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Generalized extended Marshall-Olkin family of lifetime distributions

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ABSTRACT

We introduce a new generalized family of nonnegative continuous distributions by adding two extra parameters to a lifetime distribution, called the baseline distribution, by twice compounding a power series distribution. The new family, called the lifetime power series-power series family, has a serial arrangement of parallel structures, which extends the Marshall and Olkin structure. Four special models are discussed. A mathematical treatment of the new distributions is provided, including ordinary and incomplete moments, quantile, moment generating and mean residual functions. The maximum likelihood estimation technique is used to estimate the model parameters and a simulation study is conducted to investigate the performance of the maximum likelihood estimates. Its applicability is also illustrated by means of two real data sets.

Key words: compound distribution, hazard rate function, lifetime distribution, maximum likelihood estimation, power series distribution.

1. Introduction

Classical well-known distributions, such as Weibull, gamma and Lomax distributions, are widely used for modeling data in many disciplines, including engineering, statistics, medical sciences, economics, and insurance. However, in many practical situations, they cannot provide appropriate fits on real data sets. Throughout the two last decades, several generators have been proposed in the literature to extend well-known distributions by adding one or more parameters to the baseline distribution. Since 1997, when Marshall and Olkin proposed a way to add a parameter to the lifetime distribution, by compounding with the geometric distribution, several new families of distributions have been derived by compounding the power series distribution with many other nonnegative continuous distributions to provide more flexible distributions for modeling lifetime data.

Marshall and Olkin's (1997) method was based on the lifetime of a series or parallel system with an unknown amount of components. Their work was extended by Chahkandi and Ganjali (2009), which proposed the exponential power series (EPS) distribution. Furthermore, Morais and Barreto-Souza (2011) proposed the Weibull power series (WPS) distribution containing the EPS distribution as a particular case. On the other hand, Flores et al. (2013) introduced the complementary EPS distribution, complementary to the EPS

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distribution and Munteanu et al. (2014) presented complementary WPS distribution. Some more well-known generators based on Marshal-Olkin generated family (MO-G) are Kumaraswamy Marshall-Olkin by Alizadeh et al. (2015), Beta Marshall-Olkin by Alizadeh et al. (2015), exponentiated logarithmic Marshal-Olkin by Marinhoa and Cordeiro (2016), and Marshall-Olkin alpha power family by Nassar et al. (2019).

According to Ross (2010), any system can be represented both as a series arrangement of parallel structures or as a parallel arrangement of series structures. Using this key, the purpose of this paper is to introduce a new generator of lifetime distributions by compounding a lifetime distribution with twice power series distribution, obtaining what is referred to as the LPS^2 family of distributions. The proposed family is motivated by a system consisting of serial components with each component consisting of a parallel of components. Some researchers published real examples of systems made by the serial, and parallel components are resonant converters (Kazimierczuk et al., 1993), hybrid electric bus (Xiong et al., 2009), and hybrid envelope amplifiers (Hassan et al., 2012).

The paper is organized as follows. In Section 2, we introduce the new family of distributions. Four special cases of this family are defined in Section 3. Section 4 derives some of its mathematical properties. The explicit expressions for the moments, incomplete moments, generating function, and mean residual time are given in this section. The estimation of parameters using the maximum likelihood method is investigated in Section 5. In Section 6, a simulation study is performed to show the behaviour of asymptotic biases and mean square errors of maximum likelihood estimations (MLEs). Illustrative examples of two real data sets are given in Section 7. Finally, in Section 8, we present some concluding remarks.

2. The LPS^2 family of distributions

Let $X_{i,j}$s be a sequence of independent and identically (iid) random sample from a baseline lifetime distribution for $j = 1,2,\ldots,Z_i$, $i = 1,2,\ldots,U$, with probability density function (pdf) $\pi(x;\boldsymbol{\varsigma})$ and cumulative distribution function (cdf) $\Pi(x;\boldsymbol{\varsigma})$, where $\boldsymbol{\varsigma}$ denoted the parameter vector of baseline distribution. Suppose $Z_1, Z_2, \ldots, Z_U$ are iid zero truncated power series random variables with probability mass function (pmf)

