STATISTICS IN TRANSITION new series, March 2022
Vol. 23, No. 1, pp. 129–152, DOI 10.21307/stattrans-2022-008
Received – 05.05.2020; accepted – 13.11.2020

Estimation procedures for reliability functions of Kumaraswamy-G Distributions based on Type II Censoring and the sampling scheme of Bartholomew

Aditi Chaturvedi[1], **Surinder Kumar**[2]

ABSTRACT

In this paper, we consider Kumaraswamy-G distributions and derive a Uniformly Minimum Variance Unbiased Estimator (UMVUE) and a Maximum Likelihood Estimator (MLE) of the two measures of reliability, namely $R(t) = P(X > t)$ and $P = P(X > Y)$ under Type II censoring scheme and sampling scheme of Bartholomew (1963). We also develop interval estimates of the reliability measures. A comparative study of the different methods of point estimation has been conducted on the basis of simulation studies. An analysis of a real data set has been presented for illustration purposes.

Key words: interval estimation, Kumaraswamy-G distributions, Monte-Carlo simulation, point estimation.

1. Introduction

The Kumaraswamy (Kum) distribution is widely applied to model the random phenomenon having finite lower and upper bounds, e.g.-, the height of individuals, atmospheric temperatures, hydrological data such as daily rain fall, daily stream flow, etc. The distribution was first defined by Kumaraswamy (1976, 1978). Nadarajah (2008) demonstrated that the distribution may be viewed as a special case of three parameter Beta distribution. Several other unimodal distributions can also be approximated by Kumaraswamy's distribution [See, Kumaraswamy (1980) and Ponnambalam *et al.* (2001)]. Garg (2009) studied the generalized order statistics from the Kum distribution. Jones (2009) explored the background and genesis of the Kum distribution and demonstrated some similarities and differences between the beta and Kum distributions. He highlighted several advantages of the Kum distribution over the beta distribution. In hydrology and related areas, the Kum distribution has received considerable interest [See, Sundar and Subbiah (1989), Fletcher and Ponnambalam (1996), Seifi *et al.* (2000), Ponnambalam *et al.* (2001) and Ganji *et al.* (2006)]. Sindhu *et al.* (2013) focused on Bayesian and non-Bayesian estimation for the shape parameter of the Kum distribution under Type-II censored samples.

Eldin *et al.* (2014) obtained the MLE's and Bayes estimators for the parameters of the Kum distribution under general progressive Type II censoring. Mameli (2015) propose a new generalization of the skew-normal distribution, referred to as the Kum skew-normal

[1]Corresponding Author. Department of Statistics, Babasaheb Bhimrao Ambedkar University, Lucknow, India. E-mail: caditic@gmail.com.

[2]Department of Statistics, Babasaheb Bhimrao Ambedkar University, Lucknow, India. E-mail: surinderntls@gmail.com.

distribution. He demonstrated that this new distribution is computationally more tractable than the Beta skew-normal distribution proposed by Mameli and Musio (2013). Kızılaslan and Nadar (2016) considered the Kum distribution, when the lower record values along with the number of observations following the record values (inter-record times) were observed, and derived the maximum likelihood and Bayes estimators for estimating the parameters of the distribution as well as for the future record values prediction. Dey *et al.* (2017) focussed on Bayesian and non-Bayesian estimation of multicomponent stress–strength reliability when both step and strength follow the Kum distribution with common shape parameter. Dey *et al.* (2018) considered and investigated performance of ten different frequentist approaches for estimation of parameters of Kum distribution, namely, maximum likelihood estimators, moments estimators, L-moments estimators, percentile based estimators, least squares estimators, weighted least squares estimators, maximum product of spacings estimators, Cramér–von-Mises estimators, Anderson–Darling estimators and right tailed Anderson–Darling estimators.

In recent years, a large amount of literature has been developed regarding the generalization of classical distributions. For some of the citations, one may refer to Hassan *et al.* (2020) and the references therein. Cordeiro and Castro (2011) introduced a new Kumaraswamy generalized (Kum-G) family of distributions and discussed its basic statistical properties. They mentioned that the Kum-G family of densities has ability of fitting skewed data and allows for greater flexibility of its tails. The distribution generalizes the modelling ability of the Kumaraswamy distribution and can be widely applied in many areas of engineering and biology. Nadarajah *et al.* (2012) derived simple representation for the Kum-G family of distributions as a linear combination of exponentiated distributions and studied its general properties. They obtained MLEs of its parameters and discussed its bivariate extension as well. Tamandi and Nadarajah (2016) developed maximum spacing estimation procedure for the parameters of Kum-G distribution. Kundu and Chowdhary (2018) compared the minimums of two independent and heterogeneous samples each following Kum-G distribution with respect to usual stochastic ordering and hazard rate ordering. They also established likelihood ratio ordering between the minimum order statistics for heterogeneous multiple-outlier Kum-G random variables with the same parent distribution function. Kumari *et al.* (2019) provided characterization ofthe Kum-G distribution based on record values and obtained point and interval estimates of two measures of reliability function $R(t) = P(X > t)$ and $P = P(X > Y)$ based on records. They considered two types of point estimators, namely UMVUE's and MLE's and developed procedures for testing hypotheses related to various parametric functions. Chaturvedi and Bhatnagar (2020) developed classical and preliminary test estimators for measures of reliability of the Kum-G distribution under progressive Type II censoring.

The purpose of the present paper is to extend the results of Kumari *et al.* (2019) for the cases of Type II censoring and the sampling scheme proposed by Bartholomew. Considering the Kum-G distribution, we develop UMVUE's and MLE's for the reliability functions, $R(t)$ and P. For deriving UMVUE's, we followed the approach proposed by Chaturvedi and Tomer (2003), which saves tedious and time-consuming calculation of stress–strength function. The paper is organized as follows: In Section 2, we provide point estimators and exact confidence intervals for the q^{th} power of parameter α, for $q \in (-\infty, +\infty)$, and for functions

$R(t)$ and P based on Type II censoring scheme. In Section 3, based on the sampling scheme proposed by Bartholomew (1963), the point estimators for the α^q, $R(t)$ and P are provided. In Section 4, we present findings of simulation studies followed by real data analysis in Section 5. We end with a brief set of conclusions in Section 6. Proofs of some important results can be found in the Appendix.

2. Estimation based on Type II Censoring Scheme

A random variable X is said to follow the Kumaraswamy (1980) distribution if its pdf is given by

$$f(x;\alpha,\beta)=\alpha\beta x^{\beta-1}(1-x^{\beta})^{\alpha-1};\ 0<x<1,\ \alpha,\beta>0. \tag{1}$$

Considering the complete sample case, Nadar *et al.* (2014) have obtained the estimator of P for the distribution given in (1) assuming the parameter 'β' to be common for the two distributions.

A random variable X follow Kumaraswamy-G distributions [Cordeiro and Castro (2011)], if its pdf is of the form

$$f(x;\alpha,\beta)=\alpha\beta g(x)G^{\beta-1}(x)[1-G^{\beta}(x)]^{\alpha-1};\ x>0,\ \alpha,\beta>0, \tag{2}$$

where $g(x)$ denotes the pdf of $G(x)$, α and β are the shape parameters of the Kum-G distribution.

It is to be noted that the distribution given in (2) reduces to the Kumaraswamy distribution when $G(x)=x$.

2.1. UMVUE's and MLE's of $\alpha^q, R(t)$ and P Based on Type II Censoring

Suppose 'n' items are put on a test and the test is terminated after the first 'r' ordered observations are recorded. Let us denote by $0<X_{(1)}\leq X_{(2)}\leq ...\leq X_{(r)}$, $0<r<n$, the lifetimes of first r failures. Obviously $(n-r)$ items survived until $X_{(r)}$. Here, we provide an important lemma, which will be helpful in proving the main results of this section.

***Lemma 1** Let

$$S_{(r)}=-\left[\sum_{i=1}^{r} ln\left\{1-G^{\beta}(x_i)\right\}+(n-r)ln\left\{1-G^{\beta}(x_r)\right\}\right],$$

then, $S_{(r)}$ is complete and sufficient for the Kum-G distribution (2). Moreover, the pdf of $S_{(r)}$ is given by

$$g_{S_{(r)}}(s;\alpha)=\frac{1}{\Gamma(r)}s^{r-1}\alpha^{r}\exp\{-\alpha s\},s>0,\alpha>0,r>0, \tag{3}$$

*The proof of Lemma 1 is available from the corresponding author on request.

where, $\Gamma(\cdot)$ denotes the Gamma function.

In the following theorems, we provide the UMVUEs of α^q, $R(t)$ and P, based on Type II censoring scheme and under the assumption that β is known.

Theorem 1 For $q \in (-\infty,\infty)$, the UMVUE of α^q is given by:

$$\widetilde{\alpha}^q_{II} = \begin{cases} \frac{\Gamma(r)}{\Gamma(r-q)} S_{(r)}^{-q}; & r-q>0 \\ 0; & \text{otherwise.} \end{cases}$$

Proof. From (3),

$$E\left(S_{(r)}^{-q}\right) = \frac{\Gamma(r-q)}{\Gamma(r)}\alpha^q, r>q, \tag{4}$$

and the theorem follows from Lehmann-Scheffe theorem [see Rohatgi & Saleh(2012)].
Let us write the pdf (2) as follows

$$f(x;\alpha,\beta) = \frac{\alpha\beta g(x)G^{\beta-1}(x)}{1-G^{\beta}(x)} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!}\left\{-ln(1-G^{\beta}(x))\right\}^i \alpha^i,$$

then the following Corollary straight away follows from Theorem 1.

