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# Record data from Kies distribution and related statistical inferences

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## ABSTRACT

The Kies probability model was proposed as an alternative to the extended Weibull models as it provides a more efficient fit to some real-life data sets in comparison to the aforementioned models. The paper proposes classical and Bayesian inferences for the Kies distribution based on records. Maximum likelihood estimates are studied jointly with asymptotic and bootstrap confidence intervals. Moreover, Bayes estimates, along with credible intervals are discussed assuming squared and LINEX loss functions. The proposed estimation methods have been investigated and compared via simulation studies. A real data set has been analysed for illustrative purposes.

**Key words:** Bayesian estimates, Kies distribution, maximum likelihood estimation, records.

## 1. Introduction

For its importance in many practical fields, the Weibull distribution has received the attention of several authors in the literature. Moreover, many modified versions of the Weibull distribution were developed in the literature. One of the modified versions of the Weibull distribution is known as Kies Distribution and was firstly proposed by Kies (1958). Recently, Kies distribution has received the attention of different authors, including Kumar and Dharmaja (2014), who studied some of its important statistical aspects and showed that it possess increasing, decreasing and bathtub hazard rate functions that would make it a good alternative for some versions of the extended Weibull distributions, namely the generalized Weibull (GW) distribution, modified Weibull (MW) distribution, beta Weibull (BW) distribution and beta generalized Weibull (BGW) distribution. In 2013, Kumar and Dharmaja studied the one-parameter Kies distribution as a special case, called the reduced Kies (RK) distribution, which is shown to possess certain special properties that are analogous to those of the Weibull distribution. In 2017, they proposed a generalized version of the extended reduced Kies distribution, called a modified Kies (MK) distribution, see Kumar and Dharmaja (2017a). In addition, Kumar and Dharmaja (2017b) introduced and studied an exponentiated reduced Kies distribution with two parameters.

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The cumulative distribution function (CDF), probability density function (PDF), hazard rate and cumulative hazard rate functions of the two-parameter Kies distribution K($\lambda$, $\beta$) are given by:

$$F(x;\lambda,\beta) = 1 - e^{-\lambda\left(\frac{x}{1-x}\right)^{\beta}}, \tag{1}$$

$$f(x;\lambda,\beta) = \frac{\beta\lambda x^{\beta-1}}{(1-x)^{\beta+1}} e^{-\lambda\left(\frac{x}{1-x}\right)^{\beta}}, \tag{2}$$

$$h(x;\lambda,\beta) = \frac{\beta\lambda x^{\beta-1}}{(1-x)^{\beta+1}}, \tag{3}$$

and

$$H(x;\lambda,\beta) = \lambda\left(\frac{x}{1-x}\right)^{\beta}, \tag{4}$$

respectively, where $0 < x < 1$, $\lambda > 0$ and $\beta > 0$.

The Kies distribution has a bounded range, which makes it appropriate model for fitting real data sets with a bounded range. However, there are many situations in which the observations can take values only in a limited range, like proportions, percentages or fractions. Papke and Wooldridge (1996) pointed out that variables in many economic applications such as the fraction of total weekly hours spent on working, the proportion of income spent on non-durable consumption, industry market shares, and a fraction of land area allocated to agriculture are all bounded between zero and one. Moreover, Genc (2013) indicated that when the reliability is measured as a percentage or ratio, it is important to have models defined on the unit interval in order to have reasonable results.

This paper studies classical and Bayesian inferences for the parameters of the Kies distribution based on records. Records play an important role in several fields of statistics which date back to Chandler (1952), who firstly defined and provided groundwork for mathematical theory of records. However, record statistics arise in many practical fields including hydrology, meteorology, sporting and athletic events wherein only records are usually considered, for more details and applications on records, readers may refer to Arnold et al. (1998), Ahsanullah (2004), Ahsanullah and Raqab (2006) and Ahsanullah and Nevzorov (2015).

Let $\{X_j, j \geq 1\}$ be a sequence of independent and identically distributed (iid) continuous random variables (r.v.'s) with CDF $F(x)$ and PDF $f(x)$. An observation $X_j$ is defined to be an upper record if $X_j > X_i$ for every $j > i$, and an analogous definition can be given for lower records (with the inequality being reversed). By convention, the first record $X_1$ is called the trivial record because it is an upper and a lower record value simultaneously.

The set of the upper record values is given by the r.v.'s $X_{U(k)}$ for $k \geq 1$ where
$U(1) = 1$, $U(k) = min\{j : j > U(k-1), X_j > X_{U(k-1)}\}$.

Suppose we have a random sample (not ordered) of size $n$, say $\{X_1, X_2, ..., X_n\}$, the set

$$\{X_{U(1)} = X_1, X_{U(2)}, ..., X_{U(m)}\},$$

presents a set of upper record values with size $1 \leq m \leq n$ that is obtained from the random sample. The sequence $U(k)$, $k \geq 1$ is called the sequence of upper record times. For simplicity, we denote the sequence of upper record values $\{X_{U(j)}\}_{j=1}^{m}$ by $\{Y_j\}_{j=1}^{m}$.

In this paper, we will need the following lower and upper incomplete gamma functions

$$\int_0^z t^{\alpha-1}e^{-\mu t}dt = \mu^{-\alpha}\gamma(\alpha,\mu z), \tag{5}$$

$$\int_z^\infty t^{\alpha-1}e^{-\mu t}dt = \mu^{-\alpha}\Gamma(\alpha,\mu z), \tag{6}$$

respectively. Additionally,

$$\int_z^\infty t^{-\alpha}e^{-t}dt = z^{\frac{-\alpha}{2}}e^{(\frac{-\alpha}{2})}W_{-\frac{\alpha}{2},(\frac{1-\alpha}{2})}(z), \tag{7}$$

where $W_{c_1,c_2}(g)$ is the Whittaker function, which is defined, for $|\arg(-g)| < \frac{3\pi}{2}$, as

$$W_{c_1,c_2}(g) = \frac{\Gamma(-2c_2)}{\Gamma(\frac{1}{2}-c_2-c_1)}M_{c_1,c_2}(g) + \frac{\Gamma(2c_2)}{\Gamma(\frac{1}{2}+c_2-c_1)}M_{c_1,-c_2}(g), \tag{8}$$

in which

$$M_{c_1,c_2}(g) = e^{-\frac{g}{2}}g^{c_2+\frac{1}{2}}\sum_{k=0}^{\infty}\left\{\frac{(\frac{1}{2}-c_1+c_2)_k}{(1+2c_2)_k}\frac{g^k}{k!}\right\}, \tag{9}$$

the series given in Eq. (9) converges for all finite values of g. Also, the pochhammer symbol is defined as follows:

$$(a)_k = a(a+1)(a+2)...(a+k-1) = \frac{\Gamma(a+k)}{\Gamma(a)} = \prod_{i=1}^{k}(a+i-1), \tag{10}$$

where $(a)_0 = 1$ and $(1)_k = k!$.

The rest of this paper is organized as follows: forms of the single moment and some properties for records from K($\lambda$, $\beta$) are derived in Section 2. In Section **??**, classical estimation methods are proposed for the parameters of the Kies distribution based on upper records. In Section 4, the Bayes estimators based on the squared error and linear exponential loss functions are computed using gamma priors for the two unknown parameters. Further in Section 5, we consider a real data set for illustrative purposes. In Section 6, simulation studies are carried out in order to study the performance of the proposed estimation methods. Finally, the paper is concluded in Section 7.

## 2. Distributional properties of records from Kies distribution

The aim of this section is to present some properties and derive the form of the $kth$ moment of the $m^{th}$ record from K($\lambda$, $\beta$). The PDF of the $m^{th}$ record value and the joint PDF of the $m^{th}$ and $s^{th}$ records are given, respectively, Arnold et al. (1998) by

$$f_m(y) = \frac{[H(y)]^{m-1}}{\Gamma(m)}f(y), \tag{11}$$

and

$$f_{m,s}(y,z) = \frac{[H(y)]^{m-1}}{\Gamma(m)}h(y)\frac{[H(z)-H(y)]^{s-m-1}}{\Gamma(s-m)}f(z), \tag{12}$$

where $-\infty < y < z < \infty$, $H(.)$ and $h(.)$ are the cumulative hazard and the hazard rate functions, respectively.

