left(x_1^{k+\beta}+\lambda x_1^k\right)f(x_1)[1-F(x_1)]^{R_1}dx_1$$

$$= \int_0^{x_2} x_1^{k+\beta-1}\left(x_1+\lambda x_1^{1-\beta}\right)f(x_1)[1-F(x_1)]^{R_1}dx_1.$$

Making use of the relation (1.5), we have

$$I(x_2) = \alpha\beta\int_0^{x_2} x_1^{k+\beta-1}[1-F(x_1)]^{R_1+1}dx_1.$$

Upon integrating by parts by treating $x_1^{k+\beta-1}$ for integration and $[1-F(x_1)]^{R_1+1}$ for differentiation we have

$$I(x_2) = \frac{\alpha\beta}{(k+\beta)}\left[x_2^{k+\beta}[1-F(x_2)]^{R_1+1} + (R_1+1)\int_0^{x_2} x_1^{k+\beta}[1-F(x_1)]^{R_1}f(x_1)dx_1\right]. \quad (3.4)$$

Substituting the resultant expression of $I(x_2)$ from (3.4) in (3.3), we get

$$\mu_{1:m:n}^{(R_1,R_2,\dots,R_m)^{(k+\beta)}} + \lambda\, \mu_{1:m:n}^{(R_1,R_2,\dots,R_m)^{(k)}} = \frac{\alpha\beta}{(k+\beta)}\left[(n-R_1-1)\mu_{1:m-1:n}^{(R_1+R_2+1,R_3,\dots,R_m)^{(k+\beta)}} + (R_1+1)\mu_{1:m:n}^{(R_1,R_2,R_3,\dots,R_m)^{(k+\beta)}}\right],$$

which upon rearrangement yields the relation in (3.1).

Next, for m=1, n=1,2,… and k≥0,

$$\begin{aligned}\mu_{1:1:n}^{(R_1)^{(k+\beta)}} + \lambda\, \mu_{1:1:n}^{(R_1)^{(k)}} &= A(n,0)\int_0^\infty \left(x_1^{k+\beta}+\lambda x_1^{k}\right) f(x_1)[1-F(x_1)]^{R_1}dx_1\\ &= n\int_0^\infty x_1^{k+\beta-1}\left(x_1+\lambda x_1^{1-\beta}\right)f(x_1)[1-F(x_1)]^{R_1}dx_1\\ &= n\alpha\beta\int_0^\infty x_1^{k+\beta-1}[1-F(x_1)]^{R_1+1}dx_1\\ &= n\frac{\alpha\beta}{(k+\beta)}\mu_{1:1:n}^{(n-1)^{(k+\beta)}},\end{aligned}$$

which, upon rearrangements, yields the relation in (3.2).

Remark 3.1: It may be noted that the first progressively Type-II right censored order statistic
$X_{1:m:n}^{(R_1,R_2,\dots,R_m)}$ is the same as the first usual order statistic from a sample of size n, regardless of the censoring scheme employed. This is because no censoring has taken place before this time.

Theorem 3.2: For $2 \le i \le m-1, m \le n$ and $k \ge 0$,

$$\begin{aligned}\mu_{i:m:n}^{(R_1,R_2,\dots,R_m)^{(k+\beta)}}\left[1-\frac{\alpha\beta(R_i+1)}{(k+\beta)}\right] &= \frac{\alpha\beta}{(k+\beta)}\Big[(n-R_1-R_2-\dots-R_i\\ &\quad -i)\mu_{i:m-1:n}^{(R_1,R_2,\dots,R_{i-1},R_i+R_{i+1}+1,R_{i+2},\dots,R_m)^{(k+\beta)}}\\ &\quad -(n-R_1-R_2-\dots-R_{i-1}-i\\ &\quad +1)\mu_{i-1:m-1:n}^{(R_1,R_2,\dots,R_{i-2},R_{i-1}+R_i+1,R_{i+1},\dots,R_m)^{(k+\beta)}}\Big] - \lambda\mu_{i:m:n}^{(R_1,R_2,\dots,R_m)^{(k)}}.\end{aligned} \tag{3.5}$$

Proof: From (2.3), we have

$$\mu_{i:m:n}^{(R_1,R_2,\dots,R_m)^{(k+\beta)}} + \lambda\, \mu_{i:m:n}^{(R_1,R_2,\dots,R_m)^{(k)}} = A(n,m-1) \underset{0<x_1<\dots<x_{i-1}<x_{i+1}<\dots<x_m}{\int \dots \int \dots \int} J(x_{i-1},x_{i+1}) \prod_{\substack{t=1\\ t\neq i}}^{m} f(x_t)[1-F(x_t)]^{R_t}\, dx_t, \tag{3.6}$$

where

$$J(x_{i-1},x_{i+1}) = \int_{x_{i-1}}^{x_{i+1}} x_i^{k+\beta-1}\left(x_i+\lambda x_i^{1-\beta}\right) f(x_i)[1-F(x_i)]^{R_i}dx_i\,. \tag{3.7}$$

Making use of the relation in (1.5), we have

$$J(x_{i-1},x_{i+1}) = \alpha\beta\int_{x_{i-1}}^{x_{i+1}} x_i^{k+\beta-1}[1-F(x_i)]^{R_i+1}dx_i\,. \tag{3.8}$$

Integrating by parts by treating $x_i^{k+\beta-1}$ for integration and $[1-F(x_i)]^{R_i+1}$ for differentiation, we have

$$J(x_{i-1},x_{i+1}) = \frac{\alpha\beta}{(k+\beta)}\Big[x_{i+1}^{k+\beta}[1-F(x_{i+1})]^{R_i+1} - x_{i-1}^{k+\beta}[1-F(x_{i-1})]^{R_i+1} + (R_i+1)\int_{x_{i-1}}^{x_{i+1}} x_i^{k+\beta}[1-F(x_i)]^{R_i}f(x_i)dx_i\Big]. \tag{3.9}$$

Substituting the resultant expression of $J(x_{i-1},x_{i+1})$ from (3.9) in (3.6) and simplifying, leads to (3.5).

Likewise, the following recurrence relation can also be established.

Theorem 3.3: For $m \le n$ and $k \ge 0$,

$$\mu_{m:m:n}^{(R_1,R_2,\dots,R_m)^{(k+\beta)}}\left[\frac{\alpha\beta(R_m+1)}{(k+\beta)}-1\right] = \frac{\alpha\beta}{(k+\beta)}\left[(n-R_1-R_2-\dots-R_{m-1}-m+1)\times\mu_{m-1:m-1:n}^{(R_1,R_2,\dots,R_{m-2},R_{m-1}+R_m+1)^{(k+\beta)}}\right] + \lambda\mu_{m:m:n}^{(R_1,R_2,\dots,R_m)^{(k)}}. \tag{3.10}$$

4. Recurrence relations for product moments

In this section, we shall exploit the relation (1.5) to establish recurrence relations for the product moments, defined in Eq. (2.4), of progressively Type-II right censored order statistics from the power Lomax distribution. The results are presented in the form of the following theorems.

Theorem 4.1: For $1 \le i < j < m, m \le n,$ and $r,s \ge 0$,

$$\mu_{i,j:m:n}^{(R_1,R_2,\dots,R_m)^{(r,s+\beta)}}\left[1-\frac{\alpha\beta(R_j+1)}{(s+\beta)}\right] = \frac{\alpha\beta}{(s+\beta)}\Big[(n-R_1-R_2-\dots-R_j-j)\mu_{i,j:m-1:n}^{(R_1,R_2,\dots,R_{j-1},R_j+R_{j+1}+1,R_{j+2},\dots,R_m)^{(r,s+\beta)}} - \big(n-R_1-R_2-\dots-R_{j-1}-(j-1)\big) \times \mu_{i,j-1:m-1:n}^{(R_1,R_2,\dots,R_{j-2},R_{j-1}+R_j+1,R_{j+1},\dots,R_m)^{(r,s+\beta)}}\Big] - \lambda\mu_{i,j:m:n}^{(R_1,R_2,\dots,R_m)^{(r,s)}}. \tag{4.1}$$

Proof: From (2.4), we have

$$\mu_{i,j:m:n}^{(R_1,R_2,\dots,R_m)^{(r,s+\beta)}} + \lambda\, \mu_{i,j:m:n}^{(R_1,R_2,\dots,R_m)^{(r,s)}}$$
$$= A(n,m-1) \underset{0<x_1<\cdots<x_{j-1}<x_{j+1}<\cdots<x_m<\infty}{\int \dots \int \dots \int} x_i^r J(x_{j-1},x_{j+1}) \prod_{\substack{t=1\\t\neq j}}^{m} f(x_t)[1-F(x_t)]^{R_t}\, dx_t, \tag{4.2}$$

where

$$J(x_{j-1},x_{j+1}) = \int_{x_{j-1}}^{x_{j+1}} x_j^{s+\beta-1}\left(x_j + \lambda x_j^{1-\beta}\right) f(x_j)\left[1-F(x_j)\right]^{R_j} dx_j\,. \tag{4.3}$$

Using (3.9), we get

$$J(x_{j-1},x_{j+1}) = \frac{\alpha\beta}{(s+\beta)}\begin{bmatrix} x_{j+1}^{s+\beta}\left[1-F(x_{j+1})\right]^{R_j+1} - x_{j-1}^{s+\beta}\left[1-F(x_{j-1})\right]^{R_j+1} \\ +(R_j+1)\displaystyle\int_{x_{j-1}}^{x_{j+1}} x_j^{s+\beta}\left[1-F(x_j)\right]^{R_j} f(x_j)\,dx_j \end{bmatrix}. \tag{4.4}$$

Substituting the resultant expression for $J(x_{j-1},x_{j+1})$ from (4.4) in (4.2) and simplifying, on using (2.4), we get

$$\mu_{i,j:m:n}^{(R_1,R_2,\dots,R_m)^{(r,s+\beta)}} + \lambda\, \mu_{i,j:m:n}^{(R_1,R_2,\dots,R_m)^{(r,s)}}$$
$$= \frac{\alpha\beta A(n,m-1)}{(s+\beta)} \underset{0<x_1<\cdots<x_{j-1}<x_{j+1}<\cdots<x_m<\infty}{\int \dots \int \dots \int} x_i^r x_{j+1}^{s+\beta}\left[1-F(x_{j+1})\right]^{R_j+1} \prod_{\substack{t=1\\t\neq j}}^{m} f(x_t)[1-F(x_t)]^{R_t}\, dx_t$$
$$-\frac{\alpha\beta A(n,m-1)}{(s+\beta)} \underset{0<x_1<\cdots<x_{j-1}<x_{j+1}<\cdots<x_m<\infty}{\int \dots \int \dots \int} x_i^r x_{j-1}^{s+\beta}\left[1-F(x_{j-1})\right]^{R_j+1} \prod_{\substack{t=1\\t\neq j}}^{m} f(x_t)[1-F(x_t)]^{R_t}\, dx_t$$
$$+\frac{\alpha\beta(R_j+1)}{(s+\beta)} \mu_{i,j:m:n}^{(R_1,R_2,\dots,R_m)^{(r,s+\beta)}}$$
$$= \frac{\alpha\beta A(n,m-1)}{(s+\beta)}$$
$$\times \underset{0<x_1<\cdots<x_{j-1}<x_{j+1}<\cdots<x_m<\infty}{\int \dots \int \dots \int} x_i^r x_{j+1}^{s+\beta}\left[1-F(x_{j+1})\right]^{R_j+R_{j+1}+1} f(x_{j+1})\,dx_{j+1} \prod_{\substack{t=1\\t\neq j,j+1}}^{m} f(x_t)[1-F(x_t)]^{R_t}\, dx_t$$

$$-\frac{\alpha\beta A(n,m-1)}{(s+\beta)}$$

$$\times \int \dots \int \dots \int x_i^r x_{j-1}^{s+\beta}\left[1-F(x_{j-1})\right]^{R_{j-1}+R_j+1} f(x_{j-1})dx_{j+1} \prod_{\substack{t=1\\ t\neq j-1,j}}^{m} f(x_t)[1-F(x_t)]^{R_t}\, dx_t$$

$$0< x_1 < \dots < x_{j-1} < x_{j+1} < \dots < x_m < \infty$$

$$+\frac{\alpha\beta\left(R_j+1\right)}{(s+\beta)}\mu_{i,j:m:n}^{(R_1,R_2,\dots,R_m)^{(r,s+\beta)}}$$

$$=\frac{\alpha\beta}{(s+\beta)}\Bigg[\left(n-R_1-R_2-\dots-R_j-j\right)\mu_{i,j:m-1:n}^{\left(R_1,R_2,\dots,R_{j-1},R_j+R_{j+1}+1,R_{j+2},\dots,R_m\right)^{(r,s+\beta)}}$$

$$-\Big(n-R_1-R_2-\dots-R_{j-1}$$

$$-(j-1)\Big)\,\mu_{i,j-1:m-1:n}^{\left(R_1,R_2,\dots,R_{j-2},R_{j-1}+R_j+1,R_{j+1},\dots,R_m\right)^{(r,s+\beta)}}$$

$$+\left(R_j+1\right)\mu_{i,j:m:n}^{(R_1,R_2,\dots,R_m)^{(r,s+\beta)}}\Bigg],$$

which on rearranging the terms leads to (4.1).