Corollary 1 *The UMVUE of the sampled pdf at a specified point x is:*

$$\widetilde{f}_{II}(x;\alpha,\beta) = \begin{cases} \frac{\beta g(x)G^{\beta-1}(x)}{B(1,r-1)S_{(r)}(1-G^{\beta}(x))}\left(1+\frac{ln(1-G^{\beta}(x))}{S_{(r)}}\right)^{r-2}; \\ \qquad\qquad -ln(1-G^{\beta}(x)) < S_{(r)} \\ 0; \qquad\qquad \mathit{otherwise}, \end{cases}$$

where $B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ is the Beta function.

Theorem 2 The UMVUE of $R(t)$ at a specified point t is

$$\widetilde{R}(t)_{II} = \begin{cases} \left[1+\frac{ln(1-G^{\beta}(t))}{S_{(r)}}\right]^{r-1}; & -ln(1-G^{\beta}(t)) < S_{(r)} \\ 0; & \text{otherwise.} \end{cases}$$

Proof. Using Corollary 1, we have

$$\widetilde{R}(t)_{II} = \int_t^{\infty} \frac{\beta g(x)G^{\beta-1}(x)}{B(1,r-1)S_{(r)}(1-G^{\beta}(x))}\left(1+\frac{ln(1-G^{\beta}(x))}{S_{(r)}}\right)^{r-2} dx,$$

and the result follows by substituting $\frac{-ln(1-G^{\beta}(x))}{S_{(r)}}$=v.

Let X and Y be two independent random variables following the classes of distributions $f_1(x;\alpha_1,\beta_1)$ and $f_2(y;\alpha_2,\beta_2)$, respectively, where

$$f_1(x;\alpha_1,\beta_1) = \alpha_1\beta_1 g(x)G^{\beta_1-1}(x)(1-G^{\beta_1}(x))^{\alpha_1-1};\ x>0,\ \alpha_1,\beta_1>0 \tag{5}$$

and

$$f_2(y;\alpha_2,\beta_2) = \alpha_2\beta_2 h(y)H^{\beta_2-1}(y)(1-H^{\beta_2}(y))^{\alpha_2-1};\ y>0,\ \alpha_2,\beta_2>0. \tag{6}$$

Let n items on X and m items on Y are put on a life test and the termination numbers for X and Y are r and r', respectively. Let us define

$$S_{(r)} = -\left[\sum_{i=1}^{r} ln(1-G^{\beta_1}(x_i)) + (n-r)(ln(1-G^{\beta_1}(x_r))\right]$$

and

$$T_{(r')} = -\left[\sum_{j=1}^{r'} ln(1-H^{\beta_2}(y_j)) + (m-r')(ln(1-H^{\beta_2}(y_{r'}))\right].$$

In the following theorem, we obtain the UMVUE of P.

***Theorem 3** The UMVUE of P, when X and Y belong to different family of distributions, is given by

$$\widetilde{P}_{II} = \begin{cases} \int_{z=0}^{c} \frac{1}{B(1,r'-1)} \left[1+\frac{ln\left\{1-G(H^{-1}(1-e^{-zT_{(r')}}))^{\beta_1/\beta_2}\right\}}{S_{(r)}}\right]^{r-1} \\ (1-z)^{r'-2}dz; \quad if\ G^{-1}\left\{(1-e^{-S_{(r)}})^{1/\beta_1}\right\} \leq H^{-1}\left\{(1-e^{-T_{(r')}})^{1/\beta_2}\right\} \\ \\ \int_{z=0}^{1} \frac{1}{B(1,r'-1)} \left[1+\frac{ln\left\{1-G(H^{-1}(1-e^{-zT_{(r')}}))^{\beta_1/\beta_2}\right\}}{S_{(r)}}\right]^{r-1} (1-z)^{r'-2}dz; \\ if\ G^{-1}\left\{(1-e^{-S_{(r)}})^{1/\beta_1}\right\} > H^{-1}\left\{(1-e^{-T_{(r')}})^{1/\beta_2}\right\}, \end{cases}$$

where $c = -T^{-1}ln\left[1-H\left\{G^{-1}(1-e^{-S_{(r)}})^{\beta_2/\beta_1}\right\}\right]$.

Along the lines of Theorem 3, we can easily prove the following Corollary.

Corollary 2 *The UMVUE of P, when X and Y belong to same family of distributions, i.e., when $G(\cdot)=H(\cdot)$ and $\beta_1=\beta_2$, is given by*

**The proof of Theorem 3 is available from the corresponding author on request.*

$$\widetilde{P}_{II} = \begin{cases} \frac{1}{B(1,r'-1)} \sum_{i=0}^{r'-2} (-1)^i \binom{r'-2}{i} \left(\frac{S_{(r)}}{T_{(r')}}\right)^i B(i+1,r); & S_{(r)} \leq T_{(r')} \\ \frac{1}{B(1,r'-1)} \sum_{j=0}^{r-1} (-1)^j \binom{r-1}{j} \left(\frac{T_{(r')}}{S_{(r)}}\right)^j B(j+1,r'-1); & S_{(r)} > T_{(r')}. \end{cases}$$

Using (2), the joint pdf of $x_{(1)} \leq x_{(2)} \leq ... \leq x_{(r)}$ is given by

$$h(x_{(1)}, x_{(2)}, ..., x_{(r)}; \alpha, \beta) = \frac{n!}{(n-r)!} \alpha^r \beta^r \prod_{i=1}^{r} \frac{g(x_{(i)}) G^{\beta-1}(x_{(i)})}{1 - G^{\beta}(x_{(i)})} \exp(-\alpha S_{(r)}) \tag{7}$$

It can be easily seen from (7) that the MLE of α^q based on Type II censoring is

$$\widehat{\alpha}_{II}^{q} = \left(\frac{r}{S_{(r)}}\right)^q. \tag{8}$$

From (2) and invariance property of maximum likelihood estimators, the MLE of $f(x)$ is given by

$$\widehat{f(x)}_{II} = \frac{r}{S_{(r)}} \beta g(x) G^{\beta-1}(x) \left[1 - G^{\beta}(x)\right]^{\frac{r}{S_{(r)}}-1}.$$

Similarly, using the invariance property of MLE, the MLE of $R(t)$ is given by

$$\widehat{R(t)}_{II} = \left(1 - G^{\beta}(t)\right)^{\frac{r}{S_{(r)}}}. \tag{9}$$

The MLE of P, when X and Y belong to different family of distributions, is given by

$$\widehat{P}_{II} = \int_{z=0}^{1} \left[1 - G^{\beta_1}\left\{H^{-1}(z^{1/\beta_2})\right\}\right]^{\frac{r}{S_{(r)}}} \frac{r'}{T_{(r')}} (1-z)^{\frac{r'}{T_{(r')}}-1}.$$

The MLE of P, when X and Y belong to same family of distributions, i.e.-, when $G(\cdot) = H(\cdot)$ and $\beta_1 = \beta_2$ is given by

$$\widehat{P}_{II} = \frac{r' S_{(r)}}{r' S_{(r)} + r T_{(r')}}. \tag{10}$$

2.2. Exact Confidence Intervals for α, $R(t)$ and P based on Type II Censoring

We consider the problem of constructing a two-sided confidence interval for α. The confidence interval is obtained by using pivotal quantity $2\alpha S_{(r)}$. If we define $\chi^2(\nu)$ as the value of χ^2 such that

$$P(\chi^2 > \chi^2(\delta)) = \int_{\chi^2(\delta)}^{\infty} P(\chi^2) d\chi^2 = \delta, \tag{11}$$

where $P(\chi^2)$ is the pdf of χ^2 distribution with 2r degrees of freedom, then by using the fact that $2\alpha S_{(r)} \sim \chi^2_{2r}$, the confidence interval is given by

$$P\left(\frac{\chi^2\left(1-\frac{\delta}{2}\right)}{2S_{(r)}} \leq \alpha \leq \frac{\chi^2\left(\frac{\delta}{2}\right)}{2S_{(r)}}\right) = 1-\delta, \tag{12}$$

where $\chi^2\left(\frac{\delta}{2}\right)$ and $\chi^2\left(1-\frac{\delta}{2}\right)$ are obtained by using (11). Thus, for known β, $100(1-\delta)\%$ confidence interval for α is given by

$$\left(\frac{\chi^2\left(1-\frac{\delta}{2}\right)}{2S_{(r)}}, \frac{\chi^2\left(\frac{\delta}{2}\right)}{2S_{(r)}}\right).$$

Further, for $q<0$, the confidence interval for α^q is given by

$$\left(\left(\frac{\chi^2\left(\frac{\delta}{2}\right)}{2S_{(r)}}\right)^q, \left(\frac{\chi^2\left(1-\frac{\delta}{2}\right)}{2S_{(r)}}\right)^q\right)$$

and for $q>0$, the confidence interval for α^q is given by

$$\left(\left(\frac{\chi^2\left(1-\frac{\delta}{2}\right)}{2S_{(r)}}\right)^q, \left(\frac{\chi^2\left(\frac{\delta}{2}\right)}{2S_{(r)}}\right)^q\right).$$

The problem of obtaining the confidence interval for the reliability function $R(t) = (1-G^\beta(t))^\alpha$ can be solved by noting that $R(t_\circ;\alpha)$ is a decreasing function of α. Thus, $\Psi_1(x_1,x_2,...,x_n) \leq (1-G^\beta(t_\circ))^\alpha$ is equivalent to $\alpha \leq ln\Psi_1(x_1,x_2,...,x_n)/ln(1-G^\beta(t_\circ))$ and $\Psi_2(x_1,x_2,...,x_n) \geq (1-G^\beta(t_\circ))^\alpha$ is equivalent to $\alpha \geq ln\Psi_2(x_1,x_2,...,x_n)/ln(1-G^\beta(t_\circ))$. Therefore, the expression

$$P\Big(\Psi_1(x_1,x_2,...,x_n) \leq (1-G^\beta(t_\circ))^\alpha \leq \Psi_2(x_1,x_2,...,x_n)\Big) = 1-\delta$$

is equivalent to

$$P\left(\frac{ln\Psi_2(x_1,x_2,\ldots,x_n)}{ln(1-G^\beta(t_\circ))} \leq \alpha \leq \frac{ln\Psi_1(x_1,x_2,\ldots,x_n)}{ln(1-G^\beta(t_\circ))}\right) = 1-\delta. \tag{13}$$

Comparing (12) and (13), it immediately follows that $\chi^2\left(1-\frac{\delta}{2}\right)/2S_{(r)} = ln\Psi_2(x_1,x_2,...,x_n)/ln(1-G^\beta(t_\circ))$ and $\chi^2\left(\frac{\delta}{2}\right)/2S_{(r)} = ln\Psi_1(x_1,x_2,...,x_n)/ln(1-G^\beta(t_\circ))$.