$$P(z;\theta) = \frac{b_z \theta^z}{B(\theta)},$$

for $z = 1,2,\ldots$, where b_z depends only on z and $B(\theta) = \sum_{z=1}^{\infty} b_z \theta^z < \infty$. Furthermore, suppose U is a zero truncated power series random variable with pmf

$$P(u;\lambda) = \frac{a_u \lambda^u}{A(\lambda)},$$

for $u = 1,2,\ldots$, where a_u depends only on u and $A(\lambda) = \sum_{u=1}^{\infty} a_u \lambda^u < \infty$. Consider that $X_{i,j}$s, Z_is and U are independent, we define a system, which is made of U series components, that the ith component is made of Z_i components working in parallel. Figure 1 shows an illustration of this system. Then the lifetime of the system is

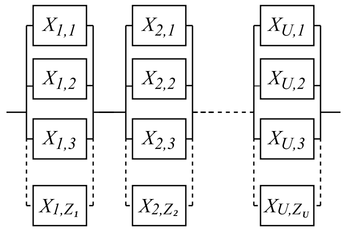


Figure 1: The system made up of series and parallel components.

$$X = \min\left\{\max\left\{X_{i,j}\right\}_{j=1}^{Z_i}\right\}_{i=1}^{U}.$$

Table 1 shows useful quantities of some members of the power series family (truncated at zero) such as the Poisson, geometric, logarithmic series, negative binomial and binomial distributions.

Table 1: Members of the power series family.

power series family	pmf	λ	extended λ after compounding	a_u	$A(\lambda)$	$I(\lambda)=\int_0^\lambda A'(v)\log\{A'(v)\}dv$
Poisson	$e^{-\lambda}\lambda^u/u!\left(1-e^{-\lambda}\right)$	$0<\lambda<\infty$	$\lambda\in(-\infty,0)\bigcup(0,+\infty)$	$1/u!$	$e^\lambda-1$	$(\lambda-1)e^\lambda+1$
Geometric	$(1-\lambda)\lambda^{u-1}$	$0<\lambda<1$	$\lambda\in(-\infty,0)\bigcup(0,1)$	1	$\lambda/(1-\lambda)$	$-2(\lambda+\log\{1-\lambda\})/(1-\lambda)$
Logarithmic series	$-\lambda^u/u\log\{1-\lambda\}$	$0<\lambda<1$	$\lambda\in(-\infty,0)\bigcup(0,1)$	$1/u$	$-\log\{1-\lambda\}$	$-\frac{1}{2}\log^2\{1-\lambda\}$
Negative Binomial	$\binom{m+u-1}{u}(1-\lambda)^m\lambda^u/1-(1-\lambda)^m$	$0<\lambda<1$	$\lambda\in(-\infty,0)\bigcup(0,1)$	$\binom{m+u-1}{u}$	$(1-\lambda)^{-m}-1$	$\log\{m\}A'(\lambda)-\frac{m+1}{m}\left\{(1-\lambda)^{-m}[m\log\{1-\lambda\}+1]-1\right\}$
Binomial	$\binom{m}{u}\lambda^u/((1+\lambda)^m-1)$	$0<\lambda<\infty$	$\lambda\in(-1,0)\bigcup(0,+\infty)$	$\binom{m}{u}$	$(1+\lambda)^m-1$	$(1+\lambda)^m\log\left\{m(1+\lambda)^{m-1}\right\}-\log\{m\}$

It could be shown that the marginal cdf of X is

$$F(x;\boldsymbol{\xi}) = 1-[A(\lambda)]^{-1}A\left(\lambda\left\{1-\frac{B(\theta\Pi(x;\boldsymbol{\varsigma}))}{B(\theta)}\right\}\right), \tag{1}$$

for $x>0$ and $\boldsymbol{\xi}=(\boldsymbol{\varsigma},\theta,\lambda)$. Hence, $S(x;\boldsymbol{\xi})=1-F(x;\boldsymbol{\xi})$ is the corresponding survival function and the pdf and hazard rate function (hrf) of LPS^2 family are defined as follows:

$$f(x;\boldsymbol{\xi}) = \frac{\lambda\theta\pi(x;\boldsymbol{\varsigma})B'(\theta\Pi(x,\boldsymbol{\varsigma}))}{A(\lambda)B(\theta)}A'\left(\lambda\left\{1-[B(\theta)]^{-1}B(\theta\Pi(x;\boldsymbol{\varsigma}))\right\}\right) \tag{2}$$

and

$$h(x;\boldsymbol{\xi}) = \frac{\lambda\theta\pi(x,\boldsymbol{\varsigma})B'(\theta\Pi(x;\boldsymbol{\varsigma}))A'\left(\lambda\left\{1-[B(\theta)]^{-1}B(\theta\Pi(x;\boldsymbol{\varsigma}))\right\}\right)}{B(\theta)A\left(\lambda\left\{1-[B(\theta)]^{-1}B(\theta\Pi(x;\boldsymbol{\varsigma}))\right\}\right)},$$

for $x>0$, respectively. Furthermore, $A'(.)$ and $B'(.)$ are the derivative of $A(.)$ and $B(.)$

functions, respectively. The hrf can be constant, decreasing, increasing, J-shaped, bathtub-shaped, and upside-down bathtub-shaped for a different type of the baseline and power series distributions (see Section 3). The LPS^2 family of distributions contains all compounded lifetime distributions, which were built by the Marshall and Olkin method. Here, some sub-models of the LPS^2 family are presented.

- When $Z_1, Z_2, \ldots = 1$ and the baseline is an exponential or a Weibull distribution, we obtain the EPS (Chahkandi and Ganjali, 2009) and WPS (Morais and Barreto-Souza, 2011) distributions respectively;
- when $U = 1$ and the baseline is an exponential or a Weibull distribution, we obtain the CEPS (Flores et al., 2013) and the max-Weibull power series (Munteanu et al., 2014) distributions respectively.

Furthermore, since

$$\lim_{\lambda\to 0} A(\lambda) = 0 \quad \text{and} \quad \lim_{\lambda\to 0} \frac{A(\lambda x)}{A(\lambda)} = x,$$

thereupon, we have

- The lifetime power series distributions with minimum structure is a special limiting case of the LPS^2 family of distributions when $\theta \to 0^+$. In general,

$$\begin{aligned}\lim_{\theta\to 0^+} F(x;\boldsymbol{\xi}) &= 1-[A(\lambda)]^{-1}\lim_{\theta\to 0} A\left(\lambda\left\{1-\frac{B(\theta\Pi(x;\boldsymbol{\varsigma}))}{B(\theta)}\right\}\right)\\ &= 1-[A(\lambda)]^{-1}A(\lambda\{1-\Pi(x;\boldsymbol{\varsigma})\});\end{aligned}$$

- The complementary lifetime power series distributions with maximum structure is a special limiting case of the LPS^2 family of distributions when $\lambda \to 0^+$. In general,

$$\lim_{\lambda\to 0^+} F(x;\boldsymbol{\xi}) = 1-\lim_{\lambda\to 0}\frac{A\left(\lambda\left\{1-\frac{B(\theta\Pi(x;\boldsymbol{\varsigma}))}{B(\theta)}\right\}\right)}{[A(\lambda)]^{-1}} = [B(\theta)]^{-1}B(\theta\Pi(x;\boldsymbol{\varsigma}));$$

- The baseline distribution is a special limiting case of this new family when $\theta, \lambda \to 0^+$

$$\lim_{\theta\to 0^+}\lim_{\lambda\to 0^+} F(x;\boldsymbol{\xi}) = \Pi(x;\boldsymbol{\varsigma}).$$

3. Some special models

In this section, we consider some special cases of the LPS^2 distribution. These special models generalize some well-known distributions in the literature. We provide four special models of this family corresponding to the baseline exponential, Weibull, Lomax (Lx), and generalized half-normal (GHN) distributions. To illustrate the flexibility of the distributions, graphs of the pdf and hrf for some selected distributions are presented.