Theorem 4.2: For $1 \le i \le m-1$ and $m \le n$ and $r,s \ge 0$,

$$\mu_{i,m:m:n}^{(R_1,R_2,\dots,R_m)^{(r,s+\beta)}}\left[\frac{\alpha\beta(R_m+1)}{(s+\beta)}-1\right]$$

$$=\frac{\alpha\beta}{(s+\beta)}\left[\left(n-R_1-R_2-\dots-R_{m-1}-(m-1)\right)\mu_{i,m-1:m-1:n}^{(R_1,R_2,\dots,R_{m-2},R_{m-1}+R_m+1)^{(r,s+\beta)}}\right]$$

$$+\lambda\mu_{i,m:m:n}^{(R_1,R_2,\dots,R_m)^{(r,s)}}. \tag{4.5}$$

Proof: The relation in (4.5) may be proved by following exactly the same steps as those used in proving Theorem 4.1.

Remark 4.1: It may be noted that Theorem 4.1 holds even for $j=i+1$, without altering the proof, provided we realize that $\mu_{i,i:m:n}^{(R_1,R_2,\dots,R_m)^{(r,s)}} = \mu_{i:m:n}^{(R_1,R_2,\dots,R_m)^{(r+s)}}$.

Remark 4.2: For the special case $R_1 = R_2 = \dots = R_m = 0$ so that $m=n$ in which case the progressively censored order statistics become the usual order statistics $X_{1:n}, X_{2:n}, \dots, X_{n:n}$, whose single moments are denoted by $\mu_{i:n}^{(k)}$ for $1 \le i \le n$ and product moments are denoted by $\mu_{i,j:n}^{(r,s)}$ for $1 \le i < j \le n$, the recurrence relations established in Sections 3 and 4 reduce to that of usual order statistics from Power Lomax distribution.

5. Recursive computational algorithm

Thomas and Wilson (1972) gave a computational method for obtaining single and product moments of progressively Type-II right censored order statistics from an arbitrary continuous distribution through a mixture form that expresses them in terms of those of the usual order statistics from a sample of size n. Utilizing the knowledge of recurrence relations obtained in Sections 3 and 4 in a systematic manner, along with the mixture formula for missing moments, one can evaluate the moments of progressively Type-II right censored order statistics from the power Lomax distribution for all sample sizes and all censoring schemes $(R_1, R_2, \dots, R_m)$ in a simple recursive way. The same has been demonstrated in the Sub- Sections 5.2 and 5.3.
First, we will derive the exact explicit forms for the single and product moments of order statistics from a given random sample $X_1, X_2, \dots, X_n$ from the power Lomax distribution.

5.1. Exact expressions for single and product moments of order statistics from power Lomax distribution

Let $X_1, X_2, \dots, X_n$ be a random sample of size n from the power Lomax distribution defined in (1.1) and let $X_{1:n} \le X_{2:n} \le \cdots \le X_{n:n}$ be the corresponding order statistics. Then the probability density function (p.d.f.) of $X_{i:n}$ $(1 \le i \le n)$ is given by:

$$f_{i:n}(x) = C_{i:n}[F(x)]^{i-1}[1-F(x)]^{n-i}f(x)\,, \quad 0 < x < \infty, \tag{5.1}$$

and the joint density function of $X_{i:n}$ and $X_{j:n}(1 \le i < j \le n)$ is given by

$$f_{i,j:n}(x,y) = C_{i,j:n}[F(x)]^{i-1}[F(y)-F(x)]^{j-i-1}\,[1-F(y)]^{n-j}f(y)f(x), \quad 0 < x < y < \infty, \tag{5.2}$$

where $f(x)$ and $F(x)$ are given by (1.1) and (1.2), respectively, and

$$C_{i:n} = \frac{n!}{(i-1)!\,(n-i)!} \text{ and } C_{i,j:n} = \frac{n!}{(i-1)!\,(j-i-1)!\,(n-j)!}.$$

Then, the single moments of order statistics $X_{i:n}(1 \le i \le n)$ are given by

$$\mu_{i:n}^{(k)} = E\left(X_{i:n}^k\right) = \int_0^{\infty} x^k f_{i:n}(x)dx\,, \quad k = 1,2,\dots\,. \tag{5.3}$$

Similarly, the product moments of $X_{i:n}$ and $X_{j:n}(1 \le i < j \le n)$ are given by

$$\mu_{i,j:n}^{(r,s)} = E\left(X_{i:n}^r X_{j:n}^s\right) = \int_0^{\infty}\int_x^{\infty} x^r y^s f_{i,j:n}(x,y)dydx\,, \quad r,s = 1,2,\dots\,. \tag{5.4}$$

Theorem 5.1: For the power Lomax distribution as given in (1.1) and for $1 \leq i \leq n$, and $k = 1,2,3, \dots$, we have

$$\mu_{i:n}^{(k)} = E\left(X_{i:n}^{k}\right) = \alpha\lambda^{k/\beta} C_{i:n} \sum_{l=0}^{i-1} \binom{i-1}{l} (-1)^{l} \frac{\Gamma\left(1+\frac{k}{\beta}\right)\Gamma\left(\alpha+\alpha n+l\alpha-\frac{k}{\beta}-\alpha i\right)}{\Gamma(\alpha+\alpha n+l\alpha+1-\alpha i)}, \tag{5.5}$$

exists for the choice of α and β such that $\alpha > \frac{k}{n\beta}$.

Proof: Using (5.1) and binomial expansion of $[1-(1-F(x))]^{i-1}$, Eq. (5.3) can be rewritten as

$$\mu_{i:n}^{(k)} = 1C_{i:n} \sum_{l=0}^{i-1} \binom{i-1}{l} (-1)^{l} \int_{0}^{\infty} x^{k}[1-F(x)]^{n-i+l} f(x)dx\,.$$

Substituting the values of $f(x)$ and $F(x)$ as given by (1.1) and (1.2), in the above equation we get

$$\mu_{i:n}^{(k)} = C_{i:n} \sum_{l=0}^{i-1} \binom{i-1}{l} (-1)^{l} \frac{\alpha\beta}{\lambda} \int_{0}^{\infty} x^{k+\beta-1} \left(1+\frac{x^{\beta}}{\lambda}\right)^{-(\alpha+\alpha n+l\alpha+1-\alpha i)} dx\,.$$

Simplifying the above integral we get the desired result as given by Eq.(5.5).

Theorem 5.2: For the power Lomax distribution as given in (1.1) and for $1 \leq i < j \leq n$, and $r, s = 1,2,3, \dots$, and $\frac{s}{\beta} \in Z^{+}$, we have

$$\mu_{i,j:n}^{(r,s)} = \alpha^{2}\lambda^{\frac{(r+s)}{\beta}} C_{i,j:n} \sum_{t=0}^{i-1} \sum_{m=0}^{j-i-1} \sum_{u=0}^{\frac{s}{\beta}} \binom{i-1}{t} \binom{j-i-1}{m} \binom{s/\beta}{u} (-1)^{t+m+u}$$
$$\times \frac{1}{c} \frac{\Gamma\left(1+\frac{r}{\beta}\right)\Gamma\left(\alpha(a+1)+c-\frac{r}{\beta}\right)}{\Gamma(\alpha(a+1)+c+1)}, \tag{5.6}$$

where

$$a = t+j-i-m-1,\ b = m+n-j \text{ and } c = \alpha(1+b)+u-\frac{s}{\beta}.$$

Proof: Using (5.2) and binomial expansion of $[F(x)]^{i-1}$ in the powers of $[1-F(x)]$, and binomial expansion of $[F(y)-F(x)]^{j-i-1}$ in the powers of $[1-F(x)]$ and $[1-F(y)]$, Eq. (5.4) can be rewritten as

$$\mu_{i,j:n}^{(r,s)} = C_{i,j:n} \sum_{t=0}^{i-1} \sum_{m=0}^{j-i-1} \binom{i-1}{t} \binom{j-i-1}{m} (-1)^{t+m}$$
$$\times \int_{0}^{\infty} \int_{x}^{\infty} x^{r} y^{s} [1-F(x)]^{t+j-i-m-1} [1-F(y)]^{m+n-j} f(x) f(y) dy dx$$
$$= C_{i,j:n} \sum_{t=0}^{i-1} \sum_{m=0}^{j-i-1} \binom{i-1}{t} \binom{j-i-1}{m} (-1)^{t+m} \int_{0}^{\infty} x^{r} [1-F(x)]^{a} f(x) I_{1}(x) dx, \tag{5.7}$$

where $a = t + j - i - m - 1, b = m + n - j$ and

$$I_1(x) = \int_x^\infty y^s\,[1 - F(y)]^b f(y)dy. \tag{5.8}$$

Substituting the values of $f(y)$ and $F(y)$ from (1.1) and (1.2),respectively in Eq.(5.8) and simplifying we get

$$I_1(x) = \alpha\lambda^{s/\beta} \sum_{u=0}^{\frac{s}{\beta}} \binom{s/\beta}{u} (-1)^u \frac{\left(1+\frac{x^\beta}{\lambda}\right)^{-c}}{c} \tag{5.9}$$

where $c = \alpha(1 + b) + u - \frac{s}{\beta}$.

Substituting the value of $I_1(x)$ from Eq. (5.9) in (5.7) and simplifying the expression by putting the values of $f(x)$ and $F(x)$ as given by (1.1) and (1.2), we get the desired result (5.6).

5.2. Recursive algorithm for single moments

Case I: When $n = 1$, then $m = 1$

In this case, we have only one progressive censoring scheme $R_1 = 0$. Thus, from Eq. (2.3) and using Eq. (5.5), we have for $\alpha > \frac{k}{\beta}$ and $k = 1,2, \dots$,

$$E\left(X_{1:1:1}^{(0)}\right)^k = \mu_{1:1:1}^{(0)^{(k)}} = \mu_{1:1}^{(k)} = E\left(X^k\right) = \alpha\lambda^{k/\beta} \frac{\Gamma(1+\frac{k}{\beta})\Gamma(\alpha-\frac{k}{\beta})}{\Gamma(\alpha+1)}. \tag{5.10}$$

Using (5.10), $\mu_{1:1:1}^{(0)^{(k)}}$ $\forall\ k = 1,2, \dots.$,can be calculated.