Therefore,

$$\Psi_1 = exp\left[ln(1-G^{\beta}(t_\circ))\frac{\chi^2\left(\frac{\delta}{2}\right)}{2S_{(r)}}\right] and \ \Psi_2 = exp\left[ln(1-G^{\beta}(t_\circ))\frac{\chi^2\left(1-\frac{\delta}{2}\right)}{2S_{(r)}}\right].$$

Thus, for known β, $(1-\delta)100\%$ confidence interval for $R(t_\circ,\alpha)$ is given by

$$\left(exp\left[ln(1-G^{\beta}(t_\circ))\frac{\chi^2\left(\frac{\delta}{2}\right)}{2S_{(r)}}\right], exp\left[ln(1-G^{\beta}(t_\circ))\frac{\chi^2\left(1-\frac{\delta}{2}\right)}{2S_{(r)}}\right]\right).$$

In order to obtain the confidence interval for P, we utilize the fact that $\frac{2\alpha_1 S_{(r)}/2r}{2\alpha_2 T_{(r')}/2r'} \sim F_{2r,2r'}$. Thus, the confidence interval for P is given by

$$P\left[\left(\frac{rT_{(r')}F(\frac{\delta}{2})}{r'S_{(r)}}+1\right)^{-1} \le \frac{\alpha_2}{\alpha_1+\alpha_2} \le \left(\frac{rT_{(r')}F(1-\frac{\delta}{2})}{r'S_{(r)}}+1\right)^{-1}\right] = 1-\delta.$$

Therefore, for known β, $(1-\delta)100\%$ confidence interval for P is given by

$$\left[\left(\frac{rT_{(r')}F(\frac{\delta}{2})}{r'S_{(r)}}+1\right)^{-1}, \left(\frac{rT_{(r')}F(1-\frac{\delta}{2})}{r'S_{(r)}}+1\right)^{-1}\right].$$

3. Estimation based on the Sampling Scheme of Bartholomew

Throughout this section, we assume that n items are put on a test and we terminate life-testing experiment at a preassigned time $t_\circ$. Suppose we carry out time-censored test where the items that fail are immediately replaced. Let $X_{(1)} \le X_{(2)} \le ... \le X_{(n)}$ be the failure times of n items under a test from (2). The test begins at time $X_{(0)} = 0$ and the system operates until $X_{(1)} = x_1$, when the first failure occurs. The failed item is replaced by a new one and the system operates until the second failure occurs at time $X_{(2)} = x_2$ and so on. The experiment is terminated at time $t_\circ$. Here, $X_{(i)}$ is the time until i^{th} failure measured from time 0.

3.1. UMVUEs and MLEs of α^q, $R(t)$ and P, based on the Sampling Scheme of Bartholomew

We, first provide an important lemma, which will be utilized in deducing UMVUE's and MLE's of α^q, $R(t)$ and P.

***Lemma 2** Let $N(t_\circ)$ be the number of failures during the interval $[0;t_\circ]$. Then,

*The proof of Lemma 2 is available from the corresponding author on request.

$$P[N(t_\circ)=r|t_\circ]=\frac{[-n\alpha\ ln\ (1-G^\beta(t_\circ))]^r}{r!}exp\left\{n\alpha\ ln\ (1-G^\beta(t_\circ))\right\}.$$

In the following theorems, we provide the UMVUEs of α^q, $R(t)$ and P, based on the sampling scheme of Bartholomew (1963).

Theorem 4 For positive integer q, the UMVUE of α^q is given by

$$\widetilde{\alpha}_I^q=\begin{cases}\frac{r!}{(r-q)!}[-n\ ln\left\{1-G^\beta(t_\circ)\right\}]^{-q}; & r-q>0\\ 0; & \text{otherwise.}\end{cases}$$

Proof. It follows from Lemma 2 and the Fisher-Neyman factorization theorem [see Rohatgi and Saleh (2012), p. 341] that r is sufficient for α. Moreover, since the distribution of r belongs to an exponential family, it is also complete [see Rohatgi and Saleh (2012), p. 347]. The theorem now follows from the result that the q^{th} factorial moment of the distribution of r is given by

$$E[r(r-1)(r-2)...(r-q+1)]=\left[-n\alpha\ ln\left\{1-G^\beta(t_\circ)\right\}\right]^q.$$

Let us write the pdf (2) as follows:

$$f(x;\alpha,\beta)=\frac{\alpha\beta g(x)G^{\beta-1}(x)}{1-G^\beta(x)}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}\left\{-ln(1-G^\beta(x))\right\}^i\alpha^i.$$

Then, the Corollary 3 straight away follows from Theorem 4.

Corollary 3 *The UMVUE of $f(x;\alpha,\beta)$ at a specified point x is*

$$\widetilde{f}_I(x;\alpha,\beta)=\begin{cases}\frac{r\beta g(x)G^{\beta-1}(x)}{\left[-n\ ln(1-G^\beta(t_\circ))\right](1-G^\beta(x))}\left(1-\frac{ln(1-G^\beta(x))}{n\ ln(1-G^\beta(t_\circ))}\right)^{r-1}; & \\ & ln(1-G^\beta(x))<n\ ln(1-G^\beta(t_\circ))\\ 0; & otherwise.\end{cases}$$

Theorem 5 The UMVUE of $R(t)$ at a specified point t is given by

$$\widetilde{R}(t)=\begin{cases}\left[1-\frac{ln(1-G^\beta(t))}{n\ ln(1-G^\beta(t_\circ))}\right]^r; & ln(1-G^\beta(t))<n\ ln(1-G^\beta(t_\circ))\\ 0; & \text{otherwise.}\end{cases}$$

Proof. Using Corollary (3),

$$\widetilde{R}(t)=\int_t^\infty\frac{r\beta g(x)G^{\beta-1}(x)}{\left[-n\ ln(1-G^\beta(t_\circ))\right](1-G^\beta(x))}\left(1-\frac{ln(1-G^\beta(x))}{n\ ln(1-G^\beta(t_\circ))}\right)^{r-1}dx,$$

and the result follows by substituting $\frac{ln(1-G^{\beta}(x))}{n\ ln(1-G^{\beta}(t_\circ))} = z$.

Let n items on X and m on Y be put on a life test, where X and Y are distributed as in (5) and (6). Let $t_\circ$ and $t_{\circ\circ}$ be the termination times for X and Y , respectively and r and r' be the number of failures before $t_\circ$ and $t_{\circ\circ}$, respectively. Obviously, using Corollary 3, the UMVUEs of $f_1(x;\alpha_1,\beta_1)$and $f_2(y;\alpha_2,\beta_2)$, based on the sampling scheme of Bartholomew is given by

$$\widetilde{f}_{1I}(x;\alpha_1,\beta_1) = \frac{r\beta_1 g(x) G^{\beta_1-1}(x)}{\left[-n\ ln(1-G^{\beta_1}(t_\circ))\right](1-G^{\beta_1}(x))}\left(1-\frac{ln(1-G^{\beta_1}(x))}{n\ ln(1-G^{\beta_1}(t_\circ))}\right)^{r-1}; \quad ln(1-G^{\beta_1}(x)) < n\ \ ln(1-G^{\beta_1}(t_\circ)) \tag{14}$$

and

$$\widetilde{f}_{2I}(y;\alpha_2,\beta_2) = \frac{r'\beta_2 h(y) H^{\beta_2-1}(y)}{\left[-m\ ln(1-H^{\beta_2}(t_{\circ\circ}))\right](1-H^{\beta_2}(y))}\left(1-\frac{ln(1-H^{\beta_2}(y)}{m\ ln(1-H^{\beta_2}(t_{\circ\circ}))}\right)^{r'-1}; \quad ln(1-H^{\beta_2}(y)) < m\ ln(1-H^{\beta_2}(t_{\circ\circ})). \tag{15}$$

***Theorem 6** The UMVUE of P is given by

$$\widetilde{P}_I = \begin{cases} r'\int_{z=0}^{c}\left[1-\frac{ln\left\{1-G^{\beta_1}(H^{-1}(1-(1-H^{\beta_2}(t_{\circ\circ}))^{mz}))^{1/\beta_2}\right\}}{n\ ln\left\{1-G^{\beta_1}(t_\circ)\right\}}\right](1-z)^{r'-1}dz; \\ G^{-1}\left\{1-(1-G^{\beta_1}(t_\circ))^n\right\}^{\frac{1}{\beta_1}} \leq H^{-1}\left\{1-(1-H^{\beta_2}(t_{\circ\circ}))^m\right\}^{\frac{1}{\beta_2}} \\ r'\int_{z=0}^{1}\left[1-\frac{ln\left\{1-G^{\beta_1}(H^{-1}(1-(1-H^{\beta_2}(t_{\circ\circ}))^{mz}))^{1/\beta_2}\right\}}{n\ ln\left\{1-G^{\beta_1}(t_\circ)\right\}}\right](1-z)^{r'-1}dz; \\ G^{-1}\left\{1-(1-G^{\beta_1}(t_\circ))^n\right\}^{\frac{1}{\beta_1}} > H^{-1}\left\{1-(1-H^{\beta_2}(t_{\circ\circ}))^m\right\}^{\frac{1}{\beta_2}}, \end{cases}$$

where $c = \frac{ln\left[1-H^{\beta_2}\left\{G^{-1}(1-(1-G^{\beta_1}(t_\circ))^n)^{1/\beta_1}\right\}\right]}{m\ ln\left\{1-G^{\beta_1}(t_\circ)\right\}}$.