**Result 2.1.** *By using Eqs. (2), (3) and (4), the PDF of the $m^{th}$ record and the joint PDF of the $m^{th}$ and $s^{th}$ records from K($\lambda$, $\beta$) given in Eqs. (11) and (12),respectively, become*

$$f_m(y) = \frac{\beta\lambda^m}{\Gamma(m)}(\frac{y}{1-y})^{m\beta}\frac{1}{y(1-y)}e^{-\lambda(\frac{y}{1-y})^{\beta}}, \tag{13}$$

$$f_{m,s}(y,z) = \frac{\lambda^s\beta^2}{\Gamma(m)}(\frac{y}{1-y})^{m\beta}\frac{1}{y(1-y)}\frac{[(\frac{z}{1-z})^{\beta}-(\frac{y}{1-y})^{\beta}]^{s-m-1}}{\Gamma(s-m)}\frac{z^{\beta-1}}{(1-z)^{\beta+1}}e^{-\lambda(\frac{z}{1-z})^{\beta}}, \tag{14}$$

*where* $0 < y < z < 1$ *and* $\lambda$, $\beta > 0$.

Using Eqs. (13) and (5), the CDF $F_m$ of the $m^{th}$ record value from the Kies distribution is given by

$$F_m(y) = \frac{\gamma(m,\lambda(\frac{y}{1-y})^{\beta})}{\Gamma(m)}, m \geq 1, \tag{15}$$

where $0 < y < 1$ and $\lambda$, $\beta \geq 0$.

**Result 2.2.** *Suppose that the random variable X follows a Kies distribution. Then, one can prove that*

$$X \overset{D}{=} \frac{(\frac{1}{\lambda}X^*)^{\frac{1}{\beta}}}{1+(\frac{1}{\lambda}X^*)^{\frac{1}{\beta}}},$$

*where D means converges in distribution and* $X^* = -\log(1-U)$ *where U is Uniform(0, 1). It is obvious that* $X^*$ *follows a standard exponential distribution. Consequently, using the result, A.4.10, Page(174) of Houchens (1984), the corresponding sequence of records can be described by*

$$Y_m \overset{D}{=} \frac{(\frac{1}{\lambda}\sum_{i=1}^{m}X_i^*)^{\frac{1}{\beta}}}{1+(\frac{1}{\lambda}\sum_{i=1}^{m}X_i^*)^{\frac{1}{\beta}}}, \tag{16}$$

*where* $\{X_i^*\}_{i=1}^{m}$ *is a sequence of* $i.i.d.$ $Exp(1)$ *random variables*

**Result 2.3.** *If the random variable X has a Kies distribution, then $k^{th}$ moment* $\mu_m^{(k)} = E(Y_m^k)$ *for the $m^{th}$ record from the Kies distribution is given by*

$$\begin{aligned}\mu_m^{(k)} = \Psi(m,\lambda,\beta,k) =& \frac{1}{\Gamma(m)}\sum_{j=0}^{\infty}(-1)^j\frac{(k)_j}{j!}\lambda^{-(\frac{k+j}{\beta})}\gamma(m+\frac{k+j}{\beta},\lambda) \\ &+ \frac{1}{\Gamma(m)}\sum_{j=0}^{\beta(m-1)}(-1)^j\frac{(k)_j}{j!}\lambda^{\frac{j}{\beta}}\Gamma(m-\frac{j}{\beta},\lambda) \\ &+ \frac{1}{\Gamma(m)}[\sum_{j=\beta(m-1)+1}^{\infty}(-1)^j\frac{(k)_j}{j!}\lambda^{\frac{m}{2}+\frac{j}{2\beta}-\frac{1}{2}}e^{\frac{m}{2}-\frac{j}{2\beta}-\frac{1}{2}} \\ &\times W_{\frac{m}{2}-\frac{j}{2\beta}-\frac{1}{2},(\frac{m}{2}-\frac{j}{2\beta})}(\lambda)].\end{aligned} \tag{17}$$

*Proof.* By using Eq. (13), the $k^{th}$ moment for the $m^{th}$ record from the Kies distribution is

$$E(Y_m^k) = \int_0^1 \frac{\beta\lambda^m}{\Gamma(m)}(\frac{y_m}{1-y_m})^{m\beta}\frac{y_m^k}{y_m(1-y_m)}e^{-\lambda(\frac{y_m}{1-y_m})^\beta}dy_m. \tag{18}$$

On substituting $(\frac{y_m}{1-y_m})^\beta = t$ in Eq(18), we get

$$E(Y_m^k) = \frac{\lambda^m}{\Gamma(m)}\int_0^\infty (\frac{t^{\frac{1}{\beta}}}{1+t^{\frac{1}{\beta}}})^k t^{m-1}e^{-\lambda t}dt.$$

On splitting the integral and expanding $(1+t^{\frac{1}{\beta}})^{-k}$ using Newton's Generalization of the binomial theorem, we get the following

$$E(Y_m^k) = \frac{\lambda^m}{\Gamma(m)}\int_0^1 \frac{t^{m+\frac{k}{\beta}-1}}{(1+t^{\frac{1}{\beta}})^k}e^{-\lambda t}dt + \frac{\lambda^m}{\Gamma(m)}\int_1^\infty \frac{t^{m+\frac{k}{\beta}-1}}{t^{\frac{k}{\beta}}(t^{\frac{-1}{\beta}}+1)^k}e^{-\lambda t}dt, \tag{19}$$

$$\begin{aligned} E(Y_m^k) = {} & \frac{\lambda^m}{\Gamma(m)}\sum_{j=0}^{\infty}\frac{(-1)^j(k)_j}{j!}\int_0^1 (t^{\frac{k+j+m\beta}{\beta}-1})e^{-\lambda t}dt \\ & + \frac{\lambda^m}{\Gamma(m)}\sum_{j=0}^{\infty}\frac{(-1)^j(k)_j}{j!}\int_1^\infty (t^{\frac{m\beta-j}{\beta}-1})e^{-\lambda t}dt, \end{aligned} \tag{20}$$

where $(.)_j$ is the Pochhammer symbol given by (10), if we put $u = \lambda t$ we get

$$\begin{aligned} E(Y_m^k) = {} & \frac{1}{\Gamma(m)}\sum_{j=0}^{\infty}\frac{(-1)^j(k)_j}{j!}\lambda^{-\frac{k+j}{\beta}}\int_0^\lambda (u^{\frac{k+j+m\beta}{\beta}-1})e^{-u}du \\ & + \frac{1}{\Gamma(m)}\sum_{j=0}^{\infty}\frac{(-1)^j(k)_j}{j!}\lambda^{\frac{j}{\beta}}\int_\lambda^\infty (u^{\frac{m\beta-j}{\beta}-1})e^{-u}du, \end{aligned} \tag{21}$$

since the exponent $m-\frac{j}{\beta}$ in the second integral carries positive and negative values, therefore, on splitting the second summation we get the following:

$$\begin{aligned} E(Y_m^k) = {} & \frac{1}{\Gamma(m)}\sum_{j=0}^{\infty}\frac{(-1)^j(k)_j}{j!}\lambda^{-\frac{k+j}{\beta}}\int_0^\lambda (u^{\frac{k+j+m\beta}{\beta}-1})e^{-u}du \\ & + \frac{1}{\Gamma(m)}\sum_{j=0}^{\beta(m-1)}\frac{(-1)^j(k)_j}{j!}\lambda^{\frac{j}{\beta}}\int_\lambda^\infty u^{m-\frac{j}{\beta}-1}e^{-u}du \\ & + \frac{1}{\Gamma(m)}\sum_{j=\beta(m-1)+1}^{\infty}\frac{(-1)^j(k)_j}{j!}\lambda^{\frac{j}{\beta}}\int_\lambda^\infty u^{-(1+\frac{j}{\beta}-m)}e^{-u}du, \end{aligned} \tag{22}$$

which leads to (17) in the light of (5), (6) and (7). □

The expected value of the $m^{th}$ record value $[E(Y_m)]$ is the first moment, which is given by:

$$\mu_m^{(1)} = \Psi(m,\lambda,\beta,1).$$

In addition, the variance of the $m^{th}$ record value is

$$var(Y_m) = \Psi(m,\lambda,\beta,2) - [\Psi(m,\lambda,\beta,1)]^2.$$

For illustrative purposes, $E(Y_m)$ and variance of some records of the Kies distribution, namely $3^{rd}$, $5^{th}$, $7^{th}$ and $10^{th}$, are computed and summarized in Tables (1) and (2) assuming different values of $\lambda$ and $\beta$. It can be observed from these tables that $E(Y_m)$(Variance) increases(decreases) with m, which is expected.