Alternatively, these moments can also be obtained by using the recurrence relation given in

Eq. (3.2) on putting n=1, i.e. by using the relation

$$\mu_{1:1:1}^{(0)^{(k+\beta)}} = \mu_{1:1}^{(k+\beta)} = \frac{\lambda(k+\beta)}{\beta(\alpha-1)-k} \mu_{1:1}^{(k)}.$$

Case II: When $n = 2$, then $m = 1$ or 2

Subcase (i): $m = 1$

We have only one progressive censoring scheme $R_1 = 1$, and in this case we have from Eq. (3.2), on putting $n = 2$,

$$\mu_{1:1:2}^{(1)^{(k+\beta)}} \left(\frac{2\alpha\beta}{k+\beta} - 1\right) = \lambda\, \mu_{1:1:2}^{(1)^{(k)}}, \tag{5.11}$$

where

$$\mu_{1:1:2}^{(1)^{(k)}} = \mu_{1:1:2}^{(k)} = \mu_{1:2}^{(k)} = 2\alpha\lambda^{k/\beta} \frac{\Gamma(1+\frac{k}{\beta})\Gamma(2\alpha-\frac{k}{\beta})}{\Gamma(2\alpha+1)}. \tag{5.12}$$

(Obtained on putting $i = 1$ and $n = 2$ in Eq. (5.5))

Using Eq. (5.12) and the recurrence relation given by Eq. (5.11) (for values of $\beta \in Z^{+}$), $\forall\ k = 1,2,\ldots.$, and $\alpha > \frac{k}{2\beta}$, $\mu_{1:1:2}^{(1)^{(k)}}$ can be calculated.

Subcase (ii): $m = 2$

We have only one progressive censoring scheme $R_1 = R_2 = 0$, In this case we have

$$E\left(X_{1:2:2}^{(0,0)}\right) = \mu_{1:2:2}^{(0,0)} = \mu_{1:2} \text{ and } E\left(X_{2:2:2}^{(0,0)}\right) = \mu_{2:2:2}^{(0,0)} = \mu_{2:2}.$$

Also, $E\left(X_{1:2:2}^{(0,0)}\right)^2 = \mu_{1:2:2}^{(0,0)^{(2)}} = \mu_{1:2}^{(2)}$ and $E\left(X_{2:2:2}^{(0,0)}\right)^2 = \mu_{2:2:2}^{(0,0)^{(2)}} = \mu_{2:2}^{(2)}$

and these values concerning ordinary order statistics can be evaluated using Eq. (5.5).

Case III: When $n = 3$, then $m = 1$ or 2 or 3

Subcase (i): $m = 1$

We have only one progressive censoring scheme $R_1 = 2$, and in this case we have from Eq. (3.2), on putting $n = 3$, we get

$$\mu_{1:1:3}^{(2)^{(k+\beta)}}\left(\frac{3\alpha\beta}{k+\beta} - 1\right) = \lambda\, \mu_{1:1:3}^{(2)^{(k)}}, \tag{5.13}$$

where

$$\mu_{1:1:3}^{(2)^{(k)}} = \mu_{1:1:3}^{(k)} = \mu_{1:3}^{(k)} = 3\alpha\lambda^{k/\beta}\frac{\Gamma(1+\frac{k}{\beta})\Gamma(3\alpha-\frac{k}{\beta})}{\Gamma(3\alpha+1)}. \tag{5.14}$$

(Obtained on putting $i = 1$ and $n = 3$ in Eq. (5.5))

Using Eq. (5.14) and the recurrence relation given by Eq. (5.13) (for values of $\beta \in Z^{+}$), $\forall\ k = 1,2,\ldots$, and $\alpha > \frac{k}{3\beta}$, $\mu_{1:1:3}^{(2)^{(k)}}$ can be calculated.

Subcase (ii): $m = 2$

We have only two progressive censoring schemes. One is $R_1 = 1$ and $R_2 = 0$ and the other is $R_1 = 0$ and $R_2 = 1$.

When $R_1 = 1$ and $R_2 = 0$

On putting $n = 3, m = 2,\ R_1 = 1$ and $R_2 = 0$ in (3.1), we get

$$\mu_{1:2:3}^{(1,0)^{(k+\beta)}}\left(1 - \frac{2\alpha\beta}{k+\beta}\right) = \frac{\alpha\beta}{k+\beta}\, \mu_{1:1:3}^{(2)^{(k+\beta)}} - \lambda\mu_{1:2:3}^{(1,0)^{(k)}}, \tag{5.15}$$

where $\mu_{1:1:3}^{(2)^{(k+\beta)}}$ can be calculated using (5.14), and $\mu_{1:2:3}^{(1,0)^{(k)}} = \mu_{1:3}^{(k)}$.

Using the recurrence relation given by Eq. (5.15) (for values of $\beta \in Z^{+}$), $\mu_{1:2:3}^{(1,0)^{(k+\beta)}}$ $\forall\ k = 1,2,\ldots$, can be calculated.

Further, on using mixture formula, we have

$$\mu_{2:2:3}^{(1,0)} = \frac{1}{2}[\mu_{2:3} + \mu_{3:3}] \text{ and}$$

$$\mu_{2:2:3}^{(1,0)^{(2)}} = \frac{1}{2}\left[\mu_{2:3}^{(2)} + \mu_{3:3}^{(2)}\right].$$

Proceeding in a similar manner $\mu_{1:2:3}^{(1,0)^{(k)}}$ and $\mu_{2:2:3}^{(1,0)^{(k)}}$ $\forall\, k = 1,2, \ldots.$ can be calculated.

When $R_1 = 0$ and $R_2 = 1$

In this case, we find that

$$E\left(X_{1:2:3}^{(0,1)}\right) = \mu_{1:2:3}^{(0,1)} = \mu_{1:3}, \qquad E\left(X_{2:2:3}^{(0,1)}\right) = \mu_{2:2:3}^{(0,1)} = \mu_{2:3},$$

$$E\left(X_{1:2:3}^{(0,1)}\right)^2 = \mu_{1:2:3}^{(0,1)^{(2)}} = \mu_{1:3}^{(2)} \text{ and } E\left(X_{2:2:3}^{(0,1)}\right)^2 = \mu_{2:2:3}^{(0,1)^{(2)}} = \mu_{2:3}^{(2)}.$$

Other moments can be obtained similarly.

Subcase (iii): $m = 3$

We have only one progressive censoring scheme $R_1 = 0,\ R_2 = 0$ and $R_3 = 0$. In this case

$$E\left(X_{1:3:3}^{(0,0,0)}\right) = \mu_{1:3:3}^{(0,0,0)} = \mu_{1:3}, \qquad E\left(X_{2:3:3}^{(0,0,0)}\right) = \mu_{2:3:3}^{(0,0,0)} = \mu_{2:3},$$

$$E\left(X_{1:3:3}^{(0,0,0)}\right)^2 = \mu_{1:3:3}^{(0,0,0)^{(2)}} = \mu_{1:3}^{(2)}, \qquad E\left(X_{2:3:3}^{(0,0,0)}\right)^2 = \mu_{2:3:3}^{(0,0,0)^{(2)}} = \mu_{2:3}^{(2)},$$

$$\text{and} \qquad E\left(X_{3:3:3}^{(0,0,0)}\right)^2 = \mu_{3:3:3}^{(0,0,0)^{(2)}} = \mu_{3:3}^{(2)}.$$

All these values can be obtained by using the result given in Eq. (5.5) for ordinary order statistics.

5.3. Recursive algorithm for product moments

To evaluate the moments of progressively Type-II right censored order statistics from Power Lomax distribution, we have considered the case for $r = s = 1$.

Case I: When $n = 2$ and $m = 2$

In this case we have only one progressive censoring scheme, i.e. $R_1 = R_2 = 0$. Thus, from Eq. (2.4), we have for $\alpha > \frac{1}{\beta}$

$$E\left(X_{1:2:2}^{(0,0)} X_{2:2:2}^{(0,0)}\right) = \mu_{1,2:2:2}^{(0,0)} = \mu_{1:2:2} = (\mu_{1:1})^2 = \left(\alpha\lambda^{1/\beta}\frac{\Gamma(1+\frac{1}{\beta})\Gamma(\alpha-\frac{1}{\beta})}{\Gamma(\alpha+1)}\right)^2,$$

(cf. Arnold et al. (1992), Eqn. (5.3.10)).

Case II: When $n = 3$ and $m = 2$

We have only two progressive censoring schemes. One is $R_1 = 1$ and $R_2 = 0$ and the other is $R_1 = 0$ and $R_2 = 1$.

When $R_1 = 1$ and $R_2 = 0$

In this case we have

$$E\left(X_{1:2:3}^{(1,0)} X_{2:2:3}^{(1,0)}\right) = \mu_{1,2:2:3}^{(1,0)} = \frac{1}{2}\left(\mu_{1,2:3} + \mu_{1,3:3}\right), \text{ from the mixture formula.}$$

When $R_1 = 0$ and $R_2 = 1$

In this case we have $E\left(X_{1:2:3}^{(0,1)} X_{2:2:3}^{(0,1)}\right) = \mu_{1,2:2:3}^{(0,1)} = \mu_{1,2:3}$.

Case III: When $n = 3$ and $m = 3$

In this case we have only one progressive censoring scheme $R_1 = R_2 = R_3 = 0$ and

$$E\left(X_{1:3:3}^{(0,0,0)} X_{2:3:3}^{(0,0,0)}\right) = \mu_{1,2:3:3}^{(0,0,0)} = \mu_{1,2:3},\ E\left(X_{1:3:3}^{(0,0,0)} X_{3:3:3}^{(0,0,0)}\right) = \mu_{1,3:3:3}^{(0,0,0)} = \mu_{1,3:3}$$

and $E\left(X_{2:3:3}^{(0,0,0)} X_{3:3:3}^{(0,0,0)}\right) = \mu_{2,3:3:3}^{(0,0,0)} = \mu_{2,3:3}.$

Likewise, using Eq. (5.6) and recurrence relations for product moments as derived in Section 4, one could proceed for higher values of n and all choices of m and $(R_1, R_2, \ldots, R_m)$.

6. Maximum Likelihood Estimators (MLEs)

Based on the observed sample $x_1 < x_2 < \cdots < x_m$ from a progressively Type-II censoring scheme, $(R_1, R_2, \ldots, R_m)$, the likelihood function can be written as

$$L(\alpha,\beta,\lambda) = A(n, m-1) \prod_{t=1}^{m} f(x_t, \alpha, \beta, \lambda)[1 - F(x_t, \alpha, \beta, \lambda)]^{R_t}; x > 0, \alpha, \beta, \lambda > 0, \quad (6.1)$$

where

$$A(n, m-1) = n(n - R_1 - 1)(n - R_1 - R_2 - 2) \ldots (n - R_1 - R_2 - \cdots - R_{m-1} - m + 1),$$

and $f(.)$ and $F(.)$ are same as defined in (1.1) and (1.2), respectively. Therefore, ignoring the additive constant the log-likelihood function is written as

$$log\big(L(\alpha,\beta,\lambda)\big) = mlog(\alpha) + mlog(\beta) - mlog(\lambda) + (\beta - 1)\sum_{t=1}^{m} log(x_t) - \sum_{t=1}^{m}(\alpha(R_t + 1) + 1)\, log\left(1 + \frac{x_t^{\beta}}{\lambda}\right). \quad (6.2)$$

To compute the MLEs of the unknown parameters α, β and λ, consider the three normal equations:

$$\frac{\partial log(L)}{\partial \alpha} = \frac{m}{\alpha} - \sum_{t=1}^{m} (1 + R_t) \, log\left(1 + \frac{{x_t}^{\beta}}{\lambda}\right) = 0,$$

$$\frac{\partial log(L)}{\partial \beta} = \frac{m}{\beta} + \sum_{t=1}^{m} log(x_t) - \sum_{t=1}^{m} \frac{(\alpha(1 + R_t) + 1){x_t}^{\beta} \, log(x_t)}{\lambda + {x_t}^{\beta}} = 0,$$

and

$$\frac{\partial log(L)}{\partial \lambda} = -\frac{m}{\lambda} + \frac{1}{\lambda} \sum_{t=1}^{m} \frac{{x_t}^{\beta}(\alpha(1 + R_t) + 1)}{\lambda + {x_t}^{\beta}} = 0,$$

whose solution provide the MLEs $\hat{\alpha}, \hat{\beta}$ and $\hat{\lambda}$.