Corollary 4 *The UMVUE of P, when X and Y belong to the same family of distributions, i.e.,* $G(\cdot) = H(\cdot)$ *with* $\beta_1 = \beta_2$ *and* $t_\circ = t_{\circ\circ}$ *is given by*

$$\widetilde{P}_I = \begin{cases} r'\sum_{i=0}^{r'-1}(-1)^i\binom{r'-1}{i}\left(\frac{n}{m}\right)^{i+1}B(i+1,r+1); & n \leq m \\ r'\sum_{j=0}^{r}(-1)^j\binom{r}{j}\left(\frac{m}{n}\right)^{j}B(j+1,r'); & n > m. \end{cases}$$

---

*The proof of Theorem 6 is available from the corresponding author on request.

It can be easily seen from Lemma 2 that the MLE of α^q based on the sampling scheme of Bartholomew (1963) is given by

$$\widehat{\alpha}_I^q = \left(\frac{-r}{n\ ln(1-G^{\beta}(t_\circ))} \right)^q. \tag{16}$$

Using (2), $R(t)$ at point t is given by

$$R(t) = (1-G^{\beta}(t))^{\alpha}. \tag{17}$$

From (17) and invariance property of MLEs, the MLE of $R(t)$ is given by

$$\widehat{R}(t)_I = [1-G^{\beta}(t)]^{\frac{-r}{n\ ln(1-G^{\beta}(t_\circ))}}. \tag{18}$$

Similarly, using the invariance property of MLE, the MLE of $f(x;\alpha,\beta)$ at a specified point x is

$$\widehat{f}_I(x;\alpha,\beta) = \frac{-r}{n\ ln\left\{1-G^{\beta}(t_\circ)\right\}} g(x)G^{\beta-1}(x)[1-G^{\beta}(x)]^{\frac{-r}{n\ ln\left\{1-G^{\beta}(t_\circ)\right\}}-1}.$$

The MLE of P, when X and Y belong to a different family of distributions, is given by

$$\widehat{P}_I = \int_{z=0}^{1} [1-G^{\beta_1}\left\{H^{-1}(z^{1/\beta_2})\right\}]^{\frac{-r}{n\ ln(1-G^{\beta_1}(t_\circ))}} \frac{-rr'}{m\ ln(1-H^{\beta_2}(t_{\circ\circ}))} \times (1-z)^{\frac{-r'}{m\ ln(1-H^{\beta_2}(t_{\circ\circ}))}} dz.$$

The MLE of P, when X and Y belongs to the same family of distributions, i.e-., $G(\cdot) = H(\cdot)$, $\beta_1 = \beta_2$ and $t_\circ = t_{\circ\circ}$, is given by

$$\widehat{P} = \frac{r'n}{r'n+rm}. \tag{19}$$

4. Simulation Study

In order to validate the results obtained in Sections 2 and 3, we first consider the Kum distribution as a particular case of the Kum-G distributions. The pdf and cdf of the Kum distribution are given by:

$$f(x;\alpha,\beta) = \alpha\beta x^{\beta-1}(1-x^{\beta})^{\alpha-1};\ 0<x<1,\ \alpha,\beta>0. \tag{20}$$

$$F(x;\alpha,\beta) = 1-(1-x^{\beta})^{\alpha}. \tag{21}$$

respectively.

4.1. Simulation Based on Type II Censoring

For comparing the performances of estimators of α^q based on Type II censoring scheme, we have generated 1000 random samples from (20) each of size $n = 50$ for (α,β)=(2,0.5), (2,1), (2,2). For each sample we arranged the data in ascending order and considered a sample of first r ($\leq n$) observations. For different values of r =10, 20, 30 and 50, we have computed average values of $\widetilde{\alpha_{II}^q}$ and $\widehat{\alpha_{II}^q}$, their corresponding bias, MSE and approximate 95% confidence interval. For $q = 1$ and 2, results are reported in Table 1. It has been observed that MSE obtained corresponding to UMVUE is much lower than MSE obtained corresponding to MLE. Thus, the performance of UMVUE of α^q for $q = 1,2$ based on Type II censoring is much better than the performance of MLE of α^q. From Table 1, we observe that as r increases, the performance improves in the sense that their MSE decreases. It is also interesting to note that, with increasing r, the two estimators come close to each other.

For comparing the performance of MLE and UMVUE of reliability function $R(t)$, the bias, MSE and 95% confidence intervals are presented in Table 2. Comparing the estimates on the basis of MSE, we observe that the MLE of $R(t)$ performs better than the UMVUE for all parametric settings. As r increases, the performance of both the estimators improve and both estimators come close to each other.

For investigating the performance of estimators of P , we have generated 1000 random samples from each of the populations X and Y with sizes (n,m) with $\beta_1 = \beta_2 = 2$ and (α_1, α_2) = (0.5,0.5),(0.5,1), (0.5, 1.5) and (1.5,2). Samples corresponding to both the populations are arranged in ascending order and first (r, r') observations are considered. For $(r, r') = (10, 10), (20, 20), (30, 25), (40, 40)$ and $(50, 50)$, we have computed average values of $\widetilde{P}$ and $\widehat{P}$, their corresponding bias, MSE and approximate 95% confidence interval and results are presented in Table 3. We observe that for all selected values of (r, r'), the MLE of P performs superior to the UMVUE of P in the sense that it has lower MSE.

4.2. Simulation Based on Sampling Scheme of Bartholomew

In order to obtain point estimates of $R(t)$ based on the sampling scheme of Bartholomew, we have generated 1000 random samples each of size 100 from (20) with $\alpha = 2$ and $\beta = 0.9$. By fixing the termination time at $t_\circ$, and replacing the failure by operating one, values of r (the number of failures before time $t_\circ$) is computed. For different termination time $t_\circ$ = 0.20, 0.50, 0.65, 0.80 and 0.90, we have computed average values of $\widetilde{R(t)}$ and $\widehat{R(t)}$, their corresponding bias, MSE and approximate 95% confidence interval. For different values of t results are presented in Table 4. It has been observed that for small values of t and small values of $t_\circ$, MLE is more efficient than UMVUE of $R(t)$. However, for large values of $t_\circ$, UMVUE becomes more efficient than MLE of $R(t)$. For large values of t and all values of $t_\circ$, both the estimators become equally efficient. The best results are obtained for $t_\circ = 0.65$ as bias and MSE are least for all values of t. This result shows the importance of termination time $t_\circ$ in the sampling scheme of Bartholomew.

Now, to investigate the performance of estimators of P based on the sampling scheme of Bartholomew, we have generated 1000 random samples from each of the population X and Y with sizes (n,m) with $\beta_1 = \beta_2 = 2$ and (α_1, α_2) = (0.5,0.75), (0.5,1), (0.5, 1.5) and (1.5,2.5). For each sample corresponding to both the population, fixing the termination time at $t_\circ = t_{\circ\circ}$ and replacing the failure by operating one, values of r (no. of failures before time $t_\circ$ in X) and values of r' (no. of failures before time $t_{\circ\circ}$ in Y) are computed. For $t_\circ = t_{\circ\circ}$= 0.50, 0.70 and 0.80, we have computed average values of $\widetilde{P_I}$ and $\widehat{P_I}$, their corresponding bias, MSE and approximate 95% confidence interval for $n > m$ and $n < m$, and results are presented in Tables 5 and 6 respectively. From Table 5, for $n > m$, it is observed that for small m when $n = 50$, UMVUE of P performs superior than MLE of P. As m increases both the estimators are equally efficient. However, for $n < m$, the results given in Table 6 show that for all n with $m = 50$, the MLE of P is superior than the UMVUE and, as n increases, both the estimators become equally efficient.

5. Real Data Study

In this section, to illustrate the usefulness of our procedure, we present real data analysis. We consider the real data set used by Kumari *et al.* (2019), originally taken from Proschan (1963). The data represent the intervals between failures (in hours) of the air conditioning system of a fleet of 13 Boeing 720 jet airplanes. Canavos and Tsokos (1971) observed that the failure time distribution of the air conditioning system for each of the planes can be well approximated by exponential distributions. We have considered the planes '7913' and '7914' for our illustrative purposes. The data are presented below:

x_1 (Plane 7914): 3, 5, 5, 13, 14, 15, 22, 22, 23, 30, 36, 39, 44, 46, 50, 72, 79, 88, 97,102, 139, 188, 197, 210.

y_1(Plane 7913): 1, 4, 11, 16, 18, 18, 18, 24, 31, 39, 46, 51, 54, 63, 68, 77, 80, 82, 97, 106, 111, 141, 142, 163, 191, 206, 216.