Table 1: Expected values and variances of records from K($\lambda$, $\beta$) with $\lambda$= 0.75 and 1

| m | $\lambda=0.75$ | | | | $\lambda=1$ | | | |
|---|---|---|---|---|---|---|---|---|
| | $\beta=0.75$ | | $\beta=2$ | | $\beta=0.75$ | | $\beta=2$ | |
| | $E(Y_m)$ | Variance | $E(Y_m)$ | Variance | $E(Y_m)$ | Variance | $E(Y_m)$ | Variance |
| 3 | 0.80600 | 0.01810 | 0.64400 | 0.00519 | 0.74800 | 0.02350 | 0.61100 | 0.00549 |
| 5 | 0.90300 | 0.00387 | 0.70800 | 0.00242 | 0.86600 | 0.00625 | 0.67800 | 0.00267 |
| 7 | 0.94000 | 0.00111 | 0.74500 | 0.00142 | 0.95500 | 0.00201 | 0.71700 | 0.00161 |
| 10 | 0.96400 | 0.00027 | 0.77900 | 0.00080 | 0.97800 | 0.00021 | 0.75400 | 0.00092 |

Table 2: Expected values and variances of records from K($\lambda$, $\beta$) with $\lambda$= 2 and 3

| m | $\lambda=2$ | | | | $\lambda=3$ | | | |
|---|---|---|---|---|---|---|---|---|
| | $\beta=0.75$ | | $\beta=2$ | | $\beta=0.75$ | | $\beta=2$ | |
| | $E(Y_m)$ | Variance | $E(Y_m)$ | Variance | $E(Y_m)$ | Variance | $E(Y_m)$ | Variance |
| 3 | 0.57000 | 0.03240 | 0.52800 | 0.00585 | 0.45300 | 0.03110 | 0.47900 | 0.00577 |
| 5 | 0.73200 | 0.01470 | 0.59900 | 0.00317 | 0.62400 | 0.01920 | 0.55000 | 0.00332 |
| 7 | 0.81600 | 0.00659 | 0.64200 | 0.00203 | 0.72600 | 0.01060 | 0.59500 | 0.00222 |
| 10 | 0.88200 | 0.00228 | 0.68500 | 0.00124 | 0.81500 | 0.00450 | 0.63900 | 0.00140 |

## 3. Classical estimation

### 3.1. Maximum likelihood estimation

Let $data = \{y_1, y_2, \ldots, y_m\}$ be the first $m$ upper record values arising from a sequence of iid K($\lambda$, $\beta$) with CDF, PDF and hazard rate being defined in Eqs. (1), (2) and (3), respectively. The likelihood function of the $data$ is given by (see Arnold et al. (1998).

$$\begin{aligned} L(data;\lambda,\beta) &= f(y_m;\lambda,\beta)\prod_{i=1}^{m-1} h(y_i;\lambda,\beta) \\ &= \beta^m\lambda^m e^{-\lambda(\frac{y_m}{1-y_m})^\beta}\prod_{i=1}^{m}\frac{y_i^{\beta-1}}{(1-y_i)^{\beta+1}}. \end{aligned} \tag{23}$$

Thus, the log-likelihood function $l(data|\lambda,\beta) = \log L(data;\lambda,\beta)$ can be written as

$$l(data|\lambda,\beta) = m\log\lambda + m\log\beta - \lambda(\frac{y_m}{1-y_m})^\beta + (\beta-1)\sum_{i=1}^{m}(\log y_i) - (\beta+1)\sum_{i=1}^{m}\log(1-y_i), \tag{24}$$

where $0 < y_1 < y_2 < ... < y_m < 1$, $\beta > 0$ and $\lambda > 0$. The following proposition shows the existence and uniqueness of the MLEs of $\lambda$ and $\beta$.

**Proposition 3.1.** *The log-likelihood function $l(data|\lambda,\beta)$ is unimodal function of $\lambda$ and $\beta$.*

*Proof.* Note that $l(data|\lambda,\beta)$ is a continuous function in $\lambda$ and $\beta$, and is strictly concave as the Hessian matrix is negative definite. Thus, $l(data|\lambda,\beta)$ is unimodal of $\lambda$ and $\beta$.
This shows the existence and uniqueness of the MLEs of the unknown parameters $\lambda$ and $\beta$. □

Substituting $R_i = \frac{y_i}{1-y_i}, i = 1,2,...,m$ and solving the following system of equations (equations 25 and 26)

$$0 = \frac{\partial l(data|\lambda,\beta)}{\partial \lambda} = \frac{m}{\lambda} - R_m^{\beta}, \tag{25}$$

$$0 = \frac{\partial l(data|\lambda,\beta)}{\partial \beta} = \frac{m}{\beta} - \lambda R_m^{\beta} \log R_m + \sum_{i=1}^{m} \log(R_i), \tag{26}$$

we immediately obtain the MLEs of $\beta$ and $\lambda$ as

$$\hat{\beta} = \frac{m}{\sum_{i=1}^{m-1} \log(\frac{R_m}{R_i})}, \tag{27}$$

and

$$\hat{\lambda} = \frac{m}{R_m^{\hat{\beta}}}. \tag{28}$$

### 3.2. Asymptotic confidence interval

Since it is not easy to derive the exact distribution of the MLEs, we cannot obtain the exact confidence intervals (CIs) for the parameters $\lambda$ and $\beta$. Consequently, the asymptotic CIs (ACIs) of the parameters are derived using the asymptotic distribution of the MLEs. To this end, we need to find the variance-covariance matrix of the MLEs. The observed information matrix of $\lambda$ and $\beta$ is given by

$$I(\lambda,\beta) = -\begin{pmatrix} \frac{\partial^2 l(data|\lambda,\beta)}{\partial^2 \lambda} & \frac{\partial^2 l(data|\lambda,\beta)}{\partial \lambda \partial \beta} \\ \frac{\partial^2 l(data|\lambda,\beta)}{\partial \beta \partial \lambda} & \frac{\partial^2 l(data|\lambda,\beta)}{\partial^2 \beta} \end{pmatrix},$$

where

$$\begin{aligned} \frac{\partial^2 l(data|\lambda,\beta)}{\partial^2 \lambda} &= -\frac{m}{\lambda^2}, \\ \frac{\partial^2 l(data|\lambda,\beta)}{\partial \lambda \partial \beta} &= \frac{\partial^2 l(data|\lambda,\beta)}{\partial \beta \partial \lambda} = -R_m^{\beta} \log R_m, \\ \frac{\partial^2 l(data|\lambda,\beta)}{\partial^2 \beta} &= -(\frac{m + \lambda \beta^2 R_m^{\beta} \log^2 R_m}{\beta^2}). \end{aligned}$$

Therefore, the approximate variance−covariance matrix for the MLE of $\theta = (\lambda, \beta)$ is given by

$$V = -\begin{pmatrix} \frac{\partial^2 l(data|\lambda,\beta)}{\partial^2\lambda} & \frac{\partial^2 l(data|\lambda,\beta)}{\partial\lambda\partial\beta} \\ \frac{\partial^2 l(data|\lambda,\beta)}{\partial\beta\partial\lambda} & \frac{\partial^2 l(data|\lambda,\beta)}{\partial^2\beta} \end{pmatrix}^{-1}_{(\lambda,\beta)=(\hat{\lambda},\hat{\beta})} = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix}, \tag{29}$$

where

$$\begin{aligned} V_{11} &= \frac{m + \lambda\beta^2 R_m^{\beta} \log^2(R_m)}{\frac{m}{\lambda^2}(m + \lambda\beta^2 R_m^{\beta}\log^2(R_m)) - \beta^2(R_m^{\beta}\log(R_m))^2} \\ V_{12} &= V_{21} = \frac{-R_m^{\beta}\log(R_m)}{\frac{m}{\lambda^2\beta^2}(m + \lambda\beta^2 R_m^{\beta}\log^2(R_m)) - (R_m^{\beta}\log(R_m))^2} \\ V_{22} &= \frac{1}{\frac{m}{\beta^2} + \lambda R_m^{\beta}\log^2(R_m) - \frac{(\lambda R_m^{\beta}\log(R_m))^2}{m}}. \end{aligned}$$

The asymptotic joint distribution of the MLEs $\hat{\lambda}$ and $\hat{\beta}$ is approximated by bivariate normal, and is given by:

$$\begin{pmatrix} \hat{\lambda} \\ \hat{\beta} \end{pmatrix} \overset{N}{\sim} \left[ \begin{pmatrix} \lambda \\ \beta \end{pmatrix}, \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix} \right]. \tag{30}$$

Hence, by replacing $\lambda$ and $\beta$ by their MLEs, we get an estimate of V as follows:

$$\hat{V} = \begin{pmatrix} \frac{m}{(R_m^{\hat{\beta}})^2}(1 + \hat{\beta}^2\log^2 R_m) & \frac{-\hat{\beta}^2\log R_m}{R_m^{\hat{\beta}}} \\ \frac{-\hat{\beta}^2\log R_m}{R_m^{\hat{\beta}}} & \frac{\hat{\beta}^2}{m}. \end{pmatrix}$$