Once MLEs of α, β and λ are obtained as $\hat{\alpha}, \hat{\beta}$ and $\hat{\lambda}$, the MLEs of $R(t)$ and $h(t)$ can be obtained using invariance property of MLEs as

$$\hat{R}(t) = \left(1 + \frac{t^{\hat{\beta}}}{\hat{\lambda}}\right)^{-\hat{\alpha}}, t > 0 \quad \text{and}$$

$$\hat{h}(t) = \frac{\hat{\alpha} \, \hat{\beta} t^{\hat{\beta}-1}}{\hat{\lambda} + t^{\hat{\beta}}} \quad , t > 0.$$

7. Simulation study

In this Section, a simulation study is conducted to observe the behaviour of the proposed method for different sample sizes, different effective sample sizes and for different censoring schemes. We have considered different sample sizes; $n = 35,40,50$; different effective sample sizes; $m = 20,25,30,35,40,50$; different censoring schemes. In all the cases we have used $\alpha = 2, \beta = 1$ and $\lambda = 2$. For a given set of n, m and a censoring scheme, using the algorithm proposed by Balakrishnan and Sandhu (1995), a sample is generated. Using the sample, the MLEs of unknown parameters α , β and λ are computed based on the method proposed in Section 6. Finally, with 1000 replications, using a program in R, the MLEs of α, β, λ , $R(t)$ and $h(t)$along with their average bias and mean square errors (MSEs) are obtained. The average bias is reported within brackets against each estimate and the results are presented in Tables 7.1, 7.2a and 7.2b.

Table 7.1. MLEs of α, β and λ along with their Average Bias and MSE for different censoring schemes, for $\alpha = 2, \beta = 1$ and $\lambda = 2$

n	m	Censoring Scheme	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$MSE(\hat{\alpha})$	$MSE(\hat{\beta})$	$MSE(\hat{\lambda})$
35	20	(3*0,5,3*0,5,3*0,5,8*0)	1.94181 (-0.0582)	1.03903 (0.03903)	1.89249 (-0.1075)	0.35059	0.20826	0.31453
35	20	(1,2,1,2,1,2*0,1,1,0,1,0,1,0,2,0,1, 1,2*0)	1.95704 (-0.0429)	1.04216 (0.04216)	1.96635 (-0.0337)	0.36253	0.20725	0.34712
35	20	(15,19*0)	2.01236 (0.01236)	1.01958 (0.01958)	2.01264 (0.01264)	0.33291	0.13837	0.33011
35	25	(5*0,5,7*0,5,11*0)	1.95215 (-0.0478)	1.02663 (0.02663)	1.93115 (-0.0689)	0.32803	0.17725	0.30247
35	25	(1, 2, 0, 2, 1, 0, 0, 1, 1, 0, 1, 0, 1, 12*0)	1.96322 (-0.0368)	1.02506 (0.02506)	1.96717 (-0.0328)	0.32532	0.14379	0.31937
35	25	(10,24*0)	2.02428 (0.02428)	1.01749 (0.01749)	2.02533 (0.02533)	0.30106	0.12965	0.32033
35	35	(35*0)	1.99586 (-0.0041)	1.00524 (0.00524)	1.99661 (-0.0034)	0.19950	0.07522	0.20118
40	25	(5,4*0,5,7*0,5,11*0)	1.96951 (-0.0305)	1.01450 (0.0145)	1.94261 (-0.0574)	0.31210	0.19161	0.29743
40	25	(1,2,1,2,1,2*0,1,1,0,1,0,1,0,2,0,1, 1,7*0)	1.96329 (-0.0367)	1.02692 (0.02692)	1.97827 (-0.0217)	0.30661	0.13456	0.30967
40	25	(15,24*0)	2.02742 (0.02742)	1.02021 (0.02021)	2.04453 (0.04453)	0.28108	0.11671	0.29954
40	30	(3*0,5,10*0,5,15*0)	1.97058 (-0.0294)	1.02329 (0.02329)	1.95472 (-0.0453)	0.30219	0.18862	0.30098
40	30	(1, 2, 1, 2, 1, 0, 0, 1, 1, 0, 1, 19*0)	1.95811 (-0.0419)	1.02032 (0.02032)	1.97122 (-0.0288)	0.28149	0.12976	0.28352
40	30	(10,20*0)	1.99483 (-0.0052)	1.01458 (0.01458)	1.99475 (-0.0052)	0.27662	0.12839	0.29211
40	40	(40*0)	1.99802 (-0.0019)	1.00227 (0.00227)	1.99777 (-0.0023)	0.19019	0.05713	0.19717
50	25	(5,4*0,5,4*0,5,4*0,5,4*0,5,4*0)	2.02323 (0.02323)	0.99753 (-0.0025)	2.04855 (0.04855)	0.28921	0.13003	0.25912
50	25	(25,24*0)	2.00972 (0.00972)	1.01253 (0.01253)	2.03109 (0.03109)	0.25413	0.11080	0.24879
50	30	(5,4*0,5,5*0,5,5*0,5,12*0)	1.99786 (0.00214)	1.02274 (0.02274)	2.02981 (0.02981)	0.24142	0.12644	0.26907
50	30	(20,29*0)	2.01786 (0.01786)	0.99015 (-0.0099)	2.02153 (0.02153)	0.22832	0.12561	0.20991
50	40	(8*0,5,8*0,5,22*0)	1.95692 (-0.0431)	1.01711 (0.01711)	1.94165 (-0.0584)	0.21131	0.12675	0.25002
50	40	(10,39*0)	2.00875 (0.00875)	1.01251 (0.01251)	2.01141 (0.01141)	0.20149	0.10939	0.19972
50	50	(50*0)	2.00123 (0.00123)	1.00135 (0.00135)	2.00191 (0.00191)	0.12615	0.03927	0.16793

Table 7.2a. MLEs of $R(t)$and $h(t)$ along with their Average Bias and MSE for different censoring schemes, for $\alpha = 2, \beta = 1, \lambda = 2$ and $t = 0.5$

$$\boldsymbol{t = 0.5; R(t) = 0.64; h(t) = 0.8}$$

n	m	Censoring Scheme	$\hat{R}(t)$	$\hat{h}(t)$	$MSE(\hat{R}(t))$	$MSE(\hat{h}(t))$
35	20	(3*0,5,3*0,5,3*0,5,8*0)	0.63804 (-0.002)	0.74962 (-0.0504)	0.01321	0.03465
35	20	(1,2,1,2,1,2*0,1,1,0,1,0,1,0,2,0,1,1,2*0)	0.64294 (0.00294)	0.76023 (-0.0398)	0.01327	0.03022
35	20	(15,19*0)	0.63772 (-0.0023)	0.75152 (-0.0485)	0.01262	0.03349
35	25	(5*0,5,7*0,5,11*0)	0.64416 (0.00416)	0.76376 (-0.0362)	0.01329	0.02173
35	25	(1, 2, 0, 2, 1, 0, 0, 1, 1, 0, 1, 0, 1 ,12*0)	0.64215 (0.00215)	0.75891 (-0.0411)	0.00981	0.02020
35	25	(10,24*0)	0.64309 (0.00309)	0.76364 (-0.0364)	0.00976	0.01964
35	35	(35*0)	0.64043 (0.00043)	0.79646 (-0.0035)	0.00512	0.01582
40	25	(5,4*0,5,7*0,5,11*0)	0.63524 (-0.0048)	0.75699 (-0.043)	0.01236	0.03126
40	25	(1,2,1,2,1,2*0,1,1,0,1,0,1,0,2,0,1,1,7*0)	0.64416 (0.00416)	0.75361 (-0.0464)	0.01483	0.03212
40	25	(15,24*0)	0.63783 (-0.0022)	0.76114 (-0.0389)	0.00933	0.02231
40	30	(3*0,5,10*0,5,15*0)	0.64176 (0.00176)	0.76083 (-0.0392)	0.00940	0.02153
40	30	(1, 2, 1, 2, 1, 0, 0, 1, 1, 0, 1 ,19*0)	0.63668 (-0.0033)	0.75993 (-0.0401)	0.00971	0.01987
40	30	(10,20*0)	0.64857 (0.00857)	0.76648 (-0.0335)	0.01050	0.01901
40	40	(40*0)	0.63928 (-0.0007)	0.79685 (-0.0032)	0.00425	0.01322
50	25	(5,4*0,5,4*0,5,4*0,5,4*0,5,4*0)	0.64424 (0.00424)	0.77405 (-0.026)	0.01393	0.01874
50	25	(25,24*0)	0.63967 (-0.0003)	0.78257 (-0.0174)	0.01009	0.01716
50	30	(5,4*0,5,5*0,5,5*0,5,12*0)	0.64205 (0.00205)	0.78017 (-0.0198)	0.01006	0.01718
50	30	(20,29*0)	0.63367 (-0.0063)	0.77948 (-0.0205)	0.00960	0.01872
50	40	(8*0,5,8*0,5,22*0)	0.64139 (0.00139)	0.78965 (-0.0104)	0.01110	0.01482
50	40	(10,39*0)	0.63778 (-0.0022)	0.78342 (-0.0166)	0.01211	0.01613
50	50	(50*0)	0.63994 (-0.00006)	0.79866 (-0.0013)	0.00303	0.01209

Table 7.2b. MLEs of $R(t)$and $h(t)$ along with their Average Bias and MSE for different censoring schemes, for $\alpha = 2, \beta = 1, \lambda = 2$ and t = 2

$$t = 2; R(t) = 0.25; h(t) = 0.5$$

n	m	Censoring Scheme	$\hat{R}(t)$	$\hat{h}(t)$	$MSE(\hat{R}(t))$	$MSE(\hat{h}(t))$
35	20	(3*0,5,3*0,5,3*0,5,8*0)	0.25230 (0.0023)	0.51557 (0.01557)	0.00866	0.06067
35	20	(1,2,1,2,1,2*0,1,1,0,1,0,1,0,2,0,1,1,2*0)	0.25098 (0.00098)	0.51532 (0.01532)	0.00834	0.05801
35	20	(15,19*0)	0.24831 (-0.0017)	0.48845 (-0.0116)	0.00902	0.04897
35	25	(5*0,5,7*0,5,11*0)	0.24792 (-0.0021)	0.50292 (0.00292)	0.00838	0.04923
35	25	(1, 2, 0, 2, 1, 0, 0, 1, 1, 0, 1, 0, 1 ,12*0)	0.24803 (-0.002)	0.50371 (0.00371)	0.00692	0.04810
35	25	(10,24*0)	0.24672 (-0.0033)	0.50354 (0.00354)	0.00689	0.04799
35	35	(35*0)	0.24976 (-0.00024)	0.50102 (0.00102)	0.00451	0.02815
40	25	(5,4*0,5,7*0,5,11*0)	0.24562 (-0.0044)	0.49653 (-0.0035)	0.00767	0.03965
40	25	(1,2,1,2,1,2*0,1,1,0,1,0,1,0,2,0,1,1,7*0)	0.25093 (0.00093)	0.48032 (-0.0197)	0.00558	0.04365
40	25	(15,24*0)	0.24847 (-0.0015)	0.48964 (-0.0104)	0.00621	0.04221
40	30	(3*0,5,10*0,5,15*0)	0.24863 (-0.0014)	0.49876 (-0.0012)	0.00606	0.03932
40	30	(1, 2, 1, 2, 1, 0, 0, 1, 1, 0, 1 ,19*0)	0.24798 (-0.002)	0.49721 (-0.0028)	0.00755	0.03123
40	30	(10,20*0)	0.24832 (-0.0017)	0.50176 (0.00176)	0.00518	0.03078
40	40	(40*0)	0.24978 (-0.0002)	0.50090 (0.0009)	0.00317	0.02120
50	25	(5,4*0,5,4*0,5,4*0,5,4*0,5,4*0)	0.24882 (-0.0018)	0.50387 (0.00387)	0.00578	0.03821
50	25	(25,24*0)	0.25017 (0.00017)	0.49536 (-0.0046)	0.00592	0.03729
50	30	(5,4*0,5,5*0,5,5*0,5,12*0)	0.24932 (-0.0007)	0.49674 (-0.0033)	0.00503	0.03655
50	30	(20,29*0)	0.25102 (0.00102)	0.48991 (-0.0101)	0.00499	0.03812
50	40	(8*0,5,8*0,5,22*0)	0.25091 (0.00091)	0.50148 (0.00148)	0.00432	0.03556
50	40	(10,39*0)	0.24965 (-0.0004)	0.49839 (-0.0016)	0.00341	0.03233
50	50	(50*0)	0.25009 (0.00009)	0.50039 (0.00039)	0.00246	0.01109