Before applying the Kolmogorov–Smirnov (KS) test, we transform the above given two data sets in the range of unit interval by using the transformation $X_i = \frac{X_i}{max(X_i)+1}$ and $Y_i = \frac{Y_i}{max(Y_i)+1}$.The two transformed data sets are given below:

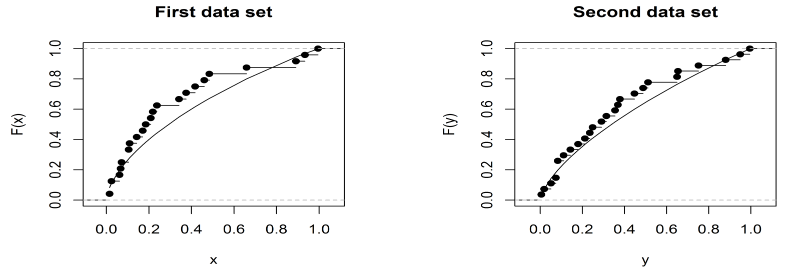
First data set x: (0.0142, 0.0237, 0.0616, 0.0664, 0.0711, 0.1043, 0.1090, 0.1422, 0.1706, 0.1848, 0.2085, 0.2180, 0.2370, 0.3412, 0.3744, 0.4171, 0.4597, 0.4834, 0.6588, 0.8910, 0.9336, 0.9953).

Second data set y: (0.0046, 0.0184, 0.0507, 0.0737, 0.0829, 0.1106, 0.1429, 0.1797, 0.2120, 0.2350, 0.2488, 0.2903, 0.3134, 0.3548, 0.3687, 0.3779, 0.4470, 0.4885, 0.5115, 0.6498, 0.6544, 0.7512, 0.8802, 0.9493, 0.9954).

We first apply the KS test to check whether the Kum distribution (20), fits the given X and Y populations. We obtain the following ML estimates of (α_1, β_1) and (α_2, β_2).

$$(\alpha_1, \beta_1)_{\text{complete data}} = (1.0728, 0.6022), \quad (\alpha_2, \beta_2)_{\text{complete data}} = (1.042, 0.6658).$$

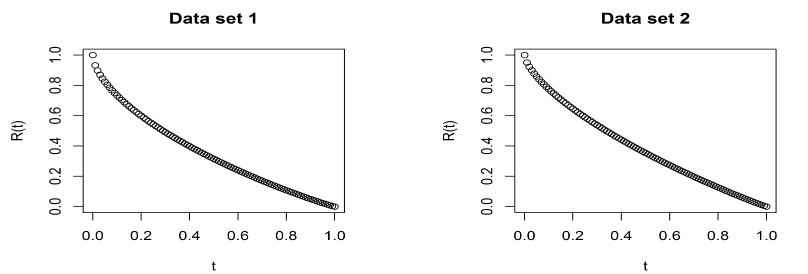
According to the KS test, we do not reject the null hypothesis that both the data observed for X $(KS = 0.18226; p = 0.4026)$ and the data observed for $Y(KS = 0.1289; p = 0.7604)$ are drawn from (20). Figure 1 confirms the good fit of (20), for these two data sets. In order to obtain the MLE of $R(t)$ and P based on Type II censoring, we first consider $r = 16$



(a) **Empirical and theoretical cdf of first data set**

(b) **Empirical and theoretical cdf of second data set**

Figure 1: **Plots of empirical and theoretical cdf**



(a) **Plot of ML Estimator of R(t) of first data set**

(b) **Plot of ML Estimator of R(t) of second data set**

Figure 2: **Plots of MLE of R(t)**

lifetimes from X population and the remaining 8 observations are considered as censored. Similarly, we consider first $r' = 20$ lifetimes from Y population and the remaining 7 observations are considered as censored. Considering the Kum distribution as a lifetime model for X-population, the MLEs of α_{1II} and β_{1II} are obtained as $\widehat{\alpha_{1II}} = 1.272$ and $\widehat{\beta_{1II}} = 0.6659$. Similarly, considering the Kum distribution as a lifetime model for Y-population, the MLEs of α_{2II} and β_{2II} are $\widehat{\alpha_{2II}} = 1.4677$ and $\widehat{\beta_{2II}} = 0.8128$. To evaluate MLE of P_{II} , we have considered the first data set as X-population and second data set as Y-population. We get $\widehat{P}_{II} = 0.5847$. For different values of t, we have evaluated MLE of $R(t)$ for X and Y populations, respectively. Results are plotted in Figure 2. In particular, for $t = 0.8$, $\widehat{R_{1II}}(t) = 0.1081$ and $\widehat{R_{2II}}(t) = 0.127$.

From Figure 2, it is clear that at initial time, the probability of survival is very high and as time increases the probability of survival decreases.

6. Conclusions

In this article, we have developed the estimation procedures for the Kum-G family of distributions based on Type II censoring and Bartholomew censoring schemes. Considerations are given to both point and interval estimations. The finite sample performance of the UMVUE's and MLE's of reliability functions and other parameters are investigated using extensive Monte Carlo experiment. The comparisons are made on the basis of MSE of the estimators. The main conclusions of the simulation experiments are as follows.

For Type II censoring, for all values of n, the UMVUE of α^q performs better than MLE of α^q. On the contrary, the performance of MLE of $R(t)$ is better than the performance of UMVUE of $R(t)$ for all selected values of t. However, for large values of r, the performance of both the estimators is quite similar. Further, as r increases, MSE corresponding to both the estimator decreases. Similarly, for estimating P, the MLE performs superior than the UMVUE.

For the sampling scheme of Bartholomew, for small values of t and $t_\circ$, MLE is more efficient than UMVUE of $R(t)$. However, for large values of $t_\circ$, UMVUE becomes more efficient than MLE of $R(t)$. For large values of t and all values of $t_\circ$, both the estimators are almost equally efficient. The best results are obtained for $t_\circ = 0.65$ as the bias and MSE are least for all values of t. This result shows the importance of termination time $t_\circ$ in the sampling scheme of Bartholomew. For comparing the performance of MLE and UMVUE of P, we observe that, when $n = 50$ and $m < n$, UMVUE outperforms MLE. As m increases both the estimators become equally efficient. On the contrary, for $n < m$ and $m = 50$ for small n, MLE of P gives better performance than UMVUE. But as m increases both the estimators become almost equally efficient.
The paper focuses on developing classical estimators for different parameters and reliability functions of Kumaraswamy-G distributions under various sampling schemes and investigating their properties. However, an interesting alternative to MLE and UMVU estimators can be provided by the empirical Bayes approach or ML-II estimators based on the robust Bayesian approach of Shrivastava *et al.* (2019). We leave exploration of this area for future work.

Acknowledgement

The authors would like to thank the Editor and the anonymous reviewer(s) for their valuable comments and suggestions on an earlier version of the manuscript. Their suggestions have greatly improved the presentation of the manuscript.

References

Bartholomew, D. J., (1963). The sampling distribution of an estimate arising in life testing, *Technometrics*, 5, pp. 361–374.

Canavos, G., Tsokos, C. P., (1971). A Study of an Ordinary and Empirical Bayes Approach of Estimation of Reliability in the Gamma Life Testing Model. *Proceedings of IEEE Symposium on Reliability*, pp. 1–17.

Chaturvedi, A., Bhatnagar, A., (2020). Development of Preliminary Test Estimators and Preliminary Test Confidence Intervals for Measures of Reliability of Kumaraswamy-G

Distributions Based on Progressive Type-II Censoring. *Journal of Statistical Theory and Practice,* 14(3), pp. 1–28.

Chaturvedi, A., Tomer, S. K., (2003). UMVU estimation of the reliability function of the generalized life distributions. *Statistical Papers*, 44(3), pp. 301–313.

Cordeiro, G. M., De Castro, M., (2011). A new family of generalized distributions. Journal of Statistical Computation and Simulation, 81(7), pp. 883–898.

Dey, S., Mazucheli, J., Anis, M. S., (2017). Estimation of reliability of multicomponent stress–strength for a Kumaraswamy distribution. *Communications in Statistics - Theory and Methods*, 46(4), pp. 1560–1572, DOI: 10.1080/03610926.2015.1022457.

Dey, S., Mazucheli, J., Nadarajah, S., (2018). Kumaraswamy distribution: different methods of estimation. *Computational and Applied Mathematics*, 37(2), pp. 2094–2111. DOI: https://doi.org/10.1007/s40314-017-0441-1

Eldin, M. M., Khalil, N., Amein, M., (2014). Estimation of Parameters of the Kumaraswamy Distribution Based on General Progressive Type II Censoring. *American Journal of Theoretical and Applied Statistics*, 3(6), pp. 217–222, DOI: 10.11648/j.ajtas.20140306.17

Fletcher, S.G., Ponnambalam, K., (1996). Estimation of reservoir yield and storage distribution using moments analysis. *Journal of Hydrology*, 182, pp. 259–275.

Ganji, A., Ponnambalam, K., Khalili, D.,Karamouz, M., (2006). Grain yield reliability analysis with crop water demand uncertainty. *Stochastic Environmental Research and Risk Assessment* , 20, pp. 259–277, DOI:https://doi.org/10.1007/s00477-005-0020-7.

Garg, M., (2009). On generalized order statistics from Kumaraswamy distribution. *Tamsui Oxford Journal of Mathematical Sciences*, 25(2), pp. 153–166.