Consequently, asymptotic $100(1-\alpha)\%$ CIs for the parameters $\lambda$ and $\beta$ are, respectively, given by:

$$(L_\lambda, U_\lambda) = (\hat{\lambda} - z_{1-\frac{\alpha}{2}}\sqrt{\hat{V}_{11}}), \hat{\lambda} + z_{1-\frac{\alpha}{2}}\sqrt{\hat{V}_{11}}), \tag{31}$$

and

$$(L_\beta, U_\beta) = (\hat{\beta} - z_{1-\frac{\tau}{2}}\sqrt{\hat{V}_{22}}, \hat{\beta} + z_{1-\frac{\tau}{2}}\sqrt{\hat{V}_{22}}), \tag{32}$$

where $z_\alpha$ is $100\alpha^{th}$ percentile of the standard normal distribution. However, some cases provide negative lower bounds of the asymptotic CI while the parameters $\lambda$ and $\beta$ are positive. In order to avoid such a case, we propose using a log−transformation for parameters in order to construct a modified asymptotic confidence intervals for $\lambda$ and $\beta$ following the lines of Ren and Gui (2020). Since, for a parameter, $\eta$, $g(\eta) = \log(\eta)$ is differentiable with $g\prime(\eta) \neq 0$, hence $Var[g(\hat{\eta})] = \frac{Var(\hat{\eta})}{\hat{\eta}^2}$. Therefore, modified asymptotic $(1-\alpha)100\%(0 < \alpha < 1)$ CIs for $\lambda$ and $\beta$ can be easily obtained, respectively, as follows:

$$\left( \frac{\hat{\lambda}}{e^{\frac{z_{1-\frac{\tau}{2}}}{\hat{\lambda}}\sqrt{V_{11}}}}, \hat{\lambda} e^{\frac{z_{1-\frac{\tau}{2}}}{\hat{\lambda}}\sqrt{V11}} \right) \text{ and } \left( \frac{\hat{\beta}}{e^{\frac{z_{1-\frac{\tau}{2}}}{\hat{\beta}}\sqrt{V_{22}}}}, \hat{\beta} e^{\frac{z_{1-\frac{\tau}{2}}}{\hat{\beta}}\sqrt{V_{22}}} \right). \tag{33}$$

### 3.3. Bootstrap method

Since the asymptotic CIs results do not perform quite well for a small sample size, the percentile Bootstrap method, which is denoted by Boot-p, is presented in this section to construct approximate CIs for $\lambda$ and $\beta$ using the following algorithm, see for example, Ahmed (2014):

Step 1) From the records $y_1, y_2, ..., y_m$, compute the MLEs $\hat{\lambda}_{ML}$ and $\hat{\beta}_{ML}$.

Step 2) Using $\hat{\lambda}_{ML}$ and $\hat{\beta}_{ML}$ that are obtained in Step 1, generate a random sample of records from K($\lambda$, $\beta$), called a bootstrap sample.

Step 3) Based on the Bootstrap sample that is obtained in Step 2, compute the corresponding MLEs $\hat{\lambda}^*$ and $\hat{\beta}^*$ of $\lambda$ and $\beta$, respectively.

Step 4) Repeat Steps (2) and (3) B-times to obtain $\{\hat{\lambda}_1^*, \hat{\lambda}_2^*, \ldots, \hat{\lambda}_B^*\}$ and $\{\hat{\beta}_1^*, \hat{\beta}_2^*, \ldots, \hat{\beta}_B^*\}$.

Step 5) Arrange $\{\hat{\lambda}_1^*, \hat{\lambda}_2^*, ..., \hat{\lambda}_B^*\}$ and $\{\hat{\beta}_1^*, \hat{\beta}_2^*, \ldots, \hat{\beta}_B^*\}$ in ascending order and obtain $\{\hat{\lambda}_{(1)}^*, \hat{\lambda}_{(2)}^*, \ldots, \hat{\lambda}_{(B)}^*\}$ and $\{\hat{\beta}_{(1)}^*, \hat{\beta}_{(2)}^*, ..., \hat{\beta}_{(B)}^*\}$.

Step 6) The approximate $100(1-\alpha)\%$ Boot-p CIs for $\lambda$ and $\beta$ are given by $\left(\hat{\lambda}^*_{(B\frac{\alpha}{2})}, \hat{\lambda}^*_{(B(1-\frac{\alpha}{2}))}\right)$ and $\left(\hat{\beta}^*_{(B\frac{\alpha}{2})}, \hat{\beta}^*_{(B(1-\frac{\alpha}{2}))}\right)$, respectively.

## 4. Bayesian estimation

In this section, we derive the posterior densities of the parameters $\beta$ and $\lambda$ based on the upper record values, then obtain the corresponding Bayes estimates of these parameters under different loss functions. Symmetric and asymmetric loss functions are considered in our study, which are squared error (SE) and linear exponential (LINEX) loss functions. The SE loss function of the parameter $\eta$ and an estimate $\hat{\eta}$ is given by:

$$L_{SE}(\hat{\eta}, \eta) = (\hat{\eta} - \eta)^2. \tag{34}$$

As the SE loss function leads to identical penalization for underestimation and overestimation, an asymmetric loss function, known as LINEX loss function, was proposed by Zellner (1986). The LINEX loss function of the parameter $\eta$ and an estimate $\hat{\eta}$ is given by:

$$L_{LINEX}(\hat{\eta}, \eta) = b[e^{\nu(\hat{\eta}-\eta)} - \nu(\hat{\eta} - \eta) - 1], \tag{35}$$

where $b > 0$ is the scale of the loss function. In our study, we assume $b = 1$. The parameter $\nu \neq 0$ indicates the shape parameter of the loss function. The LINEX loss function is affected by $\nu$, the sign of $\nu$ indicates the direction of the asymmetry, and the magnitude of $\nu$ indicates the degree of the asymmetry. It is known that assuming $\nu > 0$ means that overestimation is considered to be more costly than underestimation, while assuming $\nu < 0$ means the reverse situation, and when $\nu$ is close to zero, the LINEX loss function is almost symmetric and is approximately equal to the SE loss function. Thus, for small values of $\nu$, estimation results obtained by both LINEX and SE are close, for more details about the LINEX loss function readers may refer to Zellner (1986).

A natural choice of the priors of $\lambda$ and $\beta$ would be to assume that the two quantities are independent with gamma distributions; namely $Gamma(a_1,b_1)$ and $Gamma(a_2,b_2)$, respectively, where the hyper-parameters $a_1,a_2,b_1$ and $b_2$ are nonnegative numbers chosen to reflect prior knowledge about the parameters $\lambda$ and $\beta$.

The joint prior distribution of $\lambda$ and $\beta$ is obtained as follows:

$$g(\lambda,\beta) \propto \lambda^{a_1-1}e^{-b_1\lambda}\beta^{a_2-1}e^{-b_2\beta}. \tag{36}$$

In light of the upper record $data = \{y_1,y_2,...,y_m\}$, the joint posterior distribution of $\lambda$ and $\beta$ is obtained as follows:

$$\pi(\lambda,\beta|data) \propto L(data|\lambda,\beta)g(\lambda,\beta), \tag{37}$$

where $L(data|\lambda,\beta)$ is the likelihood function given in Eq. (23) and $g(\lambda,\beta)$ is the joint prior density that is given in Eq. (36). By substituting Eqs. (36) and (23) in Eq. (37), the joint posterior density of $\lambda$ and $\beta$ is immediately given by:

$$\pi(\lambda,\beta|data) \propto \lambda^{m+a_1-1}\beta^{m+a_2-1}e^{-\beta b_2}e^{-\lambda(b_1+(R_m)^{\beta})}\prod_{i=1}^{m}R_i^{\beta}. \tag{38}$$

It can be seen that the joint posterior distribution in Eq. (38) can be represented as follows:

$$\pi(\lambda,\beta|data) \propto \pi_1(\beta|data)\pi_2(\lambda|\beta,data), \tag{39}$$

where

$$\pi_1(\beta|data) \propto \frac{\beta^{m+a_2-1}e^{-\beta b_2}\prod_{i=1}^{m}R_i^{\beta}}{\left(b_1+R_m^{\beta}\right)^{m+a_1}}, \tag{40}$$

and $\pi_2(\lambda|\beta,data)$ is a gamma density with shape and scale parameters equal to $m+a_1$ and $\left[b_1+R_m^{\beta}\right]^{-1}$, respectively.