It should be noted that compounding of a lifetime geometric family (Marshall and Olkin, 1997) with a geometric distribution again, just expand the parameter space and its cdf doesn't change. Suppose $X_{i,j}$ is a sequence of the independent identically lifetime random variables with cdf $F(x;\boldsymbol{\varsigma})$. The cdf of a lifetime geometric-geometric family of distributions (compounding a lifetime and twice geometric distribution) with parameter $\boldsymbol{\xi} = (\boldsymbol{\varsigma}, \theta, \lambda)$ is

$$F(x;\boldsymbol{\xi}) = \frac{(1-\lambda)F(x;\boldsymbol{\varsigma})}{1-\theta+(\theta-\lambda)F(x;\boldsymbol{\varsigma})},$$

for $x > 0$. With a reparameterization $\gamma = \frac{\theta-\lambda}{1-\lambda}$, we can write

$$F(x;\boldsymbol{\varsigma},\gamma) = \frac{F(x;\boldsymbol{\varsigma})}{1-\gamma F(x;\boldsymbol{\varsigma})},$$

for $x > 0$ and $\gamma < 1$. The lifetime geometric-geometric family of distributions (LGG) is due to Marshall and Olkin (1997) with expanded geometric parameter space. On the other hand, the parameter space of truncated Poisson distribution in compound distributions could be extended to $(-\infty,0)\bigcup(0,+\infty)$, and the parameter space of truncated binomial distribution could be extended to $(-1,0)\bigcup(0,+\infty)$. A more similar extension of the parameter space may be done to power series parameters (see Table 1).

3.1. Exponential power series-power series distribution (EPS^{2}D)

The EPS^{2}D distribution is defined from (1) by taking $\Pi(x;\beta) = 1-e^{-\beta x}$. Then, its density function is given by

$$f(x) = \frac{\beta\theta\lambda e^{-\beta x}B'\left(\theta\left[1-e^{-\beta x}\right]\right)}{A(\lambda)B(\theta)}A'\left(\lambda\left\{1-[B(\theta)]^{-1}B\left(\theta\left[1-e^{-\beta x}\right]\right)\right\}\right),$$

for $x > 0$ and $\beta > 0$.

3.2. Weibull power series-power series distribution (WPS^{2}D)

The cdf and pdf of the Weibull distribution with scale parameter β and shape parameter α are given by $\Pi(x;\alpha,\beta) = 1-e^{-\beta x^{\alpha}}$ and $\pi(x;\alpha,\beta) = \alpha\beta x^{\alpha-1}e^{-\beta x^{\alpha}}$, respectively. The WPS^{2}D pdf follows by inserting these expressions in (2) as

$$f(x) = \frac{\alpha\beta\theta\lambda x^{\alpha-1}e^{-\beta x^{\alpha}}B'\left(\theta\left[1-e^{-\beta x^{\alpha}}\right]\right)}{A(\lambda)B(\theta)}A'\left(\lambda\left\{1-[B(\theta)]^{-1}B\left(\theta\left[1-e^{-\beta x^{\alpha}}\right]\right)\right\}\right),$$

for $x > 0$, $\alpha > 0$ and $\beta > 0$. Figures 2 and 3 display the pdf and hrf of the Weibull geometric-Poisson distribution (WGPD) for some selected parameter values.