From Table 7.1, we observe that for complete samples, MLEs of α, β and λ are very nearly unbiased and can be regarded as good estimators. It is also observed that for complete samples, as the sample size n increases the average MSE decreases. In addition, the MSE generally decreases as the failure information m increases, and for all the censoring schemes the MSE of the estimates is quite small and can be used in all practical situations. Here one has to make a trade-off between the precision of the estimation method and the cost of the experiment. Also, from Tables 7.2a and 7.2b, it is observed that for the MLEs of $R(t)$ and $h(t)$, the MSE generally decreases as the failure information m increases. In addition, for the complete samples, as the sample size n increases the average MSE decreases.

8. Conclusion

Some recurrence relations between the single and the product moments of progressively Type-II right censored order statistics from the power Lomax distribution have been derived, which would assist us to compute the moments of progressively Type-II right censored order statistics for every n and for different censoring arrangements $(R_1, R_2, \dots, R_m)$, $m \leq n$. The recursive algorithm is presented with the help of which the single and product moments of progressively Type-II right censored order statistics from the power Lomax distribution can be easily obtained. Further, a maximum likelihood approach is used to estimate the parameters of the power Lomax distribution, which are further used to estimate the reliability characteristics. A Monte Carlo method is used to simulate the data and to compare the performance of the estimates for different censoring schemes.

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Towards a target employment rate within age and gender groups

Stanisław Jaworski[1], **Zofia Zielińska-Kolasińska**[2]

ABSTRACT

Quarterly employment rates in European countries are analysed in terms of the likelihood of achieving a specific employment rate within age and gender groups in a five-year horizon. The German employment rate serves as a benchmark for this research. The likelihood is estimated by a Monte-Carlo simulation based on the class of exponential smoothing models. The research presents a pessimistic prognosis of employment rates in European countries with respect to young and partly to older workers.

Key words: employment rate, exponential smoothing, forecasting, state space approach.

1. Introduction

The European Employment Strategy dates back to 1997, when the Member States of the European Union committed themselves to establishing a set of common goals and tasks in the field of employment policy. Its main goal was to create more and better jobs throughout the European Union. Consequently, national governments have proposed and agreed common employment policy priorities and objectives. Governments have committed to annual reports on the implementation of the Employment Guidelines and an assessment of the Key Employment and Social Indicators.

In 2010, the European Council adopted the Europe 2020 strategy. One of the main targets of this strategy at the European Union level was to raise, by the year 2020, the employment rate of the population aged 20-64 years to at least 75%. The year 2010 was a key milestone in the evolution of the European Employment Strategy because European cooperation on the economic and employment policy had faced the global economic and financial crisis during 2007-09. The crisis has slowed the economic development of many countries until 2013. Despite the global crisis the majority of European countries achieved their targets.

Two age groups in the labour force are of particular interest: people aged 15-24 and 55-64. The population of the European Union is currently experiencing an ageing process, and predictions in this area suggest that this process will accelerate during 2019-2050 (Corselli-Nordblad et al. (2020a)). Demographic ageing means the proportion of people of working age in the EU is shrinking. Hence, the employment rate of older and young people is among

[1]Department of Econometrics and Statistics, Warsaw University of Life Sciences, Poland. E-mail: stanislaw_jaworski@sggw.edu.pl, ORCID: https://orcid.org/0000-0002-6169-2886.

[2]Department of Mathematics, Siedlce University of Natural Sciences and Humanities, Poland. E-mail: zofia.zielinska-kolasinska@uph.edu.pl, ORCID: https://orcid.org/0000-0001-8845-758X.

the main policy objectives that European Union has adopted in recent years. A considerable portion of young people aged 15-24 years in the EU is economically inactive. The European Union supports the process of reducing youth unemployment by creating various programs, like *The Youth Guarantee* (European Commission (2018)), *Youth Employment Initiative* (European Commission (2012)), *Skills Agenda for Europe* (European Commission (2020b)) and *Investing in Europe's Youth* (European Commission (2017)). Moreover, the European Commission suggests that the most pressing concern of policy-makers is to encourage older people to remain in the labour market for as long as possible (Corselli-Nordblad et al. (2020a)). A key reason for increasing retirement age is to ensure the financial sustainability of the state pension programme. Moreover, the retirement-aged include workers with extensive experience who can contribute to generating new jobs and make net contribution to GDP. There is also evidence that meaningful and appropriate work benefits older people. Paid work increases their incomes, helping them to achieve higher standards of living. In response, many countries have decided to increase the retirement age. But there is an example of doing the opposite. The topic of the age at which Poles retire returned in the election campaign of 2020. The retirement age was raised in 2012 by the government of Donald Tusk. According to the law then in force, men were to reach the target retirement age of 67 in 2020, and women 20 years later – in 2040. In 2016, the government of Beata Szydło restored the act to its pre-2012 version.

The aim of the work is to estimate the probability of achieving given employment rates within age and gender groups across European countries in a five-year horizon on the basis of employment time series forecasts. German 2019 employment rates reduced proportionally by 5% are taken as the base rates. We compare them accordingly with employment rates of other member states of the European Union. The choice of these base rates is justified to some extent. The applied analysis required comparisons with reasonably high rates. Moreover, the German labour market belongs to the strongest in Europe and experienced robust performance in the financial crisis of 2007-2009. Therefore, the choice seems reasonable as benchmarks of better and satisfactory employment rates for many member states of the European Union. The estimates are based on quarterly employment rates from Eurostat.

2. Research methodology

The probability of achieving a given employment rate is estimated by Monte-Carlo simulation by the use of exponential smoothing methods. Although the methods have been around since the 1950s, a modelling framework incorporating procedures for model selection was not developed until the nineties. Ord et al. (1997), Hyndman et al. (2002) and Hyndman et al. (2005) have shown that all exponential smoothing methods are optimal forecasts from innovations state space models. The innovations state space approach provides prediction intervals, maximum likelihood estimation and procedures for model selection. The theoretical background of the methods can be found in Hyndman et al. (2008b). Software implementation of the methods is described in Hyndman et al. (2008a), where for each of the fifteen exponential smoothing methods the authors specify two possible innovations state space models, one corresponding to a model with additive errors and the other to a model with multiplicative errors. The models are combinations of components such as the trend, seasonal, and irregular or error components. The trend component can be additive, additive damped, multiplicative and multiplicative damped. The seasonal component can be additive or multiplicative. The general model involves a state vector $x_t = (l_t, b_t, s_t, s_{t-1}, \dots, s_{t-m+1})'$ and state space equations of the form

$$\begin{aligned} y_t &= w(x_{t-1}) + r(x_{t-1})\varepsilon_t \\ x_t &= f(x_{t-1}) + g(x_{t-1})\varepsilon_t, \end{aligned}$$

where y_t, $t = 1, \dots, T$, denote observations, $\{\varepsilon_t\}$ is a Gaussian white noise process with mean zero and variance σ^2. Components l_t, b_t and s_t are unobservable and represent the trend, slope and seasonality respectively. The number of seasons in a period, denoted by m, is a given number. Functions w, r, f, g are known and have a specific form (see Hyndman et al. (2008a)).

As an illustrative example of this class of models let us consider the known Holt's model defined by the equations:

$$\begin{aligned} \text{Level:} \quad & y_t = \alpha y_t + (1-\alpha)(l_t + b_{t-1}) \\ \text{Growth:} \quad & b_t = \beta^\star(l_t - l_{t-1}) + (1-\beta^\star)b_{t-1} \end{aligned}$$

These equations can be rewritten to the following form:

$$\begin{aligned} y_t &= \begin{pmatrix} 1 & 1 \end{pmatrix} x_{t-1} + \varepsilon_t \\ x_t &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} x_{t-1} + \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \varepsilon_t \end{aligned}$$

where $\beta = \alpha\beta^\star$, $x_t = (l_t, b_t)'$, $\mu_t = \hat{y}_t = l_{t-1} + b_{t-1}$ and $\varepsilon_t = y_t - \mu_t$ denotes the on-step forecast error at time t. The model is fully specified once we state the distribution of the error term ε_t. Note that in this case $w_t(x_{t-1}) = \begin{pmatrix} 1 & 1 \end{pmatrix} x_{t-1}$, $r(x_{t-1}) = 1$, $f(x_{t-1}) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} x_{t-1}$ and $g(x_{t-1}) = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$. Generally, a penalised method based on the in-sample fit Akaike's Information Criterion is used in the work to choose appropriate models for the investigated employment series. To estimate the probability of achieving the given target employment rate the set of 1000 sample paths that are future to and conditional on the data is produced. Then, the number of paths crossing the target is calculated. Thus, we estimate probability $P(y_t > \text{target})$ for a given t. The maximum estimate error in this case does not exceed 3%.

3. An example: the Visegrád Group

The Visegrád Group is a cultural and political alliance of four countries of Central Europe (the Czech Republic, Hungary, Poland and Slovakia), all of which are members of the EU and of NATO, with the aim to advance mutual co-operation in military, cultural, economic, climate and energy matters and to further their integration to the EU. All of the countries belonged to the former Eastern Block. The employment rates of these countries and Germany are shown in Figure 1.