Subsequently, the Bayes estimate of any function of $\lambda$ and $\beta$, say $\eta(\lambda,\beta)$, under SE and LINEX loss functions separately are respectively given by:

$$\hat{\theta}_{BS} = \frac{\int_0^\infty\int_0^\infty \eta(\lambda,\beta)\pi_1(\beta|data)\pi_2(\lambda|\beta,data)d\beta d\lambda}{\int_0^\infty\int_0^\infty \pi_1(\beta|data)\pi_2(\lambda|\beta,data)d\beta d\lambda}, \tag{41}$$

and

$$\hat{\theta}_{BL} = -\frac{1}{\nu}\log\left(\frac{\int_0^\infty\int_0^\infty e^{-\nu\eta(\lambda,\beta)}\pi_1(\beta|data)\pi_2(\lambda|\beta,data)d\beta d\lambda}{\int_0^\infty\int_0^\infty \pi_1(\beta|data)\pi_2(\lambda|\beta,data)d\beta d\lambda}\right). \tag{42}$$

Unfortunately, the Bayes estimates in Eqs. (41) and (42) cannot be derived in explicit forms. Therefore, we propose to approximate the Bayes estimates and the corresponding credible intervals by using an importance sampling technique as suggested by Chen and Shao (1999). Similar procedure was used, for example, by Chen et al. (2000), Kundu and Pradhan (2009), Pradhan and Kundu (2009), Pradhan and Kundu (2011) and Bayoud (2016).

It can be easily seen that the marginal posterior of $\beta$ in Eq. (40) can be rewritten as follows:

$$\pi_1(\beta|data) \propto g_1(\beta|data)g_2(\beta), \tag{43}$$

where $g_1(\beta|data)$ is a gamma density with shape and scale parameters equal to $(m+a_2)$

and $\frac{1}{b_2}$, respectively, and

$$g_2(\beta) = \frac{\prod_{i=1}^{m} R_i^{\beta}}{\left(b_1 + R_m^{\beta}\right)^{m+a_1}}. \tag{44}$$

Now, we propose the following algorithm, along the line of Kundu and Pradhan (2009), to compute the approximate Bayes estimates and to construct the associated credible intervals for the parameters $\beta$ and $\lambda$.

Let $data = \{y_1, y_2, \ldots, y_m\}$ be a set of $m$ upper records and let $a_i$ and $b_i, (i = 1,2)$ be pre-assumed hyper-parameters chosen based on prior information about the underlying parameters $\beta$ and $\lambda$.

Step 1) Generate a random sample of size $M$ from the gamma density function $g_1(\beta|data)$, say $\{\beta_1, \beta_2, ..., \beta_M\}$;

Step 2) For each $\beta_j$, generate $\lambda_j$ from the gamma density function $\pi_2(\lambda|\beta_j, data)$, say $\{\lambda_1, \lambda_2, ..., \lambda_M\}$;

Step 3) Compute $g_2(\beta_i)$, for $j = 1, 2, ..., M$;

Step 4) Under the SEL function, a simulation consistent estimate of $\eta(\lambda, \beta)$ can be obtained using the importance sampling technique as:

$$\hat{\eta}_{BS}(\lambda, \beta) = \frac{\sum_{j=1}^{M} \eta(\lambda_j, \beta_j)\, g_2(\beta_j)}{\sum_{j=1}^{M} g_2(\beta_j)}.$$

Hence, $\hat{\beta}_{BS} = \frac{\sum_{j=1}^{M} \beta_j\, g_2(\beta_j)}{\sum_{j=1}^{M} g_2(\beta_j)}$ and $\hat{\lambda}_{BS} = \frac{\sum_{j=1}^{M} \lambda_j\, g_2(\beta_j)}{\sum_{j=1}^{M} g_2(\beta_j)}$.

Step 5) Under the LINEX function, a simulation consistent estimate of $\eta(\lambda, \beta)$ can be obtained using the importance sampling technique as:

$$\hat{\theta}_{BL} = \hat{\eta}_{BL}(\lambda, \beta) = -\frac{1}{v} \log \frac{\sum_{j=1}^{M} e^{-v\eta(\lambda_j, \beta_j)} g_2(\beta_j)}{\sum_{j=1}^{M} g_2(\beta_j)}.$$

Hence, $\hat{\beta}_{BL} = -\frac{1}{v} \log \frac{\sum_{j=1}^{M} e^{-v\beta_j} g_2(\beta_j)}{\sum_{j=1}^{M} g_2(\beta_j)}$ and $\hat{\lambda}_{BL} = -\frac{1}{v} \log \frac{\sum_{j=1}^{M} e^{-v\lambda_j} g_2(\beta_j)}{\sum_{j=1}^{M} g_2(\beta_j)}$.

Step 6) Compute

$$w_j = \frac{g_2(\beta_j)}{\sum_{j=1}^{M} g_2(\beta_j)} for j = 1, 2, \ldots, M;$$

Step 7) Arrange the set $\{(\beta_1, w_1), (\beta_2, w_2), ..., (\beta_M, w_M)\}$ as $\left\{(\beta_{(1)}, w_{[1]}), (\beta_{(2)}, w_{[2]}), \ldots, (\beta_{(M)}, w_{[M]})\right\}$, where $\beta_{(1)} \leq \beta_{(2)}, \ldots, \leq \beta_{(M)}$ are order statistics of $\beta_j$ from the sample of size $M$ obtained in Step (1) with $w[k]$ being the value of $w_i$'s associated with $k^{th}$ order statistic of $\beta_i$'s, say $\beta_{(k)}$.
Similarly, we obtain $\left\{(\lambda_{(1)}, w_{[1]}), (\lambda_{(2)}, w_{[2]}), \ldots, (\lambda_{(M)}, w_{[M]})\right\}$, which are order statistics of $\lambda_j$ from the sample of size $M$ obtained in Step (2) and $w[k]$ as defined above.

Step 8) The $100(1-\alpha)\%$ credible interval (CrI) for $\eta$ is given by $(\hat{\eta}_{(100\frac{\alpha}{2})}, \hat{\eta}_{(100(1-\frac{\alpha}{2}))})$, where $\hat{\eta}_{(100p)}$ is a simulation consistent Bayes estimate for $\eta$, which is given by $\eta_{(M_p)}$ such that $M_p$ is the integer satisfying:

$$\sum_{j=1}^{M_p} w_{[j]} \leq p < \sum_{j=1}^{M_p+1} w_{[j]}.$$

**Remark 4.1.** *Since $\hat{\beta}_{BS}$ and $\hat{\lambda}_{BS}$ are unique Bayes estimates for $\beta$ and $\lambda$, respectively, then they are admissible based on Theorem 2.4 of Lehmann and Casella (1998).*

**Remark 4.2.** *Since $\hat{\beta}_{BL}$ and $\hat{\lambda}_{BL}$ are unique Bayes estimates for $\beta$ and $\lambda$, respectively, then they are admissible based on Theorem 2.4 of Lehmann and Casella (1998).*

## 5. Data analysis

In this section, record statistics from a real data set obtained from K($\lambda$, $\beta$) are analyzed in order to illustrate the proposed estimation methods. All the computations are performed using Mathematica codes.

### 5.1. Real data: total annual rainfall

In this example, we analyze the total annual rainfall (in inches) during 25 years from 1984-2008 recorded at Los Angeles Civic Center. This data is given below, see http : // www.laalmanac.com/weather/we08aa.php:

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 12.82 | 17.86 | 7.66 | 2.48 | 8.08 | 7.35 | 11.99 | 21.00 | 7.36 |
| 8.11 | 24.35 | 12.44 | 12.40 | 31.01 | 9.09 | 11.57 | 17.94 | 4.42 |
| 16.42 | 9.25 | 37.96 | 13.19 | 3.21 | 13.53 | 9.08 | | |

This data set was studied by Tarvirdizade and Ahmadpour (2016). Firstly, all observations have been divided over 100 in order to transform them to be in $(0,1)$, the support of K($\lambda$, $\beta$). Then, the well-known Kolmogorov-Smirnov (K-S) goodness of fit test is used to test whether the Kies distribution adequately fits this data set or not. The MLEs of $\lambda$ and $\beta$ have been computed based on the complete sample numerically using Newton Raphson method to be 11.1410 and 1.4171, respectively. The corresponding K-S test statistic and the associated P-value are equal to 0.1674 and 0.4851, respectively. Accordingly, one cannot reject the hypothesis that the data set comes from K($\lambda$, $\beta$).
It can be easily seen that the upper records obtained from this data set are: 0.1282, 0.1786, 0.2100, 0.2435, 0.3101, 0.3796.
Based on these records, the MLEs, 95% ACIs, Bayes estimates and the corresponding 95% credible intervals are computed for the underlying parameters $\lambda$ and $\beta$. To study how sensitive are the Bayes estimates for the choice of the hyper-parameters, the following priors are considered: *Prior* $0: a_1 = b_1 = a_2 = b_2 = 0$, *Prior* $1: a_1 = 24, b_1 = 2, a_2 = 7, b_2 = 5$, and *Prior* $2: a_1 = 12, b_1 = 1, a_2 = 12, b_2 = 9$.
Tables (3) and (4) summarize the results of point and interval estimates, respectively, based on both the classical and the Bayesian approaches.