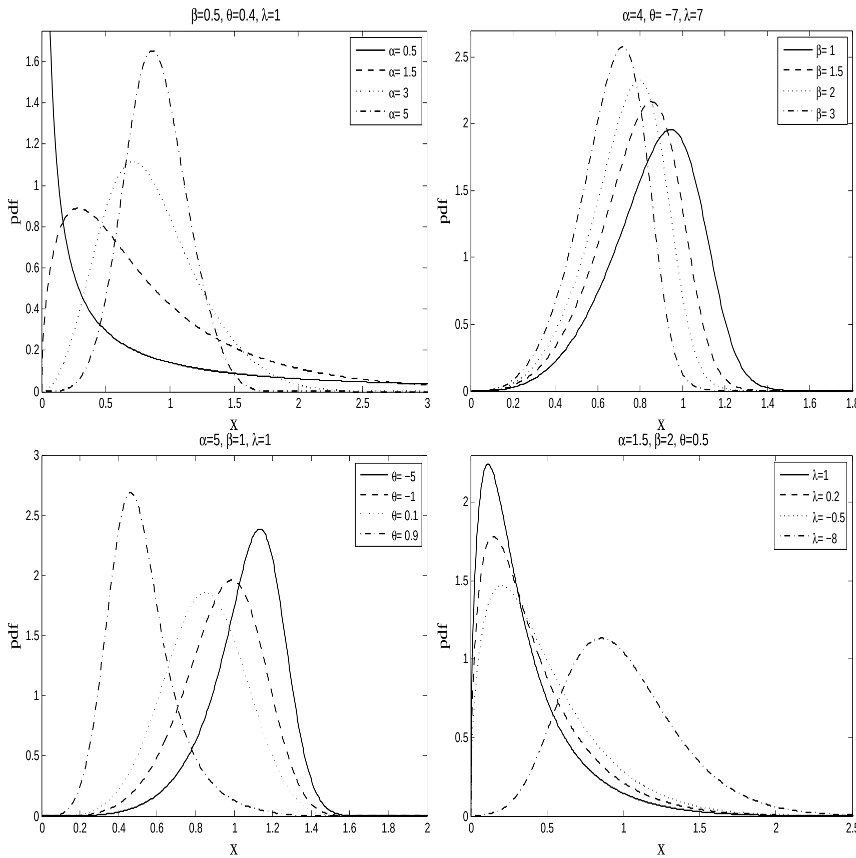


Figure 2: Graphs of the WGPD pdf for some selected values of the parameters.

3.3. Lomax power series-power series distribution (LxPS^{2}D)

The LxPS^{2}D distribution is defined from (2) by taking $\Pi(x;\beta) = 1-[1+\beta x]^{-\alpha}$ for the cdf of the Lomax distribution with parameters α and β. The LxPS^2 pdf is given by

$$f(x) = \frac{\alpha\beta\theta\lambda B'\left(\theta\left\{1-[1+\beta x]^{-\alpha}\right\}\right)}{A(\lambda)B(\theta)[1+\beta x]^{\alpha+1}} A'\left(\lambda\left\{1-[B(\theta)]^{-1}B\left(\theta\left\{1-[1+\beta x]^{-\alpha}\right\}\right)\right\}\right),$$

for $x > 0$, $\alpha > 0$ and $\beta > 0$.

3.4. Generalized half-normal power series-power series distribution (GHNPS^{2}D)

Cooray and Ananda (2008) introduced generalized half-normal distribution with cdf and pdf

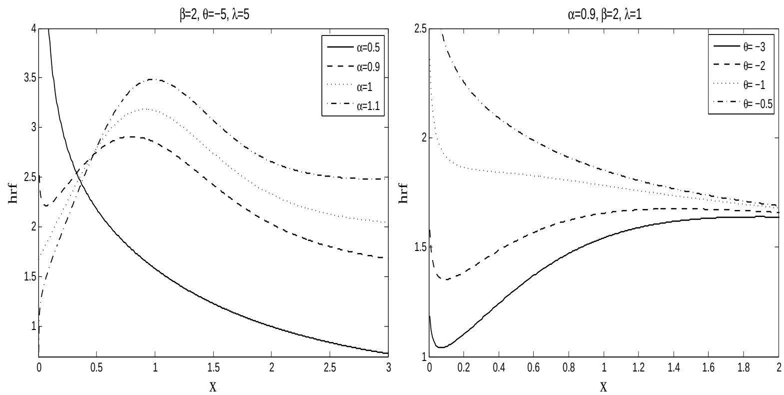


Figure 3: Graphs of the WGPD hrf for some selected values of the parameters.

$$\Pi(x) = 2\Phi(\beta x^{\alpha}) - 1 \text{ and } \pi(x) = \sqrt{\frac{2}{\pi}}\alpha\beta x^{\alpha-1}e^{-\frac{1}{2}(\beta x^{\alpha})^2},$$

respectively. The GHNPS^{2}D pdf follows by inserting these expressions in (2) as

$$f(x) = \frac{\sqrt{\frac{2}{\pi}}\alpha\beta\theta\lambda x^{\alpha-1}B'(\theta[2\Phi(\beta x^{\alpha}) - 1])}{A(\lambda)B(\theta)e^{\frac{1}{2}(\beta x^{\alpha})^2}}A'\left(\lambda\left\{1 - [B(\theta)]^{-1}B(\theta[2\Phi(\beta x^{\alpha}) - 1])\right\}\right),$$

for $x > 0$, $\alpha, \beta > 0$ and $\Phi(.)$ denotes the cdf of standard normal distribution.