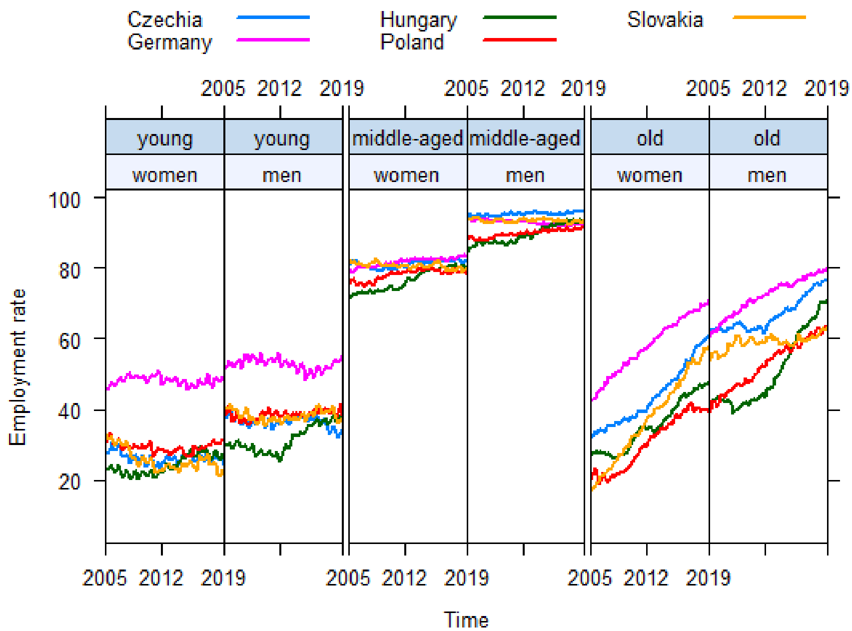

Figure 1: Quarterly time series of employment rates of Germany and the member countries of the Visegrád Group with respect to gender and age groups: 2005-2019

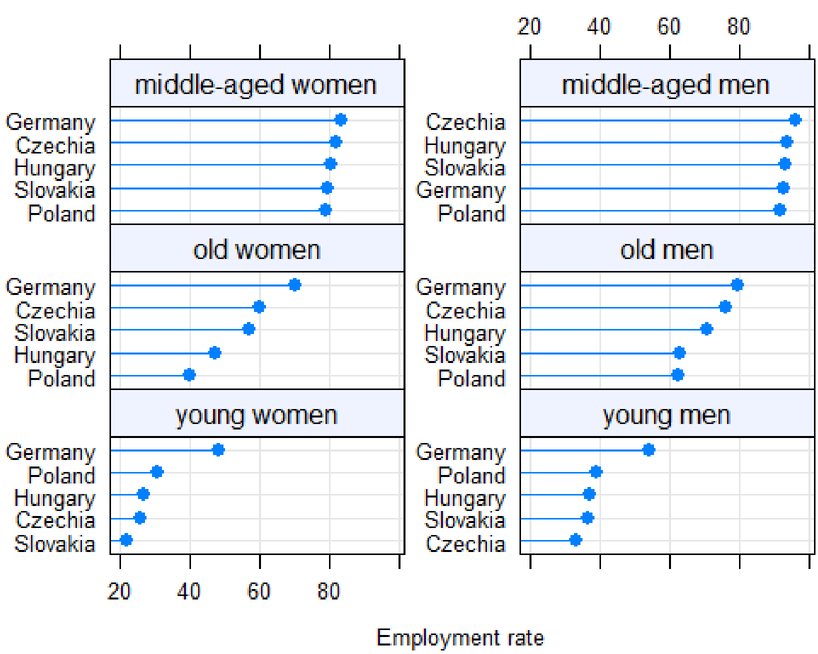

Figure 2: The comparison of the 2019 employment rates of Germany and the member countries of the Visegrád Group with respect to gender and age groups

Note that middle-aged workers (25-54 years old) have similar employment rates. There are much greater differences between the employment rates in the group of young (15-24 years old) and old people (55-64 years old). This can be seen in the comparison of Germany against different countries of the Visegŕad Group. The German employment rate is at least as high as those of other countries. We want to find out the probabilities of the member countries of the Visegrád Group achieving the given level of 95% German employment rate. Figure 2 depicts 2019 employment rates of Germany and the member countries of the Visegrád Group with respect to gender and age groups. It can be taken to mean that differences between probabilities of achieving the given levels will be smaller in young and old workers. For example, the difference in the group of young women between Slovak and German employment rates is relatively large (over 20%). The Slovak employment time series has been well under 30% since 2010 and was declining in the last four years. Thus, the probability of reaching the German employment rate is expected to be low in the case of young Slovak women.

Let us consider the forecast of the Slovak employment rate for this group. The point forecast is given in blue in Figure 3. Three further forecasts were simulated, smoothed and added in red into Figure 3. The forecasts are represented by three smoothed lines starting at 2020. In order to estimate the probability of crossing the given level, there were 1000 such lines generated altogether. The forecasts were smoothed to avoid the short-term influences of seasonal pattern.

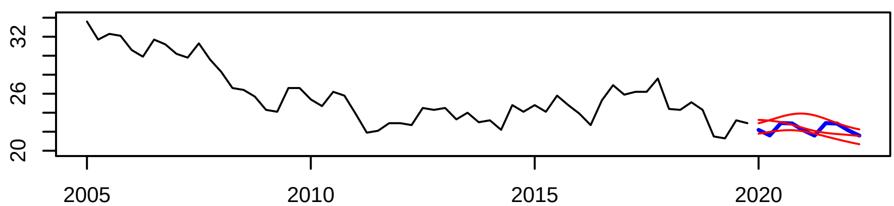

Figure 3: Forecast of the employment rate for young Slovak women

The procedure was repeated for the rest of the member countries of the Visegrád Group. The prognosis for young people is not optimistic: neither probability based on prediction for four countries and years 2020-2024 exceeds 1%. The results for old people are better but not in all of the countries (Table 1). The results for middle-aged people are the best (Table 2). In this case, most estimates are equal to one.

In general, most middle-aged people are employed. The employment rate of older workers increased between 2005 and 2019 but the pace of growth was not uniform in the considered countries. Universally older workers bring a level of experience, critical thinking and sheer knowledge that cannot be taught. In some industries it takes a decade or longer for workers to gain the technical skills necessary to master the job. Despite these advantages, the labour market of many countries does not offer suitable jobs for older people. There are many reasons for this but discussing them is beyond the scope of the work. In fact, not all countries are expected to improve the employment rate of older workers.

Table 1: Forecasts of the probabilities of reaching the 95% German employment rates by the member countries of the Visegrád Group with respect to gender; results for old people

Gender	Country	2020	2021	2022	2023	2024
Female	Czechia	0.13	0.84	0.98	1.00	1.00
	Hungary	0.00	0.00	0.00	0.00	0.00
	Poland	0.00	0.00	0.00	0.00	0.00
	Slovakia	0.02	0.45	0.80	0.93	0.97
Male	Czechia	0.99	0.99	1.00	1.00	1.00
	Hungary	0.51	0.87	0.96	0.98	0.98
	Poland	0.00	0.00	0.00	0.03	0.10
	Slovakia	0.00	0.00	0.00	0.00	0.00

Table 2: Forecasts of the probabilities of reaching the 95% German employment rates by the member countries of the Visegrád Group with respect to gender; results for middle-aged people

Gender	Country	2020	2021	2022	2023	2024
Female	Poland	0.40	0.42	0.44	0.46	0.45
	Slovakia	0.70	0.67	0.64	0.63	0.62
	Other countries	1.00	1.00	1.00	1.00	1.00
Male	All countries	1.00	1.00	1.00	1.00	1.00

Compared to old people, young workers tend to have fewer general work skills and less specific human capital relevant to the particular firm employing them (Bell et al. (2011)). They are faced with wages that provide inadequate compensation for the loss of benefits and with suboptimal commutes. The lack of mobility and limited job search area severely limits their job opportunities.

Many factors influence the employment rate. An interesting discussion may be found in Matthews et al. (2014). Each of the European countries has its own specific factors lowering its employment rate. As the set of common goals and tasks in the field of employment policy is formulated and agreed on, the main goal for every country is to create more and better jobs. This will increase the employment rate. The reasonable measure of the rise should be based on the level itself. The idea of model-based estimate of achieving the specific level representing the one of the strongest labour markets in Europe holds this criterion and takes into account the structure of employment rate time series. The benchmark in this approach is the strong German economy, which is the largest contributor to the EU budget.

4. Employment Rates Forecasts

The following countries were analysed: Austria, Belgium, Bulgaria, Croatia, Cyprus, Czechia, Denmark, Estonia, Finland, France, Greece, Spain, Ireland, Lithuania, Luxemburg, Latvia, Malta, Netherlands, Germany, Poland, Portugal, Romania, Slovakia, Slovenia, Sweden, Hungary, the United Kingdom, Italy. The model-based estimates of achieving the 95% German employment rates were calculated as in the previous section. The results are as follows.

Results for young women and men.

1. The estimated probabilities are less than 0.05 for Belgium, Bulgaria, Czechia, Greece, Spain, France, Croatia, Italy, Cyprus, Latvia, Hungary, Poland, Portugal, Romania, Slovenia, Slovakia.

2. The estimated probabilities are equal to at least 0.98 for Denmark, Malta, Netherlands, Austria, Finland, Sweden, the United Kingdom.

3. For other countries the estimates are in Table 3. With the exception of Ireland, the estimates are increasing.

Table 3: Model-based estimates of achieving the 95% German employment rates; results for young workers from countries with estimates greater or equal to 0.05 and less than 0.98

Gender	Country	2020	2021	2022	2023	2024
Female	Estonia	0.53	0.63	0.69	0.72	0.75
	Ireland	0.39	0.30	0.24	0.19	0.21
	Lithuania	0.00	0.00	0.02	0.10	0.23
Male	Estonia	0.38	0.46	0.51	0.58	0.61
	Ireland	0.12	0.10	0.06	0.06	0.06
	Lithuania	0.00	0.00	0.03	0.06	0.09

Most of the calculated estimates do not exceed 5%. Only 25% of European countries have the estimates greater than 98%. Three countries are between these two extreme cases. This shows that the European countries are highly differentiated.

Results for middle-aged women and men.

Estimates for middle-aged people are shown in Figure 4. In this case, estimates for most countries are very close to one, therefore only those that are less than 0.9 are displayed. Generally, most middle-aged people in Europe are employed. Employment rates are as high as in Germany and in some cases they are slightly greater. Therefore, the estimates are close to one.

Results for old-aged women and men.

The case of old workers is rather complex. For some of the countries the probabilities of reaching the 95% of the appropriate German employment rates levels are very low (e.g. for Poland, Romania and Greece) (Table 4). On the other hand, there are countries with a high probability of achieving it (Table 4). Some countries show the probability steeply increasing in the group of women or men (Figure 5).

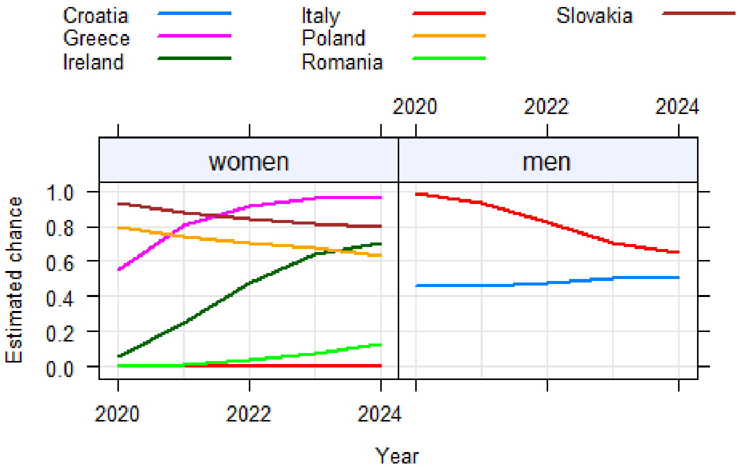

Figure 4: Model-based estimates of achieving the 95% German employment rates; results for middle-aged people from countries with estimates less than 0.9

Table 4: Countries with the investigated probabilities less than 0.3 are marked with × and greater than 0.8 are marked with ⊗

Country	Women	Men	Country	Women	Men
Greece	×	×	Czechia	⊗	⊗
Croatia	×	×	Denmark	⊗	⊗
Italy	×		Estonia	⊗	
Hungary	×	×	Latvia	⊗	
Malta	×		Lithuania	⊗	⊗
Austria	×		Netherlands	⊗	⊗
Poland	×	×	Finland	⊗	⊗
Romania	×	×	Sweden	⊗	
Belgium		×	United	⊗	
Spain		×	Cyprus		⊗
France		×	Hungary		⊗
Slovenia		×			
Slovakia		×			

An interesting case is France. The country is one of the largest contributors to the EU budget, like Germany, but its employment rate is less than in Germany and the investigated probability is very low in both gender groups (mostly less than 2%). This shows how difficult it is to resolve the dispute about the reasons for the given employment levels to differ. Differences in German and French labour-market institutions are significant. The two countries have different economic strategies that affect employment rate and level. Application of collective bargaining at the firm level allows for more flexibility in Germany (Herzog-Stein et al. (2013), Möller (2010)). However, the higher resilience and flexibility of the German labour market comes at the price of higher market-income inequality and poverty across individuals and age groups (Hartung et al. (2018), Cléaud et al. (2019)).