Table 3: Estimates for $\lambda$ and $\beta$ based on the real data set

| Parameter | MLE. | Bayes Estimates | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | *Prior* 0 | | | *Prior* 1 | | | | *Prior* 2 | | | |
| | | LINEX | | | SE. | LINEX | | | SE. | LINEX | | |
| | | $\nu=-0.01$ | $\nu=0.5$ | $\nu=2$ | | $\nu=-0.01$ | $\nu=0.5$ | $\nu=2$ | | $\nu=-0.01$ | $\nu=0.5$ | $\nu=2$ |
| $\lambda$ | 12.0148 | 12.1468 | 8.3301 | 4.8332 | 12.0110 | 12.0330 | 11.0390 | 9.2047 | 12.0510 | 12.0940 | 10.2930 | 7.8200 |
| $\beta$ | 1.4135 | 1.4152 | 1.3363 | 1.1582 | 1.4370 | 1.4376 | 1.4076 | 1.3304 | 1.3695 | 1.3700 | 1.3462 | 1.2823 |

Table 4: 95% ACIs and CrIs of $\lambda$ and $\beta$ based on the real data set

| Parameter | ACI | CrI | | |
|---|---|---|---|---|
| | | *Prior* 0 | *Prior* 1 | *Prior* 2 |
| $\lambda$ | (4.5358, 31.8260) | (4.4128, 23.3846) | (8.5085, 16.3880) | (6.7847, 18.7840) |
| $\beta$ | (0.6350, 3.1463) | (0.5187, 2.7489) | (0.9224, 2.2084) | (0.9052, 2.0203) |

## 6. Simulation study

In this section, a simulation study is conducted to evaluate the performance of the proposed estimation methods based on Kies record data. Simulations are performed using three sets of parameter values $(\lambda = 1, \beta = 2)$, $(\lambda = 2, \beta = 1)$ and $(\lambda = \beta = 2)$, mainly to compare the MLEs with the Bayes estimators and also to explore their effects on different parameter values. A given number $m$ of upper records are generated from K($\lambda$, $\beta$) using Eq. (16). The MLEs and the approximate Bayes estimates are computed using the importance sampling procedure. Bayes estimates are computed under the SE and LINEX loss functions assuming the following priors, which are assumed based on the considered cases:
*Prior* 0: $a_1 = 0$, $b_1 = 0$, $a_2 = 0$, $b_2 = 0$.
For $\lambda = 1$, $\beta = 2$:
*Prior* 1: $a_1 = 2$, $b_1 = 2$, $a_2 = 16$, $b_2 = 8$ and *Prior* 2: $a_1 = 4$, $b_1 = 4$, $a_2 = 8$, $b_2 = 4$.
For $\lambda = 2$, $\beta = 1$:
*Prior* 3: $a_1 = 4$, $b_1 = 2$, $a_2 = 8$, $b_2 = 8$ and *Prior* 4: $a_1 = 8$, $b_1 = 4$, $a_2 = 16$, $b_2 = 16$.
For $\lambda = 2$, $\beta = 2$:
*Prior* 5: $a_1 = 8$, $b_1 = 4$, $a_2 = 8$, $b_2 = 4$ and *Prior* 6: $a_1 = 10$, $b_1 = 5$, $a_2 = 10$, $b_2 = 5$.

These priors are proposed so as $\lambda$ has the same mean but different variances, similarly for $\beta$. The main purpose of this is to reflect the sensitivity of our inferences to the choice of the hyper-parameters. The shape parameter of LINEX loss function $\nu$ is assumed to equal -0.01, 0.5 and 2, separately.

Simulation studies are performed with $M = 1000$ iterations using Mathematica codes. The mean squared error (MSE) of the proposed MLEs and Bayes estimates is computed. The point estimation results are reported in Tables (5), (6) and (7) assuming the true parameters are $(\lambda = 1, \beta = 2)$, $(\lambda = 2, \beta = 1)$ and $(\lambda = \beta = 2)$, respectively, assuming $m = 5$, 6, 7 and 8. Further, the performance of the proposed classical CIs and Bayes CrIs are studied in terms of the average length (AL) and the coverage probability (CP). Tables (8), (9) and (10) present the ALs and CPs of the 95% ACIs, Boot-p CIs and CrIs for $\lambda$ and $\beta$ assuming $m = 5$, 6, 7 and 8.

Tables (5), (6) and (7) show that the performance of the Bayes estimates is better than that of the MLEs for both parameters in terms of MSEs. It can be also seen that the informative Bayes estimates under LINEX loss function with positive $\nu$ outperform the other estimates in most considered cases. However, non-informative Bayes estimates and the MLEs perform almost the same in most considered cases, but for positive $\nu$ the non-informative Bayes estimates under LINEX loss function outperform, in terms of the MSE, the MLEs. As expected, the Bayes estimates under some prior assumptions compete the corresponding Bayes estimates under other priors. For example, the MSEs of the Bayes estimates under *Prior* 4 are getting smaller than their counterparts under *Prior* 3. It is evident that all Bayes estimates under the informative priors behave better than the MLEs and the non-informative Bayes estimates. Clearly, the MSE of the proposed estimates decreases as $m$ increases for both $\lambda$ and $\beta$.

In view of interval estimation, Tables (8), (9) and (10) summarize the ALs and CPs of ACIs, Boot-p CIs and CrIs of $\lambda$ and $\beta$ when $(\lambda,\beta) = (1,2),(2,1)$ and $(2,2)$, respectively. The informative Bayes credible intervals are superior to the ACIs and the Boot-p CIs in the sense of coverage probability optimality criterion. It is noteworthy that the coverage probabilities of the Bayes credible intervals are generally well matched to their nominal levels. However, non informative Bayes credible intervals and ACIs are superior to the Boot-p CIs as they produce higher coverage probability with less average lengths. In general, there is a clear evidence that the informative credible intervals is the most valid method as it gives the highest simulated coverage probabilities comparing the intervals established by the classical approach.

Table 5: Average and MSE Values of the MLEs and Bayes estimates when $\lambda = 1$ and $\beta = 2$

| m | Parameter | Criterion | MLE. | Bayes Estimates | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | *Prior* 0 | | | | *Prior* 1 | | | | *Prior* 2 | | |
| | | | | LINEX | | | SE. | LINEX | | | SE. | LINEX | | |
| | | | | $\nu=-0.01$ | $\nu=0.5$ | $\nu=2$ | | $\nu=-0.01$ | $\nu=0.5$ | $\nu=2$ | | $\nu=-0.01$ | $\nu=0.5$ | $\nu=2$ |
| m=5 | $\beta$ | Average | 4.285 | 3.730 | 3.005 | 2.813 | 2.143 | 2.144 | 2.097 | 1.971 | 2.199 | 2.201 | 2.129 | 1.952 |
| | | MSE | 12.258 | 9.9641 | 3.0226 | 0.6083 | 0.0504 | 0.0507 | 0.0370 | 0.0236 | 0.1420 | 0.1431 | 0.1062 | 0.0650 |
| | $\lambda$ | Average | 0.550 | 0.491 | 0.672 | 0.808 | 0.941 | 0.961 | 0.913 | 0.804 | 1.015 | 1.038 | 0.987 | 0.866 |
| | | MSE | 0.670 | 0.834 | 0.598 | 0.412 | 0.1192 | 0.1099 | 0.1005 | 0.1019 | 0.0905 | 0.0911 | 0.0785 | 0.0746 |
| m=6 | $\beta$ | Average | 2.776 | 2.995 | 2.671 | 2.324 | 2.092 | 2.093 | 2.052 | 1.941 | 2.160 | 2.162 | 2.101 | 1.951 |
| | | MSE | 4.9816 | 5.1280 | 2.0570 | 0.5392 | 0.0350 | 0.0352 | 0.0271 | 0.0221 | 0.1177 | 0.1183 | 0.0938 | 0.0642 |
| | $\lambda$ | Average | 0.939 | 0.936 | 0.950 | 0.982 | 1.048 | 1.053 | 1.004 | 0.893 | 1.015 | 1.038 | 0.987 | 0.866 |
| | | MSE | 0.6109 | 0.6132 | 0.5151 | 0.3858 | 0.1100 | 0.1049 | 0.0887 | 0.0749 | 0.0905 | 0.0911 | 0.0785 | 0.0746 |
| m=7 | $\beta$ | Average | 2.492 | 2.501 | 2.237 | 2.101 | 1.984 | 1.984 | 1.949 | 1.857 | 2.155 | 2.157 | 2.1050 | 1.979 |
| | | MSE | 2.4238 | 2.4667 | 1.2455 | 0.4004 | 0.0304 | 0.0305 | 0.0228 | 0.0210 | 0.093 | 0.094 | 0.0737 | 0.0503 |
| | $\lambda$ | Average | 1.158 | 1.205 | 1.121 | 1.108 | 1.048 | 1.073 | 1.023 | 0.899 | 0.955 | 0.998 | 0.949 | 0.833 |
| | | MSE | 0.5639 | 0.5658 | 0.4832 | 0.3647 | 0.0802 | 0.0841 | 0.0694 | 0.0549 | 0.0828 | 0.0831 | 0.0751 | 0.0727 |
| m=8 | $\beta$ | Average | 2.376 | 2.421 | 2.181 | 2.115 | 2.0922 | 2.0828 | 2.051 | 1.968 | 2.114 | 2.115 | 2.074 | 1.977 |
| | | MSE | 0.7529 | 0.8120 | 0.6131 | 0.3737 | 0.0153 | 0.0154 | 0.0103 | 0.0072 | 0.0501 | 0.0504 | 0.0383 | 0.0218 |
| | $\lambda$ | Average | 1.100 | 1.112 | 1.081 | 1.072 | 1.051 | 1.086 | 1.037 | 0.921 | 0.993 | 1.050 | 1.000 | 0.880 |
| | | MSE | 0.5416 | 0.5430 | 0.4717 | 0.3597 | 0.0787 | 0.0745 | 0.0624 | 0.0541 | 0.0791 | 0.0606 | 0.0688 | 0.0683 |