4. Some useful properties

In this section, we derive some useful structural properties of the LPS^2 distributions. These include the two useful linear representations for (1) and (2) (Section 4.1), the r-th moment, moment generating function and mean residual lifetime (Section 4.2), the quantiles (Section 4.3).

4.1. Two useful linear representations

Let X be a LPS^2 random variable with parameters $\boldsymbol{\xi} = (\boldsymbol{\varsigma}, \theta, \lambda)$. Using the binomial expansion and $A'(\lambda) = \sum_{u=1}^{\infty} ua_u\lambda^{u-1}$, the cdf and pdf of X can be expanded as

$$F(x;\boldsymbol{\xi}) = \sum_{k=1}^{\infty}\sum_{j=0}^{\infty}\phi_{k,j}\Pi(x;\boldsymbol{\varsigma})^{k+j} \quad (3)$$

and

$$f(x;\boldsymbol{\xi}) = \pi(x,\boldsymbol{\varsigma}) \sum_{k=1}^{\infty} \sum_{j=0}^{\infty} \varphi_{k,j} \Pi(x;\boldsymbol{\varsigma})^{k+j-1}, \tag{4}$$

for $x > 0$, where $\phi_{k,j} = \phi_{k,j}(\theta,\lambda)$ and $\varphi_{k,j} = \varphi_{k,j}(\theta,\lambda) = (k+j)\phi_{k,j}$. For further details, see Appendix.

4.2. Moment properties

First, we derive the r-th moment for a random variable X. Therefore, the r-th moment of $X \sim \text{LPS}^2(\boldsymbol{\varsigma},\theta,\lambda)$ is given by

$$\begin{aligned} \mu'_r &= E[X^r] = \sum_{k=1}^{\infty} \sum_{j=0}^{\infty} \varphi_{k,j} \int_0^{\infty} x^r \pi(x,\boldsymbol{\varsigma}) \Pi(x)^{k+j-1} dx \\ &= \sum_{k=1}^{\infty} \sum_{j=0}^{\infty} \varphi_{k,j} M(r,k+j-1), \end{aligned}$$

for $r > 0$, where $M(s,k+j-1)$ is the $(s,k+j-1)$th probability weighted moment (PWM) of baseline distribution defined by Greenwood et al. (1979) as follows:

$$M(i,j) = E\left[X^i \Pi(X)^j\right] = \int_0^{+\infty} x^i [\Pi(x)]^j d\Pi(x).$$

The moment generating function (mgf) of the LPS^2 family of distributions is given by

$$\begin{aligned} M_X(t) &= E\left[e^{tX}\right] = E\left[\sum_{s=0}^{\infty} \frac{(tX)^s}{s!}\right] = \sum_{s=0}^{\infty} \frac{t^s}{s!} E[X^s] \\ &= \sum_{s,j=0}^{\infty} \sum_{k=1}^{\infty} \frac{\varphi_{k,j}}{s!} M(s,k+j-1) t^s. \end{aligned}$$

Given the survival to time x_0, the residual life is the period from x_0 until the time of failure. From (4), the mean residual lifetime of the LPS^2 distribution is given by

$$\begin{aligned} m(x_0) &= E[X - x_0 | X > x_0] = [S(x_0;\boldsymbol{\xi})]^{-1} \int_{x_0}^{\infty} v f(v) dv - x_0 \\ &= [S(x_0;\boldsymbol{\xi})]^{-1} \sum_{k=1}^{\infty} \sum_{j=0}^{\infty} \varphi_{k,j} M_{x_0}(1,k+j-1) - x_0, \end{aligned}$$

where the upper incomplete probability weighted moment was defined as

$$M_{x_0}(i,j) = \int_{x_0}^{\infty} x^i [\Pi(x)]^j d\Pi(x).$$

4.3. Quantiles

If U is a uniform $[0,1]$ random variable then

$$X = \Pi^{-1}\left(\frac{1}{\theta}B^{-1}\left(B(\theta)\left[1-\frac{1}{\lambda}A^{-1}(UA(\lambda))\right]\right)\right)$$