Hassan, A. S., Sabry, M. A., Elsehetry, A. M., (2020). A New Family of Upper-Truncated Distributions: Properties and Estimation. *Thailand Statistician*, 18(2), pp. 196–214.

Johnson, N. L., & Kotz, S., (1970). Distributions in Statistics, vol. III: Continuous Univariate Distributions. New York: Houghton-Mifflin.

Jones, M. C., (2009). Kumaraswamy's distribution: A beta-type distribution with some tractability advantages. *Statistical Methodology*, 6(1), pp. 70–81, DOI: https://doi.org/10.1016/j.stamet.2008.04.001.

Kizilaslan, F., Nadar, M., (2016). Estimation and prediction of the Kumaraswamy distribution based on record values and inter-record times. *Journal of Statistical Computation* *and Simulation*, 86(12), pp. 2471–2493, DOI: 10.1080/00949655.2015.111 9832.

Kumaraswamy, P., (1976). Sinepower probability density function. *Journal of Hydrology*, 31, pp. 181–184.

Kumaraswamy, P., (1978). Extended sinepower probability density function. *Journal of Hydrology*, 37, pp. 81–89.

Kumaraswamy, P., (1980). A generalized probability density function for double-bounded random process. *Journal of Hydrology*, 46, pp. 79–88.

Kundu, A., Chowdhury, S., (2018). Ordering properties of sample minimum from Kumaraswamy-G random variables. *Statistics*, 52(1), pp. 133–146, DOI: 10.1080/02 331888.2017.1353516.

Kumari T., Chaturvedi, A., Pathak, A., (2019). Estimation and Testing Procedures for the Reliability Functions of Kumaraswamy-G Distributions and a Characterization Based on Records. *Journal of Statistical Theory and Practice*, 13(1), DOI:10.1007/s42519-018-0014-7.

Mameli, V., Musio, M., (2013). A Generalization of the Skew-Normal Distribution: The Beta Skew-Normal. *Communications in Statistics - Theory and Methods*, 42(12), pp. 2229–2244, DOI: 10.1080/03610926.2011.607530.

Mameli, V., (2015). The Kumaraswamy skew-normal distribution. *Statistics & Probability Letters*, 104, pp. 75–81.

Nadar, M., Kizilaslan, F., Papadopoulos, A., (2014). Classical and Bayesian estimation of $P(Y < X)$ for Kumaraswamy's distribution. *Journal of Statistical Computation and Simulation*, 84:7, pp. 1505–1529, DOI: 10.1080/00949655.2012.750658.

Nadarajah, S., (2008). On the distribution of Kumaraswamy. *Journal of Hydrology*, 348(3-4), pp. 568–569, DOI: 10.1016/j.jhydrol.2007.09.008.

Nadarajah, S., Cordeiro, G. M., Ortega, E. M. M., (2012). General results for the Kumaraswamy-G distribution. *Journal of Statistical Computation and Simulation*, 82(7), pp. 951–979, DOI: 10.1080/00949655.2011.562504.

Patel, J. K., Kapadia, C. H., Owen D. B., (1976). Handbook of Statistical Distributions. *Marcel Dekker, New York.*

Ponnambalam, K., Seifi, A., Vlach, J., (2001). Probabilistic design of systems with general distributions of parameters. *International Journal of Circuit Theory and Applications*, 29, pp. 527–536, DOI: https://doi.org/10.1002/cta.173.

Proschan, F., (1963). Theoretical explanation of observed decreasing failure rate. *Technometrics*,15, pp. 375–383.

Rohatgi, V. K., Saleh, A. K. Md. E., (2012). An introduction to probability and statistics. Wiley, New York.

Seifi, A., Ponnambalam, K., Vlach, J., (2000). Maximization of Manufacturing Yield of Systems with Arbitrary Distributions of Component Values. *Annals of Operations Research* ,99, pp. 373–383, DOI: 10.1023/A:1019288220413.

Shrivastava, A., Chaturvedi, A., Bhatti, M. I., (2019). Robust Bayesian analysis of a multivariate dynamic model. *Physica A: Statistical Mechanics and its Applications*, 528, 121451.

Sindhu, T.N., Feroze, N., Aslam, M., (2013).Bayesian Analysis of the Kumaraswamy Distribution under Failure Censoring Sampling Scheme. *International Journal of Advanced Science and Technology*, 51, pp. 39–58.

Sundar, V., Subbiah, K., (1989). Application of double bounded probability density function for analysis of ocean waves. *Ocean Engineering*, 16(2), pp. 193–200.

Tamandi, M., Nadarajah, S., (2016). On the Estimation of Parameters of Kumaraswamy-G Distributions. *Communications in Statistics - Simulation and Computation*, 45(10), pp. 3811–3821, DOI: 10.1080/03610918.2014.957840.

Appendix

Table 1: **UMVUEs and MLEs of α^q for different values of β**

$r->$		**10**		**20**		**30**		**50**	
β	q	$\widetilde{\alpha^q}$	$\widehat{\alpha^q}$	$\widetilde{\alpha^q}$	$\widehat{\alpha^q}$	$\widetilde{\alpha^q}$	$\widehat{\alpha^q}$	$\widetilde{\alpha^q}$	$\widehat{\alpha^q}$
0.5	1	2.0374	2.2638	2.0153	2.1214	2.0136	2.083	1.998	2.0388
		0.0374	0.2638	0.0153	0.1214	0.0136	0.083	-0.002	0.0388
		0.5007	0.686	0.2287	0.2679	0.1554	0.173	0.0845	0.0895
		(1.994, 2.081)	(2.215, 2.312)	(1.986,2.045)	(2.09, 2.153)	(1.989, 2.038)	(2.058, 2.108)	(1.98,2.016)	(2.02, 2.057)
	2	4.0384	5.6089	4.0632	4.7523	4.0159	4.4511	4.0527	4.3077
		0.0384	1.6089	0.0632	0.7523	0.0159	0.4511	0.0527	0.3077
		11.1695	24.132	4.535	6.7641	2.6135	3.4138	1.4052	1.6791
		(3.831,4.246)	(5.321,5.897)	(3.931,4.195)	(4.598, 4.907)	(3.916, 4.116)	(4.34, 4.562)	(3.979,4.126)	(4.23,4.386)
1	1	2.0183	2.2425	2.0079	2.1136	2.0127	2.0821	1.9999	2.0407
		0.0183	0.2425	0.0079	0.1136	0.0127	0.0821	-1e-04	0.0407
		0.5733	0.7661	0.2212	0.2579	0.147	0.1639	0.0845	0.0897
		(1.971,2.065)	(2.19,2.295)	(1.979,2.037)	(2.083,2.144)	(1.989,2.036)	(2.058,2.107)	(1.982,2.018)	(2.022,2.059)
	2	4.1504	5.7644	4.0609	4.7496	4.0158	4.451	3.9988	4.2504
		0.1504	1.7644	0.0609	0.7496	0.0158	0.451	-0.0012	0.2504
		11.6885	25.6167	4.9774	7.3657	2.5661	3.3555	1.4462	1.6966
		(3.939, 4.362)	(5.47,6.058)	(3.923,4.199)	(4.588,4.911)	(3.917,4.115)	(4.341, 4.561)	(3.924, 4.073)	(4.171,4.33)
2	1	2.0028	2.2254	2.0186	2.1248	1.9962	2.065	2.0149	2.056
		0.0028	0.2254	0.0186	0.1248	-0.0038	0.065	0.0149	0.056
		0.4796	0.6429	0.2259	0.2655	0.1414	0.1555	0.0845	0.0909
		(1.96,2.046)	(2.178,2.273)	(1.989,2.048)	(2.094, 2.156)	(1.973, 2.019)	(2.041,2.089)	(1.997,2.033)	(2.038,2.074)
	2	3.9382	5.4697	3.9687	4.6417	3.9592	4.3883	3.9561	4.205
		-0.0618	1.4697	-0.0313	0.6417	-0.0408	0.3883	-0.0439	0.205
		10.4358	22.2834	3.726	5.5074	2.3075	2.9835	1.369	1.5866
		(3.738,4.138)	(5.192,5.748)	(3.849,4.088)	(4.502,4.782)	([1] 3.865,4.053)	(4.284,4.493)	(3.884,4.029)	(4.128,4.282)

Note: The 1st, 2nd, 3rd row represents the average estimates, average bias, MSE and the 4th row represents the confidence interval.