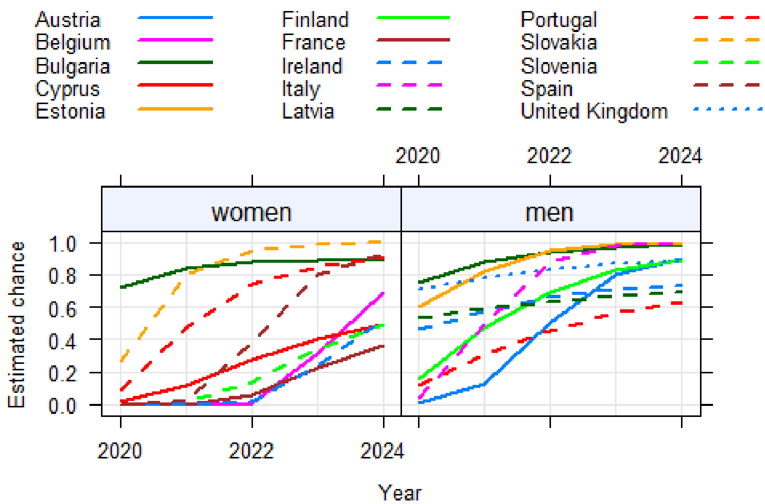

Figure 5: Model-based estimates of achieving the 95% German employment rates; results for old-aged people from countries with estimates between 0.3 and 0.8

Although Germany's growth model has allowed it to benefit from the strong post-2008 financial crisis recovery in the global economy, it also makes it more exposed to swings in the global cycle. France's growth model has relied more on domestic demand. Together with a larger public sector, this has helped to smooth out economic cycles, but has also implied some losses in cost competitiveness and a significantly higher tax burden (Cléaud et al. (2019)).

5. Summary

The idea of model-based probability estimate of reaching the specified employment rate level represented by one of the strongest labour markets in Europe and at the same time one of the largest contributors to the EU budget, i.e. Germany, was applied. This approach takes into account the structure of employment rate time series and allows to forecasting the probability of achieving the given level. The research comprises groups by age and gender. The prognosis is based on the data from 2005 to 2019. The data embracing the year 2020 are not included in the analysis. The one-year series relate to the global spread of a new disease and appear too short to be included into a statistical model. According to the knowledge of the authors no similar approach to the issue has been proposed.

The obtained results are not optimistic for young people. Low probability of reaching the German employment rates levels results from the past covering of 2005-2019 time period. It means that the time series consist of low rates (for example in the case of Bulgaria, Italy or Greece) or are not expected to increase fast enough as in the case of Cyprus, Latvia, Slovenia and other countries with relatively high employment rates. This is true for Belgium, Bulgaria, Czechia, Greece, Spain, France, Croatia, Italy, Cyprus, Latvia, Hungary, Poland, Portugal, Romania, Slovenia and Slovakia. The 2019 employment rates of these countries are shown in Figure 6.

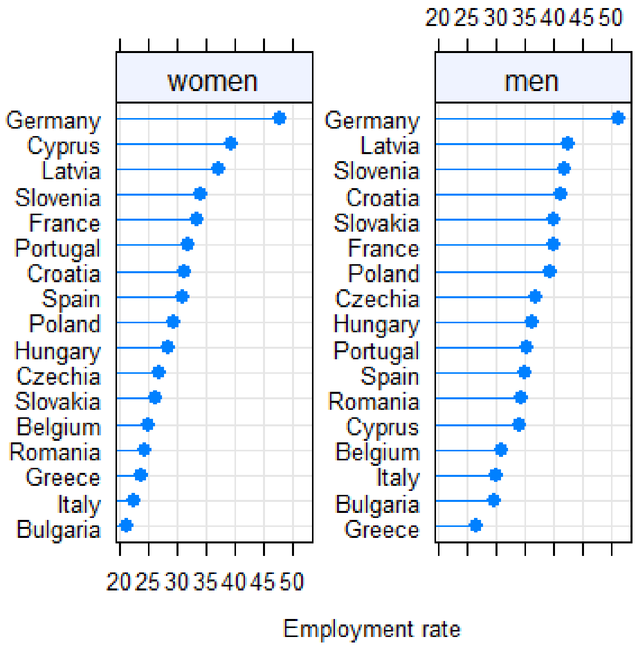

Figure 6: The 2019 employment rates for young people

The employment rates for old people are more optimistic. The rates are increasing and that is reflected by the increasing probability of approaching the levels of German employment rates. Slovakia, Portugal, Spain and Belgium show high growth rates among women. Their probabilities of approaching the 95% German employment rates levels are expected to rise above 80% no later than in 2024. Among men it includes Estonia, Italy, Finland and Austria (see Figure 5).

The employment rates for old people may be strongly influenced by the pandemic. A large body of research has established health as a significant factor affecting the labour market participation of older people, with those in poorer health more likely not to be employed (van den Berg et al. (2008), van Rijn et al. (2014)). Figure 7 shows the distribution of difference between employment rates of 2020:Q2 and of 2019:Q2. About half of the European countries show a decrease in employment rates among old-aged people since the beginning of the pandemic. The distribution is shifted to the left compared to the corresponding data of a previous year. Differences in employment levels are more varied for women, as can be seen from the different scale of distributions.

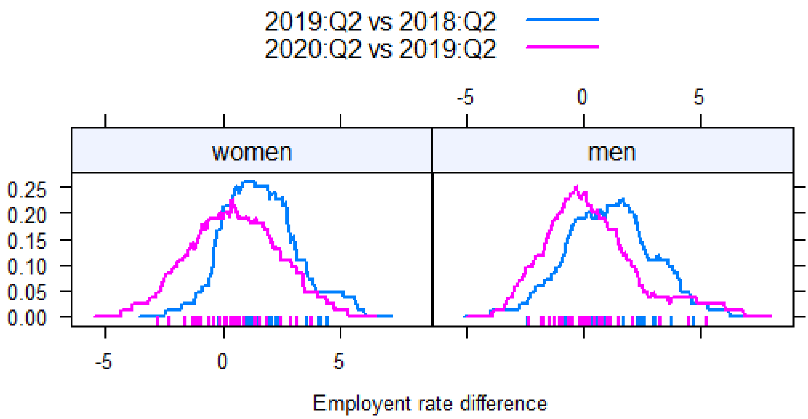

Figure 7: Employment rate difference distribution 2020:Q2 vs 2019:Q2 and 2019:Q2 vs 2018:Q2 for old people across European countries.

The German Institute for Economic Research (DIW Berlin) said that Germany's economic output could return to pre-crisis levels toward the end of 2021. The prognosis for Europe is less optimistic. Autumn 2020 Economic Forecast of European Commission claims that output in both the euro area and the EU is not expected to recover its pre-pandemic level in 2022. Hence, the forecasts contained in this article should be postponed by at least two years.

The low employment rate of young people is very disadvantageous from a social point of view. Already in the nineteenth century, Gustav Le Bon wrote: *The conditions of success in life are the possession of judgment, experience, initiative, and character – qualities which are not bestowed by books* (Le Bon (2015)). This message is still relevant today (compare with Bell et al. (2011) and Standing (2016)).

The increasing employment rate of older people is implied by demographic ageing. The European Commission suggests that policy makers should have incentivized older workers to remain longer in the labour market. Good health is for many an imperative for working beyond the pensionable age. The employment rate for old people depends strongly on the quality of health, quality of work, benefit system and other factors.

Low employment rates of young and old workers are connected with the phenomenon of a growing number of people with transient roles in the labour market, short spells of employment interspersed with unemployment spells of varied length that do not make life meaningful. Standing (2016) argues that this class of workers, called Precariat, will cause instability in society. Presence of a large Precariat raises the spectre of populist movements to restrict the movement of labour and to promote disenfranchisement of various vulnerable groups. The populist movements go against the mainstream media, universities, the political class, banks, and also against widely accepted truisms such as global warming and the benefits of free trade (Dustmann et. al (2017)).

Some implications of low employment were outlined. The issue is complex and requires detailed research. The aim of the article is not to explain the complexity of the labour market, but to state the pessimistic prognosis about employment rates in European countries. It seems that governments of many countries should revise their economic strategies affecting labour market if they want to achieve satisfactory employment rates

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STATISTICS
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Statistics Poland

An International Journal of the Polish Statistical Association and Statistics Poland

In Memoriam Professor Jan Kordos

On behalf of the Statistics in Transition new series' Editorial Board and Editorial Office, I regret to announce the death of Professor Jan Kordos – the founder of our journal and its long-time Editor-in-Chief (1993-2007).

Professor Kordos was a scientist whose passion for statistics distinguished him as an outstanding animator of broadly understood scientific life on many levels, where he also played leadership roles: as a long-term President of the Polish Statistical Association (1985-1994), as an academic teacher devoted to statistics and practitioner in various areas of its application – at home and abroad – and in the management of official statistics, including the function of the Vice-President of the Central Statistical Office (1992-1996).

Professor Kordos studied at the Jagiellonian University and the University of Wrocław, where he graduated from (in 1955) in mathematics under supervision of Hugo Steinhaus. He received a PhD in Econometrics from the University of Economics in Katowice in 1965. From the beginning of his professional career to almost the last years of his active life, he devoted himself to teaching at universities, mainly as a professor at SGH Warsaw School of Economics and Warsaw Management University.

Professor Kordos also trained experts abroad – including in the field of agricultural statistics in China (at the end of the 1980s) and in Nepal (Kathmandu, 1991). In 1994-1996, he served as a World Bank consultant for household surveys and implemented new household budget survey systems in Latvia and Lithuania. His publications included four books and over three hundred articles and other articles.

The community of statisticians in Poland and abroad is deeply saddened by the loss of one of its most outstanding and active representatives.

I express my deepest condolences to the family and relatives of Professor Jan Kordos.

Professor Włodzimierz Okrasa
Editor-in-Chief
Statistics in Transition new series

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STATISTICS
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An International Journal of the Polish Statistical Association and Statistics Poland

Many Congratulations to Professor Partha Lahiri

On behalf of the *Statistics in Transition new series (SiTns)* and its Editorial Board and Associate Editors,

I would like to congratulate Professor Partha Lahiri for receiving the Prize for Outstanding Contribution to Small Area Estimation (SAE Award) by the International Statistical Institute; it was given to Professor Lahiri at the SAE 2021 Conference on Big Data and Small Area Estimation in September 2021.

Professor Lahiri's contribution to statistics, especially to Small Area Estimation is well recognized since he has not only advanced research in the relevant domains but also inspired several researchers to publish in SiTns on topics directly or indirectly influenced by his own research.

Receiving the prestigious and much deserved award is a great achievement, which honours Professor Lahiri's hard work, determination and contribution to the development of the world statistics.

We are proud of our cooperation and grateful to Professor Lahiri also for the contribution to the increasing importance and recognition of our journal.

I would like to wish Professor Partha Lahiri many more such honours in the years to come.

Professor Włodzimierz Okrasa
Editor-in-Chief
Statistics in Transition new series

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About the Authors

Afzaal Mehreen completed her BS in Statistics, from International Islamic University, Islamabad, Pakistan. She completed her MPhil in Statistics, from Riphah International University, Islamabad, Pakistan. Her Research work is on the topic, "A study on exponentiated Gompertz distribution under Bayesian discipline using informative priors". Her research interests include, Bayesian Statistics and data analysis. Her supervisor Prof. Dr. Muhammad Aslam has more than 150 research publications in internationally reputed journals.