is a LPS^2 random variable, where $\Pi^{-1}(.)$ is the inverse of baseline cdf. Furthermore, $A^{-1}(.)$ and $B^{-1}(.)$ are the inverse of $A(.)$ and $B(.)$ functions, respectively. It follows that the ωth quantile of the LPS^2 distributions is

$$x_{\omega} = \Pi^{-1}\left(\frac{1}{\theta}B^{-1}\left(B(\theta)\left[1-\frac{1}{\lambda}A^{-1}(\omega A(\lambda))\right]\right)\right).$$

The effects of the parameters on the skewness of random variable X can be shown based on quantiles. The Bowley skewness (Kenney and Keeping, 1962), also known as the quantile skewness coefficient, is defined by

$$B = \frac{x_{0.75}+x_{0.25}-2x_{0.5}}{x_{0.75}-x_{0.25}}.$$

Figure 4 graphs the Bowleie's measure for the WGPD distribution. The graph indicates the variability of this measures on the α, β, θ and λ parameters.

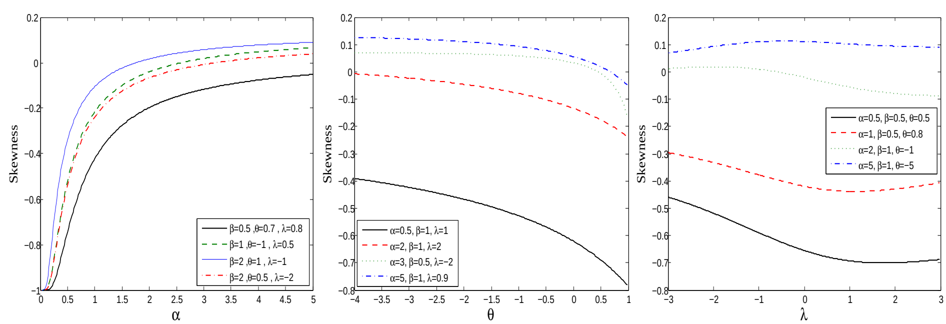


Figure 4: Graphs of skewness based on the quantiles of the WGPD distribution.

5. Estimation of the parameters

In this section, we determine the maximum likelihood estimates (MLEs) of the parameters of the LPS^2 family of distributions from complete samples only. Let $X = (X_1, X_2, \ldots, X_n)$ be a random sample from the LPS^2 distribution with observed values $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and parameters $\boldsymbol{\xi} = (\boldsymbol{\varsigma}, \theta, \lambda)$. The log-likelihood function is given by

$$\begin{aligned}\ell(\boldsymbol{\xi} \mid \mathbf{x}) &= n\log\theta + n\log\lambda - n\log[A(\lambda)] - n\log[B(\theta)] + \sum_{i=1}^{n}\log[\pi(x_i;\boldsymbol{\varsigma})] \\ &\quad + \sum_{i=1}^{n}\log\left[A'\left(\lambda\left\{1-[B(\theta)]^{-1}B(\theta\Pi(x_i;\boldsymbol{\varsigma}))\right\}\right)\right]. \end{aligned} \tag{5}$$

By differentiating (5) with respect to $\boldsymbol{\varsigma}$, θ and λ, and then equating these derivative to zero, we obtain the components of score vector $U_n(\boldsymbol{\xi}) = \left(\frac{\partial\ell}{\partial\boldsymbol{\varsigma}}, \frac{\partial\ell}{\partial\theta}, \frac{\partial\ell}{\partial\lambda}\right)$, where

$$\begin{aligned}\frac{\partial\ell}{\partial\boldsymbol{\varsigma}} &= \sum_{i=1}^{n}\frac{\pi_{\boldsymbol{\varsigma}}(x_i;\boldsymbol{\varsigma})}{\pi(x_i;\boldsymbol{\varsigma})} \\ &\quad - \frac{\lambda\theta}{B(\theta)}\sum_{i=1}^{n}\frac{\Pi_{\boldsymbol{\varsigma}}(x_i;\boldsymbol{\varsigma})B'\left(\theta\Pi_{\boldsymbol{\varsigma}}(x_i;\boldsymbol{\varsigma})\right)A''\left(\lambda\left\{1-[B(\theta)]^{-1}B(\theta\Pi(x_i;\boldsymbol{\varsigma}))\right\}\right)}{A'\left(\lambda\left\{1-[B(\theta)]^{-1}B(\theta\Pi(x_i;\boldsymbol{\varsigma}))\right\}\right)},\end{aligned}$$