Table 2: **UMVUEs and MLEs of $R(t)$**

$r->$		**10**		**20**		**30**		**50**	
$t\downarrow$	$R(t)\downarrow$	$\widetilde{R(t)}$	$\widehat{R(t)}$	$\widetilde{R(t)}$	$\widehat{R(t)}$	$\widetilde{R(t)}$	$\widehat{R(t)}$	$\widetilde{R(t)}$	$\widehat{R(t)}$
0.15	0.3754	0.3773	0.3602	0.3762	0.3672	0.377	0.3709	0.3724	0.3688
		0.0019	-0.0152	8e-04	-0.0082	0.0016	-0.0045	-0.003	-0.0066
		0.0141	0.0128	0.0067	0.0064	0.0045	0.0044	0.0029	0.0029
		(0.37,0.385)	(0.353, 0.367)	(0.371,0.381)	(0.362,0.372)	(0.373,0.381)	(0.367,0.375)	(0.369, 0.376)	(0.365,0.372)
0.20	0.3056	0.3071	0.2941	0.3055	0.2986	0.3086	0.3038	0.3082	0.3053
		0.0016	-0.0115	0	-0.007	0.003	-0.0018	0.0027	-2e-04
		0.0134	0.0118	0.0061	0.0058	0.0045	0.0043	0.0024	0.0023
		(0.3,0.314)	(0.287,0.301)	(0.301,0.31)	(0.294,0.303)	(0.304,0.313)	(0.3,0.308)	(0.305, 0.311)	(0.302,0.308)
0.25	0.25	0.2512	0.2422	0.2536	0.2485	0.2489	0.2456	0.2477	0.2457
		0.0012	-0.0078	0.0036	-0.0015	-0.0011	-0.0044	-0.0023	-0.0043
		0.0124	0.0105	0.0058	0.0053	0.004	0.0038	0.0024	0.0023
		(0.244,0.258)	(0.236,0.249)	(0.249,0.258)	(0.244,0.253)	(0.245,0.253)	(0.242, 0.249)	(0.245,0.251)	(0.243,0.249)
0.30	0.2046	0.2058	0.2004	0.2031	0.2002	0.2062	0.2041	0.203	0.2018
		0.0012	-0.0041	-0.0015	-0.0044	0.0016	-5e-04	-0.0015	-0.0028
		0.0115	0.0097	0.0054	0.005	0.0034	0.0032	0.0022	0.0021
		(0.199,0.212)	(0.194,0.207)	(0.199,0.208)	(0.196, 0.205)	(0.203, 0.21)	(0.201,0.208)	(0.2,0.206)	(0.199,0.205)
0.4	0.1351	0.1371	0.1375	0.1344	0.1347	0.1344	0.1345	0.1366	0.1367
		0.002	0.0024	-6e-04	-4e-04	-7e-04	-6e-04	0.0016	0.0016
		0.0075	0.0062	0.0036	0.0033	0.0024	0.0023	0.0014	0.0013
		(0.132, 0.142)	(0.133,0.142)	(0.131,0.138)	(0.131,0.138)	(0.131,0.137)	(0.132,0.137)	(0.134,0.139)	(0.134,0.139)
0.6	0.0508	0.0501	0.0567	0.0505	0.054	0.0503	0.0527	0.0499	0.0513
		-7e-04	0.0059	-3e-04	0.0032	-5e-04	0.0019	-9e-04	5e-04
		0.0022	0.0021	0.0011	0.0011	8e-04	8e-04	4e-04	4e-04
		(0.047, 0.053)	(0.054,0.06)	(0.048,0.053)	(0.052,0.056)	(0.049,0.052)	(0.051,0.054)	(0.049, 0.051)	(0.05,0.053)

Note: The 1st, 2nd, 3rd row represents the average estimates, average bias, MSE and the 4th row represents the confidence interval.

Table 3: **UMVUEs and MLEs of P**

$\alpha_1->$	**0.5**		**0.5**		**0.5**		**1.5**	
$\alpha_2->$	**0.5**		**1**		**1.5**		**2**	
$P->$	**0.5**		**0.6666667**		**0.75**		**0.625**	
$(r,r')\downarrow$	$\widetilde{P}$	$\widehat{P}$	$\widetilde{P}$	$\widehat{P}$	$\widetilde{P}$	$\widehat{P}$	$\widetilde{P}$	$\widehat{P}$
(10,10)	0.4917	0.492	0.6673	0.6601	0.7521	0.7428	0.6231	0.6175
	-0.0083	-0.008	6e-04	-0.0066	0.0021	-0.0072	-0.0019	-0.0075
	0.0133	0.0121	0.0102	0.0095	0.0072	0.007	0.0118	0.0109
	(0.485,0.499)	(0.485, 0.499)	(0.661, 0.674)	(0.654,0.666)	(0.747,0.757)	(0.738,0.748)	(0.616,0.63)	(0.611, 0.624)
(20,20)	0.5004	0.5004	0.6643	0.6607	0.7485	0.7439	0.6252	0.6223
	4e-04	4e-04	-0.0024	-0.006	-0.0015	-0.0061	2e-04	-0.0027
	0.0062	0.0059	0.0053	0.0052	0.0037	0.0037	0.0055	0.0053
	(0.496,0.505)	(0.496,0.505)	(0.66,0.669)	(0.656,0.665)	(0.745,0.752)	(0.74,0.748)	(0.621, 0.63)	(0.618,0.627)
(30,25)	0.5	0.5008	0.666	0.6641	0.7492	0.7464	0.6249	0.6236
	0	8e-04	-6e-04	-0.0026	-8e-04	-0.0036	-1e-04	-0.0014
	0.0048	0.0046	0.0035	0.0034	0.0027	0.0027	0.0043	0.0041
	(0.496,0.504)	(0.497, 0.505)	(0.662, 0.67)	(0.66,0.668)	(0.746,0.752)	(0.743,0.75)	(0.621,0.629)	(0.62,0.628)
(40,40)	0.5028	0.5027	0.6695	0.6677	0.7499	0.7475	0.6243	0.6229
	0.0028	0.0027	0.0029	0.001	-1e-04	-0.0025	-7e-04	-0.0021
	0.003	0.0029	0.0025	0.0024	0.0017	0.0017	0.0027	0.0027
	(0.499,0.506)	(0.499,0.506)	(0.666,0.673)	(0.665,0.671)	(0.747, 0.752)	(0.745,0.75)	(0.621,0.628)	(0.62, 0.626)
(50,50)	0.4998	0.4998	0.6653	0.6638	0.7502	0.7483	0.6238	0.6227
	-2e-04	-2e-04	-0.0014	-0.0028	2e-04	-0.0017	-0.0012	-0.0023
	0.0024	0.0024	0.002	0.002	0.0014	0.0014	0.0022	0.0022
	(0.497,0.503)	(0.497, 0.503)	(0.663,0.668)	(0.661,0.667)	(0.748,0.753)	(0.746,0.751)	(0.621,0.627)	(0.62, 0.626)

Note: The 1st, 2nd, 3rd row represents the average estimates, average bias, MSE and the 4th row represents the confidence interval.

Table 4: **UMVUEs and MLEs of $R(t)$ based on the Sampling Scheme of Bartholomew**

$t_\circ ->$		**0.20**		**0.50**		**0.65**		**0.80**		**0.90**	
$t\downarrow$	$R(t)\downarrow$	$\widetilde{R(t)}$	$\widehat{R(t)}$	$\widetilde{R(t)}$	$\widehat{R(t)}$	$\widetilde{R(t)}$	$\widehat{R(t)}$	$\widetilde{R(t)}$	$\widehat{R(t)}$	$\widetilde{R(t)}$	$\widehat{R(t)}$
0.15	0.3754	0.3545	0.3561	0.3555	0.3562	0.3661	0.3666	0.3994	0.3998	0.448	0.4483
		-0.0209	-0.0193	-0.0199	-0.0192	-0.0094	-0.0088	0.024	0.0244	0.0726	0.0729
		0.0019	0.0018	0.0015	0.0014	0.0011	0.0011	0.0017	0.0017	0.0063	0.0063
		(0.352,0.357)	(0.354,0.358)	(0.353,0.358)	(0.354,0.358)	(0.364,0.368)	(0.365,0.369)	(0.397,0.401)	(0.398,0.402)	(0.446,0.45)	(0.446,0.45)
0.20	0.3056	0.2839	0.2856	0.2836	0.2845	0.2974	0.298	0.3275	0.328	0.3794	0.3798
		-0.0217	-0.0199	-0.0219	-0.0211	-0.0082	-0.0075	0.0219	0.0224	0.0738	0.0742
		0.0019	0.0018	0.0014	0.0014	0.0011	0.0011	0.0016	0.0016	0.0066	0.0066
		(0.282,0.286)	(0.283,0.288)	(0.282,0.286)	(0.283,0.286)	(0.295,0.299)	(0.296,0.3)	(0.325, 0.33)	(0.326,0.33)	(0.377,0.381)	(0.378, 0.382)
0.25	0.25	0.2306	0.2325	0.2289	0.2298	0.2431	0.2438	0.2733	0.2738	0.3209	0.3213
		-0.0194	-0.0175	-0.0211	-0.0202	-0.0069	-0.0062	0.0233	0.0238	0.0709	0.0713
		0.0015	0.0015	0.0013	0.0012	9e-04	9e-04	0.0016	0.0016	0.0061	0.0062
		(0.228,0.233)	(0.23,0.235)	(0.227,0.231)	(0.228,0.232)	(0.241,0.245)	(0.242,0.246)	(0.271,0.275)	(0.272,0.276)	(0.319,0.323)	(0.319,0.323)
0.35	0.1668	0.1486	0.1507	0.1494	0.1505	0.1594	0.1602	0.1863	0.1869	0.2321	0.2326
		-0.0182	-0.0161	-0.0173	-0.0163	-0.0074	-0.0066	0.0195	0.0201	0.0653	0.0658
		0.0012	0.0012	0.001	9e-04	7e-04	7e-04	0.0011	0.0012	0.0051	0.0052
		(0.147, 0.15)	(0.149,0.153)	(0.148,0.151)	(0.149,0.152)	(0.158,0.161)	(0.159,0.162)	(0.185,0.188)	(0.185,0.189)	(0.23,0.234)	(0.231, 0.234)
0.45	0.1084	0.0951	0.0972	0.0962	0.0972	0.1044	0.1052	0.1245	0.1251	0.1619	0.1625
		-0.0133	-0.0112	-0.0121	-0.0111	-0.004	-0.0032	0.0161	0.0167	0.0536	0.0541
		7e-04	7e-04	6e-04	6e-04	5e-04	5e-04	8e-04	8e-04	0.0036	0.0036
		(0.094,0.097)	(0.096,0.099)	(0.095,0.098)	(0.096,0.099)	(0.103,0.106)	(0.104,0.106)	(0.123,0.126)	(0.124,0.126)	(0.16, 0.164)	(0.161,0.164)
0.55	0.0668	0.0572	0.059	0.0573	0.0582	0.0638	0.0646	0.0797	0.0803	0.1103	0.1109
		-0.0096	-0.0077	-0.0095	-0.0086	-0.0029	-0.0022	0.0129	0.0135	0.0436	0.0441
		4e-04	4e-04	3e-04	3e-04	2e-04	2e-04	5e-04	5e-04	0.0024	0.0024
		(0.056,0.058)	(0.058,0.06)	(0.056,0.058)	(0.057,0.059)	(0.063,0.065)	(0.064,0.066)	(0.079, 0.081)	(0.079, 0.081)	(0.109,0.112)	(0.11,0.112)
0.7	0.0267	0.0217	0.023	0.0217	0.0224	0.0249	0.0254	0.0347	0.0352	0.0526	0.0531
		-0.005	-0.0037	-0.0049	-0.0043	-0.0018	-0.0013	0.008	0.0085	0.0259	0.0264
		1e-04	1e-04	1e-04	1e-04	1e-04	1e-04	2e-04	2e-04	9e-04	9e-04
		(0.021,0.022)	(0.022,0.024)	(0.021,0.022)	(0.022,0.023)	(0.024,0.025)	(0.025,0.026)	(0.034,0.035)	(0.035,0.036)	(0.052,0.053)	(0.052,0.054)