Table 6: Average and MSE Values of the MLEs and Bayes estimates when $\lambda = 2$ and $\beta = 1$

| m | Parameter | Criterion | MLE. | Bayes Estimates | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | *Prior* 0 | | | | *Prior* 3 | | | | *Prior* 4 | | |
| | | | | LINEX | | | SE. | LINEX | | | SE. | LINEX | | |
| | | | | $\nu=-0.01$ | $\nu=0.5$ | $\nu=2$ | | $\nu=-0.01$ | $\nu=0.5$ | $\nu=2$ | | $\nu=-0.01$ | $\nu=0.5$ | $\nu=2$ |
| m=5 | $\beta$ | Average | 1.626 | 1.703 | 1.405 | 1.357 | 1.068 | 1.069 | 1.049 | 0.998 | 1.046 | 1.046 | 1.034 | 1.002 |
| | | MSE | 1.5740 | 1.5929 | 0.9497 | 0.3181 | 0.0285 | 0.0286 | 0.0246 | 0.0181 | 0.0123 | 0.0123 | 0.0109 | 0.0087 |
| | $\lambda$ | Average | 1.784 | 1.762 | 1.708 | 1.798 | 2.003 | 2.014 | 1.888 | 1.617 | 2.039 | 2.044 | 1.960 | 1.760 |
| | | MSE | 1.7627 | 1.7681 | 1.5471 | 1.4310 | 0.2154 | 0.2184 | 0.1907 | 0.2612 | 0.1028 | 0.1164 | 0.1006 | 0.1278 |
| m=6 | $\beta$ | Average | 1.444 | 1.398 | 1.208 | 1.198 | 1.0785 | 1.079 | 1.062 | 1.017 | 1.026 | 1.026 | 1.016 | 0.988 |
| | | MSE | 0.5791 | 0.5814 | 0.3471 | 0.2047 | 0.0250 | 0.0251 | 0.0241 | 0.0148 | 0.0105 | 0.0105 | 0.0097 | 0.0085 |
| | $\lambda$ | Average | 1.944 | 1.925 | 1.901 | 1.899 | 2.083 | 2.106 | 1.980 | 1.702 | 2.019 | 2.021 | 1.941 | 1.747 |
| | | MSE | 1.7591 | 1.752 | 1.4231 | 1.390 | 0.2113 | 0.2168 | 0.1888 | 0.2477 | 0.0892 | 0.0971 | 0.0893 | 0.1277 |
| m=7 | $\beta$ | Average | 1.315 | 1.297 | 1.2577 | 1.189 | 1.038 | 1.038 | 1.024 | 0.986 | 1.040 | 1.041 | 1.031 | 1.006 |
| | | MSE | 0.2623 | 0.2541 | 0.1914 | 0.1310 | 0.0172 | 0.0173 | 0.0154 | 0.0124 | 0.0094 | 0.0094 | 0.0085 | 0.0068 |
| | $\lambda$ | Average | 1.536 | 1.621 | 1.781 | 1.812 | 1.914 | 1.977 | 1.866 | 1.618 | 1.988 | 1.985 | 1.910 | 1.729 |
| | | MSE | 1.5867 | 1.5973 | 1.2577 | 1.0010 | 0.1879 | 0.1387 | 0.1543 | 0.2381 | 0.0881 | 0.0969 | 0.0853 | 0.1197 |
| m=8 | $\beta$ | Average | 1.174 | 1.179 | 1.128 | 1.118 | 1.044 | 1.044 | 1.033 | 1.003 | 1.032 | 1.0321 | 1.024 | 1.001 |
| | | MSE | 0.1703 | 0.1715 | 0.1128 | 0.0891 | 0.0131 | 0.0132 | 0.0116 | 0.0087 | 0.0093 | 0.0093 | 0.0082 | 0.0066 |
| | $\lambda$ | Average | 1.771 | 1.684 | 1.702 | 1.779 | 1.898 | 1.899 | 1.818 | 1.604 | 2.027 | 2.046 | 1.969 | 1.780 |
| | | MSE | 1.0664 | 1.0981 | 0.9087 | 0.8727 | 0.16697 | 0.1272 | 0.1394 | 0.2344 | 0.0879 | 0.0884 | 0.0810 | 0.1136 |

Table 7: Average and MSE Values of the MLEs and Bayes estimates when $\lambda = 2$ and $\beta = 2$

| m | Parameter | Criterion | MLE. | Bayes Estimates | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | *Prior* 0 | | | | *Prior* 5 | | | | *Prior* 6 | | |
| | | | | LINEX | | | SE. | LINEX | | | SE. | LINEX | | |
| | | | | $\nu=-0.01$ | $\nu=0.5$ | $\nu=2$ | | $\nu=-0.01$ | $\nu=0.5$ | $\nu=2$ | | $\nu=-0.01$ | $\nu=0.5$ | $\nu=2$ |
| m=5 | $\beta$ | Average | 3.517 | 3.621 | 2.971 | 2.607 | 2.181 | 2.182 | 2.111 | 1.932 | 1.997 | 1.998 | 1.942 | 1.801 |
| | | MSE | 6.8171 | 7.0197 | 2.5659 | 0.5555 | 0.1649 | 0.1658 | 0.1264 | 0.1123 | 0.1077 | 0.1080 | 0.0990 | 0.1084 |
| | $\lambda$ | Average | 3.145 | 3.208 | 2.748 | 2.510 | 1.966 | 1.953 | 1.869 | 1.672 | 2.050 | 2.057 | 1.989 | 1.815 |
| | | MSE | 4.3416 | 4.5390 | 1.9262 | 1.1788 | 0.1269 | 0.1373 | 0.1335 | 0.1872 | 0.0992 | 0.1061 | 0.0931 | 0.1076 |
| m=6 | $\beta$ | Average | 3.054 | 2.841 | 2.641 | 2.4871 | 2.090 | 2.091 | 2.027 | 1.862 | 2.039 | 2.040 | 1.989 | 1.863 |
| | | MSE | 3.5818 | 3.6457 | 1.7527 | 0.4956 | 0.1401 | 0.1407 | 0.1170 | 0.1073 | 0.1070 | 0.1076 | 0.0978 | 0.0988 |
| | $\lambda$ | Average | 1.695 | 1.642 | 1.773 | 1.893 | 1.983 | 1.991 | 1.909 | 1.710 | 2.066 | 2.055 | 1.988 | 1.820 |
| | | MSE | 2.2436 | 2.2749 | 1.4881 | 1.0301 | 0.0876 | 0.0837 | 0.0814 | 0.1401 | 0.0806 | 0.0772 | 0.0674 | 0.0856 |
| m=7 | $\beta$ | Average | 2.509 | 2.612 | 2.331 | 2.210 | 2.110 | 2.111 | 2.062 | 1.942 | 2.166 | 2.167 | 2.118 | 1.996 |
| | | MSE | 2.1891 | 2.2171 | 1.2571 | 0.3701 | 0.1061 | 0.1066 | 0.0869 | 0.0730 | 0.0920 | 0.0929 | 0.0895 | 0.0515 |
| | $\lambda$ | Average | 1.704 | 1.698 | 1.710 | 1.724 | 2.062 | 2.071 | 1.996 | 1.804 | 2.081 | 2.062 | 1.989 | 1.807 |
| | | MSE | 0.8909 | 0.8931 | 0.6138 | 0.4003 | 0.0700 | 0.0830 | 0.0662 | 0.0832 | 0.0621 | 0.0667 | 0.0649 | 0.0761 |
| m=8 | $\beta$ | Average | 2.962 | 2.979 | 2.651 | 2.341 | 2.139 | 2.140 | 2.089 | 1.957 | 2.112 | 2.112 | 2.078 | 1.990 |
| | | MSE | 2.1486 | 2.2129 | 1.1741 | 0.3540 | 0.09485 | 0.0952 | 0.0799 | 0.0680 | 0.0562 | 0.0565 | 0.0451 | 0288 |
| | $\lambda$ | Average | 1.7852 | 1.704 | 1.803 | 1.814 | 1.944 | 1.890 | 1.825 | 0.0636 | 0.0798 | 0.0801 | 0.0677 | 0.0408 |
| | | MSE | 0.8812 | 0.8921 | 0.6013 | 0.3124 | 0.0697 | 0.0821 | 0.0651 | 0.0831 | 0.0608 | 0.0652 | 0.0611 | 0.0753 |