Agu Friday Ikechukwu is an Assistant Lecturer at the Department of Statistics, Faculty of Physical Sciences, University of Calabar, Calabar, Nigeria. His research interests are distribution theory with a focus on heavy-tailed distributions, copula, data science, and R-programming. He has published over 20 research papers in international/national journals, attended international/national conferences/workshops. Agu Friday Ikechukwu is an active member of many scientific professional bodies. Currently, he is an active reviewer to three international scientific journals in the field of statistics.

Ali Kareem Ahmed Ibrahim is a PhD researcher at the Department of Mathematical Statistics, Faculty of Graduate Studies for Statistical Research, Cairo University, Giza, Egypt. He has developed an early interest in probability distributions, parameter estimation and sampling techniques. In 2015, he acquired the Master's degree from the Faculty of Graduate studies and Statistical Research.

Alizadeh Morad is an Assistant Professor at the Department of Statistics, Faculty of Intelligent Systems Engineering and Data Science, Persian Gulf University, Bushehr, Iran. His main areas of interest include univariate continuous distributions, lifetime distributions and related inference. He has published over 130 papers.

AL-Olaimat Nesreen M. is a part-time instructor at the Department of Mathematics, The University of Jordan. She obtained a PhD degree in mathematical statistics from the University of Jordan in 2021. Her main areas of interest are statistical inference and adaptive statistical methods.

Aslam Muhammad is a Full Professor of Statistics Department of Mathematics and Statistics, Riphah International University, Pakistan. He has served for 25 years at the Quaid-i-Azam University, Islamabad in Pakistan. He is an expert of Bayesian statistical inference, mathematical statistics and applied statistics. He has published over 100 articles in reputed local and international journals.

Assar Salwa M. is an Associate Professor at the Mathematical Statistics, Faculty of Graduate Studies for Statistical Research (FGSSR), Cairo University, Egypt. Her main areas of interest include family of distributions, stochastic processes, statistical inference and data analysis in particular. Salwa Assar has published many research papers in international/national journals and conferences.

Bayoud Husam A. is an Associate Professor at the Department of Mathematics and Statistics, College of Sciences and Humanities at Fahad bin Sultan University. His main areas of interest include bioequivalence and bioavailability studies, Bayesian analysis, sequential tests, information theory and entropy measures, Life tests with censored data and Goodness of fit tests. Currently, he is a member of the editorial boards of Journal of Statistics: Advances in Theory and Applications, Bioequivalence & Bioavailability International Journal and International Journal of Applied Research.

Bhatti Muḥammad Ishaq, is a Full Professor of Finance, Financial Econometrics and Statistics at La Trobe University, Melbourne, Australia and jointly hold Distinguished Professorship at Minhaj University, Lahore Pakistan. His research interests are in financial statistics, Islamic finance, and statistical inference. Professor Bhatti is a member elect of the International Statistical Institute and ISOSS. He has published over 150 articles in international journals, eleven books and edited 28 books. He is the member of the editorial board of various journals and Editor-in-chief of J. statistical theory and Application.

Bulatova Olena is a Doctor of Economics, professor, First Vice-Rector, Mariupol State University. Her main areas of interest include investigation of theoretical and practical issues of the development of world economy, international economic relations, focusing on topical issues of integration processes and regionalization of world economy, development of international and regional competitiveness, interregional cooperation, European integration line of Ukrainian policy in terms of globalization. Prof. O. Bulatova has published over 200 research papers in international/national journals and conferences. She is a member of the Scientific Council of the Ministry of Education and Science of Ukraine (Section of Economics). Currently, she is a member of the editorial boards of Bulletin of Mariupol State University, Series: Economics (Ukraine), Journal of European Economy (Ukraine), Research Papers in Economics and Finance (Poland).

Chakraborty Subrata is a Full Professor in the Department of Statistics under the Faculty of science and engineering in the Dibrugarh University, Assam, India. Currently, he is the Head of the Department. His main research interests are probability distribution theory and applications, reliability analysis. Professor Chakraborty has published over 100 research papers in international/national journals. Professor Chakraborty is an active member of many scientific associations.

Danish Faizan is currently working as an Assistant Professor (Statistics) in Sher-e-Kashmir University of Agricultural Sciences & Technology of Jammu, J&K, India and has done PhD from the same university. He has worked as a Postdoctoral Fellow in New York University, New York, USA. Dr. Danish has around 2 years of teaching experience and published around 25 research papers in reputed journals. Further, he has worked as a Biostatiscian under Research Consultation Services, Doha Qatar. His has research expertise in sampling theory, mathematical, applied statistics and biostatistics. Dr. Danish has proposed several methods for obtaining stratification points utilizing the classical technique as well as Mathematical Programming approach. Dr. Danish is well versed with Statistical Software: R, STATA, SPSS, Python Matlab, Fortran 77, O.P STAT, WINDOSTAT, Mathematica, etc., and have completed several online courses related to software from prominent global universities.

Eghwerido Joseph Thomas is a Lecture at the Department of Statistics, Federal University of Petroleum Resources Effurun Delta State, Nigeria. His research interests are probability theory, distribution theory, data science, spatial statistics and data analysis in particular. Joseph Thomas Eghwerido (PhD) has published over 50 research papers in international/national journals and conferences. He has also published a book. Dr. Eghwerido is an active member of many scientific and statistical professional bodies.

Gurgul Henryk is a Full Professor of Economics at the AGH University of Science and Technology, Department of Management. His research interests include the application of mathematics in economics, in particular input-output models and financial econometrics. Professor Gurgul has published over 200 research papers in international and national journals (including 32 papers in journals included in JCR). Professor Gurgul is also an active member of several international scientific societies, including International Statistical Institute (Elected Member).

Hassan Amal Soliman is a Professor of Statistics at the Department of Mathematical Statistics, Faculty of Graduate Studies for Statistical Research at Cairo University, Egypt. She received her PhD degree in Statistics from the Faculty of Graduate Studies for Statistical Research (Institute of Statistical Studies & Research (previously)), Cairo University, Egypt. Her main research interests are: probability distributions, record values, ranked set sampling, stress-strength models, accelerated life tests, and goodness of fit tests and information measures. Professor Amal Soliman Hassan has published over 120 research papers in international/national journals and conferences.

Hastenteufel Jessica is a Professor for Business Administration at IU International University of Applied sciences, Department of Business and Management. Simultaneously she holds the position of a private lecturer at the Department of Economics, Faculty of Human and Business Sciences, Saarland University in Saarbruecken. Her main areas of interest include: digital banking, sustainable banking, SME financing, and financial markets.

Jaworski Stanisław is an Assistant Professor of the Department of Econometrics and Statistics, Warsaw University of Life Sciences. His research interests include statistical inference, data analysis, forecasting the unemployment and employment rate.

Longford Nicholas T. was until recently a Senior Statistician in the School of Public Health, Imperial College London, England. Now he is the Director of SNTL Statistics Research and Consulting, reverting to his role in years 2004-15. He is an author of seven monographs, including Statistics for Making Decisions (Chapman and Hall/CRC, 2021). His areas of interest include causal analysis of observational studies, institutional performance (statistical audit), missing data and statistical computing. He is an Editor of Wiley StatsRef and a reviewer for Mathematical Reviews.

Nagy Heba Fathy is a Lecturer at the Department of Mathematical Statistics, Faculty of Graduate Studies for Statistical Research, Cairo University, Giza, Egypt. She developed an early interest in record values, ranked set sampling techniques, entropy estimation and stress-strength reliability. Heba Fathy Nagy has published over 9 research papers in international/national journals and conferences. In 2019, she acquired the PhD from the Faculty of Graduate studies and Statistical Research.

Osaulenko Oleksandr is the Rector of the National Academy of Statistics, Accounting and Audit, Doctor of Public Administration, Professor, Corresponding Member of the National Academy of Sciences of Ukraine, Honored Economist of Ukraine. During 1996-2014, he headed the national statistical office of Ukraine. He is an author of over 240 scientific works, including 21 monographs and handbooks on the problems of statistics, public administration and information security.

Rahman Umme Habibah is a PhD research scholar in Statistics (Assam University, Silchar). Her areas of interest are probability distribution, time series, forecasting and data analysis in particular. She completed her master's degree from Department of Statistics, Gauhati University, Assam. She has published seven research articles in different reputed journals. She has also published two book chapters. She is teaching time series and distribution theory for post-graduation students.

Raqab M. Z. is a Full Professor of the Department of Mathematics at the University of Jordan. He received his PhD from the Ohio State University. His research interests include statistical models involving ordered data analysis, optimal evaluations of statistical functionals, biostatistical applications on censoring data, reliability theory, intensive computer studies. He has over 150 publications and received various research awards during the last two decades (among them, Khalifa Award for Education (2012), Hamdi Mango Award for Distinguished Researcher (2012), Abdul-Hameed Shoman

Prize (2001), TWAS Prize to Youngest Scientists (1998)). In this line, he is a member of the editorial board in several international journals.

Reznikova Nataliia is a Full Professor of Economics, Department of World Economy and International Economic Relations, Institute of International Relations Taras Shevchenko National University of Kyiv. She has extensive experience in research and practical work in the field of international economic policy, regulation of international economic relations, international trade, analysis of strategies of economic development of countries, the development of mechanisms, incentive tools and instruments for the promotion of investments and innovations. She is an author of over 400 scientific articles indexed in international scientific databases.

Rizvi S. E. H. started his career as a Lecturer at IAAS, Tribhuvan University, Nepal in 1987. Presently, he is working as a Professor (Statistics)/Dean, Faculty of Basic Sciences at SKUAST-Jammu, India, where he joined as an Assistant Professor in 1989. He has over 34 years of teaching experience at UG&PG levels with active involvement in management and planning of various academic activities. He has vast research experience in the field of Sampling Theory & Applied Statistics and supervised MSc/PhD students of Statistics. Besides, as a member of advisory committee of MSc/PhD students of agriculture & allied sciences he guided them regarding planning of experiments and analysis of research data. Prof. Rizvi has a considerable number of research publications in journals of national and international repute and is associated in research projects in different capacities.

Sehgal Shikha is an Assistant Professor at the Department of Statistics, Faculty of Mathematical Sciences, University of Delhi. Her research interests include order statistics, multivariate analysis, reliability, censoring, statistical inference and data analysis. Dr. Sehgal has published many research papers in international/national journals and conferences.

Shanker Rama is an Associate Professor and Head of the Department of Statistics, Assam University, Silchar, Assam, India. His research interests are distribution theory, statistical inference, lifetime data, and biostatistics. He has published around 175 research papers in international and national journals of statistics, mathematics and biostatistics. He has introduced several new lifetime distributions in statistics. He has published one book titled "An Introduction to Calculus for Business and Economics for the undergraduate and postgraduate students of Economics, Management and Finance". He was the founder editor-in-chief of Eritrean Journal of Science and Engineering, published from the Eritrea Institute of technology, Eritrea. He is a member of the editorial boards of several journals of statistics and members of some statistical societies.

Wójtowicz Tomasz is an Assistant Professor at the AGH University of Science and Technology, Department of Management. The main areas of his research interest include efficiency of financial markets, relationships between stock markets, and asset pricing. He is an author and co-author of 49 scientific papers published in domestic and international journals. Six of them is included in JCR.

Zakharova Olha is an Associate Professor at the Economics and International Economic Relations Department, Faculty of Law and Economics, Mariupol State University, PhD in Economic Sciences. Her main areas of interest include: modelling and forecasting of global economic development, the problems of assessment of economic security level, economic and mathematical models, statistical analysis. She has published over 110 research papers in international/national journals and conferences. She is also a co-author of five collective monographs.

Zielińska-Kolasińska Zofia is a research and teaching assistant at the Siedlce University of Natural Sciences and Humanities. Her research interests are Bayesian methods in actuarial science and forecasting the unemployment and employment rates.

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