Note: The 1st, 2nd, 3rd row represents the average estimates, average bias, MSE and the 4th row represents the confidence interval.

Table 5: **UMVUEs and MLEs of P based on the Sampling Scheme of Bartholomew**

$\alpha_1->$	**0.5**		**0.5**		**0.5**		**1.5**	
$\alpha_2->$	**0.75**		**1**		**1.5**		**2.5**	
$P->$	**0.6**		**0.6666667**		**0.75**		**0.625**	
$t_\circ = t_{\circ\circ} \downarrow$	$\widetilde{P}$	$\widehat{P}$	$\widetilde{P}$	$\widehat{P}$	$\widetilde{P}$	$\widehat{P}$	$\widetilde{P}$	$\widehat{P}$
				$(n=50)>(m=35)$				
0.50	0.6011	0.5944	0.6535	0.6482	0.7314	0.7278	0.6001	0.5978
	0.0011	-0.0056	-0.0132	-0.0185	-0.0186	-0.0222	-0.0249	-0.0272
	0.0166	0.0169	0.0123	0.0128	0.0077	0.0079	0.0048	0.0049
	(0.593,0.609)	(0.586,0.602)	(0.647, 0.66)	(0.641,0.655)	(0.726,0.737)	(0.722,0.733)	(0.596,0.604)	(0.594,0.602)
0.70	0.5915	0.5885	0.6408	0.6383	0.7028	0.701	0.5763	0.575
	-0.0085	-0.0115	-0.0258	-0.0283	-0.0472	-0.049	-0.0487	-0.05
	0.0058	0.0059	0.0054	0.0055	0.0054	0.0056	0.0038	0.004
	(0.587,0.596)	(0.584,0.593)	(0.637,0.645)	(0.634,0.643)	(0.699,0.706)	(0.697, 0.704)	(0.574,0.579)	(0.573,0.577)
0.80	0.5781	0.5759	0.6235	0.6217	0.6784	0.6769	0.5602	0.5591
	-0.0219	-0.0241	-0.0431	-0.045	-0.0716	-0.0731	-0.0648	-0.0659
	0.0039	0.0041	0.0046	0.0048	0.0072	0.0074	0.0051	0.0053
	(0.574,0.582)	(0.572,0.58)	(0.62, 0.627)	(0.618,0.625)	(0.676,0.681)	(0.674, 0.68)	(0.558,0.562)	(0.557,0.561)
				$(n=50)>(m=45)$				
0.50	0.5962	0.5945	0.6564	0.655	0.7261	0.7252	0.6012	0.6006
	-0.0038	-0.0055	-0.0103	-0.0116	-0.0239	-0.0248	-0.0238	-0.0244
	0.013	0.0131	0.012	0.012	0.0079	0.008	0.0044	0.0044
	(0.589,0.603)	(0.587, 0.602)	(0.65,0.663)	(0.648,0.662)	(0.721,0.731)	(0.72,0.731)	(0.597,0.605)	(0.597,0.604)
0.70	0.5893	0.5886	0.639	0.6383	0.7029	0.7024	0.5757	0.5753
	-0.0107	-0.0114	-0.0277	-0.0283	-0.0471	-0.0476	-0.0493	-0.0497
	0.0056	0.0056	0.0053	0.0053	0.0052	0.0052	0.0038	0.0038
	(0.585, 0.594)	(0.584, 0.593)	(0.635,0.643)	(0.634,0.642)	(0.699,0.706)	(0.699,0.706)	(0.573,0.578)	(0.573,0.578)
0.80	0.5781	0.5776	0.626	0.6255	0.6801	0.6797	0.5598	0.5596
	-0.0219	-0.0224	-0.0407	-0.0411	-0.0699	-0.0703	-0.0652	-0.0654
	0.0038	0.0038	0.0042	0.0042	0.0068	0.0068	0.0051	0.0052
	(0.575,0.582)	(0.574,0.581)	(0.623, 0.629)	(0.622, 0.629)	(0.677,0.683)	(0.677,0.682)	(0.558,0.562)	(0.558,0.561)

Note: The 1st, 2nd, 3rd row represents the average estimates, average bias, MSE and the 4th row represents the confidence interval.

Table 6: **UMVUEs and MLEs of P based on the Sampling Scheme of Bartholomew**

$\alpha_1 ->$	**0.5**		**0.5**		**0.5**		**1.5**	
$\alpha_2 ->$	**0.75**		**1**		**1.5**		**2.5**	
$P ->$	**0.6**		**0.6666667**		**0.75**		**0.625**	
$t_\circ = t_{\circ\circ} \downarrow$	$\widetilde{P}$	$\widehat{P}$	$\widetilde{P}$	$\widehat{P}$	$\widetilde{P}$	$\widehat{P}$	$\widetilde{P}$	$\widehat{P}$
				$(n=35) < (m=50)$				
0.50	0.5958	0.6024	0.6516	0.6568	0.734	0.7375	0.6031	0.6055
	-0.0042	0.0024	-0.0151	-0.0099	-0.016	-0.0125	-0.0219	-0.0195
	0.0165	0.0164	0.0137	0.0135	0.009	0.0088	0.0054	0.0053
	(0.588,0.604)	(0.594,0.61)	(0.644,0.659)	(0.65,0.664)	(0.728,0.74)	(0.732,0.743)	(0.599,0.607)	(0.601, 0.61)
0.70	0.5859	0.5889	0.6374	0.6399	0.7034	0.7052	0.576	0.5773
	-0.0141	-0.0111	-0.0293	-0.0268	-0.0466	-0.0448	-0.049	-0.0477
	0.007	0.0069	0.0064	0.0062	0.0062	0.006	0.004	0.0039
	(0.581,0.591)	(0.584,0.594)	(0.633,0.642)	(0.635,0.644)	(0.699,0.707)	(0.701,0.709)	(0.573,0.578)	(0.575, 0.58)
0.80	0.5758	0.578	0.6241	0.626	0.678	0.6795	0.5567	0.5578
	-0.0242	-0.022	-0.0426	-0.0407	-0.072	-0.0705	-0.0683	-0.0672
	0.0043	0.0041	0.0054	0.0053	0.0078	0.0076	0.0056	0.0055
	(0.572,0.58)	(0.574,0.582)	(0.62,0.628)	(0.622,0.63)	(0.675,0.681)	(0.676,0.683)	(0.555, 0.559)	(0.556, 0.56)
				$(n=45) < (m=50)$				
0.50	0.5949	0.5966	0.6594	0.6607	0.7375	0.7384	0.601	0.6016
	-0.0051	-0.0034	-0.0073	-0.0059	-0.0125	-0.0116	-0.024	-0.0234
	0.0153	0.0153	0.0117	0.0116	0.0082	0.0081	0.0043	0.0043
	(0.587,0.603)	(0.589,0.604)	(0.653,0.666)	(0.654,0.667)	(0.732,0.743)	(0.733,0.744)	(0.597,0.605)	(0.598,0.605)
0.70	0.5761	0.5764	0.6374	0.638	0.7013	0.7018	0.5769	0.5773
	-0.0489	-0.0486	-0.0293	-0.0286	-0.0487	-0.0482	-0.0481	-0.0477
	0.0039	0.0039	0.0056	0.0056	0.0055	0.0055	0.0036	0.0036
	(0.574, 0.579)	(0.574, 0.579)	(0.633, 0.642)	(0.634, 0.642)	(0.698,0.705)	(0.698,0.705)	(0.575,0.579)	(0.575,0.58)
0.80	0.577	0.5775	0.6254	0.6258	0.6791	0.6795	0.559	0.5593
	-0.023	-0.0225	-0.0413	-0.0408	-0.0709	-0.0705	-0.066	-0.0657
	0.004	0.004	0.0044	0.0044	0.0072	0.0071	0.0053	0.0053
	(0.573,0.581)	(0.574,0.581)	(0.622,0.629)	(0.623,0.629)	(0.676,0.682)	(0.677,0.682)	(0.557,0.561)	(0.557,0.561)

Note: The 1st, 2nd, 3rd row represents the average estimates, average bias, MSE and the 4th row represents the confidence interval.