Table 8: ALs and CPs of 95% CIs of $\lambda = 1$ and $\beta = 2$

| Cases | | ACI | | Boot-p | | CrIs | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | *Prior* 0 | | *Prior* 1 | | *Prior* 2 | |
| *m* | | $\beta$ | $\lambda$ | $\beta$ | $\lambda$ | $\beta$ | $\lambda$ | $\beta$ | $\lambda$ | $\beta$ | $\lambda$ |
| m=5 | CP | 0.80 | 0.93 | 0.70 | 0.81 | 0.93 | 0.83 | 0.97 | 0.96 | 0.99 | 0.95 |
| | AL | 3.8612 | 6.0170 | 7.5359 | 12.7930 | 4.8621 | 2.3738 | 4.8741 | 2.1439 | 6.9520 | 1.8891 |
| m=6 | CP | 0.88 | 0.94 | 0.76 | 0.85 | 0.94 | 0.78 | 0.98 | 0.96 | 0.99 | 0.95 |
| | AL | 2.6151 | 5.8688 | 4.3292 | 7.0043 | 4.2388 | 2.1614 | 5.0343 | 2.0812 | 7.3202 | 1.8049 |
| m=7 | CP | 0.88 | 0.95 | 0.82 | 0.87 | 0.95 | 0.77 | 0.99 | 0.97 | 0.99 | 0.96 |
| | AL | 2.2432 | 5.6800 | 2.9700 | 4.9216 | 3.8525 | 2.0112 | 5.2122 | 1.9840 | 7.7010 | 1.7304 |
| m=8 | CP | 0.88 | 0.95 | 0.83 | 0.94 | 0.96 | 0.72 | 0.99 | 0.99 | 0.99 | 0.96 |
| | AL | 2.0205 | 5.4132 | 2.5839 | 4.9142 | 3.6011 | 1.8501 | 5.3762 | 1.9293 | 8.0700 | 1.5527 |

Table 9: ALs and CPs of 95% CIs of $\lambda = 2$ and $\beta = 1$

| Cases | | ACI | | Boot-p | | CrIs | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | *Prior* 0 | | *Prior* 3 | | *Prior* 4 | |
| $m$ | | $\beta$ | $\lambda$ | $\beta$ | $\lambda$ | $\beta$ | $\lambda$ | $\beta$ | $\lambda$ | $\beta$ | $\lambda$ |
| m=5 | CP | 0.84 | 0.93 | 0.71 | 0.77 | 0.89 | 0.81 | 0.96 | 0.97 | 0.99 | 0.93 |
| | AL | 3.3471 | 6.4309 | 6.2641 | 28.2670 | 2.4086 | 4.1827 | 3.4144 | 3.8074 | 2.4416 | 3.3268 |
| m=6 | CP | 0.86 | 0.93 | 0.73 | 0.83 | 0.91 | 0.79 | 0.97 | 0.98 | 0.99 | 0.94 |
| | AL | 2.6966 | 5.8499 | 4.7969 | 7.5577 | 2.4037 | 3.5744 | 3.6622 | 3.6126 | 2.4941 | 3.2894 |
| m=7 | CP | 0.86 | 0.95 | 0.75 | 0.83 | 0.92 | 0.79 | 0.98 | 0.98 | 0.99 | 0.95 |
| | AL | 2.3408 | 5.6125 | 4.0902 | 6.7176 | 2.0447 | 3.2721 | 3.8511 | 3.3761 | 2.6494 | 3.2165 |
| m=8 | CP | 0.90 | 0.97 | 0.79 | 0.86 | 0.93 | 0.78 | 0.99 | 0.98 | 0.99 | 0.99 |
| | AL | 1.9024 | 5.5002 | 2.7860 | 4.6248 | 1.8546 | 2.9352 | 3.9861 | 3.0051 | 2.6921 | 3.0488 |

Table 10: ALs and CPs of 95% CIs of $\lambda = 2$ and $\beta = 2$

| Cases | | ACI | | Boot-p | | CrIs | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | *Prior* 0 | | *Prior* 5 | | *Prior* 6 | |
| $m$ | | $\beta$ | $\lambda$ | $\beta$ | $\lambda$ | $\beta$ | $\lambda$ | $\beta$ | $\lambda$ | $\beta$ | $\lambda$ |
| m=5 | CP | 0.85 | 0.91 | 0.72 | 0.82 | 0.91 | 0.91 | 0.96 | 0.97 | 0.99 | 0.95 |
| | AL | 6.7013 | 6.4842 | 13.3060 | 12.9420 | 5.6959 | 4.4634 | 7.0471 | 3.2719 | 6.1916 | 3.1703 |
| m=6 | CP | 0.89 | 0.91 | 0.79 | 0.83 | 0.91 | 0.90 | 0.97 | 0.98 | 0.99 | 0.95 |
| | AL | 5.2326 | 5.8095 | 9.5716 | 6.3452 | 4.6384 | 4.0721 | 7.4476 | 3.1824 | 6.4401 | 3.0779 |
| m=7 | CP | 0.89 | 0.96 | 0.80 | 0.83 | 0.94 | 0.88 | 0.99 | 0.97 | 0.99 | 0.96 |
| | AL | 4.5255 | 5.6413 | 7.4424 | 5.5048 | 4.1055 | 3.6944 | 7.7343 | 3.0254 | 6.7930 | 3.0302 |
| m=8 | CP | 0.93 | 0.98 | 0.82 | 0.88 | 0.94 | 0.86 | 0.99 | 0.98 | 0.99 | 0.98 |
| | AL | 3.9058 | 5.6261 | 5.9419 | 4.9526 | 3.5840 | 3.3658 | 8.1000 | 2.9724 | 7.0417 | 2.8790 |

# 7. Conclusion

In this paper, classical and Bayesian inferences were proposed for the two-parameter Kies distribution based on upper records. Some distributional properties of the Kies distribution based on records were studied. Uniqueness and existence of the MLEs were discussed. Asymptotic and bootstrap confidence intervals were constructed. In the context of Bayesian estimation, the Bayes estimates of the parameters cannot be obtained in explicit forms. So, approximate Bayes estimates along with their associated credible intervals were obtained by employing importance sampling technique under SE and LINEX loss functions assuming non-informative and informative priors for both parameters. The performance of the different estimation methods was assessed via Monte Carlo simulations. Generally, from the simulation study, it was concluded that the proposed informative Bayes estimates outperform the classical estimates in all considered cases. However, non-informative Bayesian and the classical estimation methods perform almost the same under SE and LINEX under small $\nu$, while better results of the Bayesian methods are obtained under LINEX assuming other positive values of $\nu$. Classical confidence intervals (asymptotic and Boot-P) and Bayes credible intervals were also constructed for the unknown parameters. It is clearly evident that the Bayes credible intervals compete the classical confidence intervals in terms of the coverage probability in all cases. It was also noticed that the Asymptotic CI outperforms the Boot-p CI in all cases. Finally, a real data set was analyzed for illustrative purposes.

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