$$\begin{aligned}\frac{\partial\ell}{\partial\theta} &= \frac{n}{\theta} - \frac{nB'(\theta)}{B(\theta)} - \lambda\sum_{i=1}^{n}\frac{B(\theta)\Pi(x_i;\boldsymbol{\varsigma})B'\left(\theta\bar{G}(x_i;\boldsymbol{\varsigma})\right) - B'(\theta)B(\theta\Pi(x_i;\boldsymbol{\varsigma}))}{[B(\theta)]^2} \\ &\quad \times\frac{A''\left(\lambda\left\{1-[B(\theta)]^{-1}B(\theta\Pi(x_i;\boldsymbol{\varsigma}))\right\}\right)}{A'\left(\lambda\left\{1-[B(\theta)]^{-1}B(\theta\Pi(x_i;\boldsymbol{\varsigma}))\right\}\right)}\end{aligned}$$

and

$$\begin{aligned}\frac{\partial\ell}{\partial\lambda} &= \frac{n}{\lambda} - \frac{nA'(\lambda)}{A(\lambda)} \\ &\quad + \sum_{i=1}^{n}\frac{\left\{1-[B(\theta)]^{-1}B\left(\theta\bar{G}(x_i;\boldsymbol{\varsigma})\right)\right\}A''\left(\lambda\left\{1-[B(\theta)]^{-1}B(\theta\Pi(x_i;\boldsymbol{\varsigma}))\right\}\right)}{A'\left(\lambda\left\{1-[B(\theta)]^{-1}B(\theta\Pi(x_i;\boldsymbol{\varsigma}))\right\}\right)},\end{aligned}$$

where

$$\Pi_{\boldsymbol{\varsigma}}(x_i;\boldsymbol{\varsigma}) = \frac{\partial\Pi(x_i;\boldsymbol{\varsigma})}{\partial\boldsymbol{\varsigma}} \text{ and } \pi_{\boldsymbol{\varsigma}}(x_i;\boldsymbol{\varsigma}) = \frac{\partial\pi(x_i;\boldsymbol{\varsigma})}{\partial\boldsymbol{\varsigma}}.$$

The maximum likelihood estimates, $\widehat{\boldsymbol{\xi}}$ of $\boldsymbol{\xi} = (\boldsymbol{\varsigma},\theta,\lambda)$ are obtained by solving the nonlinear equations $U_n(\boldsymbol{\xi}) = \left(\frac{\partial\ell}{\partial\boldsymbol{\varsigma}}, \frac{\partial\ell}{\partial\theta}, \frac{\partial\ell}{\partial\lambda}\right) = \mathbf{0}$. These equations have no closed form and the values of the parameters $\boldsymbol{\varsigma}$, θ and λ must be found by using iterative methods. To solve these equations, it is usually more convenient to use nonlinear optimization methods such as the Newton-Raphson, quasi-Newton, or Nelder-Mead procedures. The Adequacy Model package version 1.0.8 available in the R programming language was used for numerical maximization in the data examples Section 7. For interval estimation of $(\boldsymbol{\varsigma},\theta,\lambda)$ and hypothesis tests on these parameters, we obtain the observed information matrix since the expected information matrix is very complicated and requires numerical integration. The $(p+2)\times(p+2)$ observed information matrix $J_n(\boldsymbol{\xi})$, where p is the dimension of the parameter vector $\boldsymbol{\varsigma}$, becomes

$$J_n(\boldsymbol{\xi}) = \begin{pmatrix} \frac{\partial^2 \ell}{\partial \boldsymbol{\varsigma}^2} & \frac{\partial^2 \ell}{\partial \boldsymbol{\varsigma} \partial \theta} & \frac{\partial^2 \ell}{\partial \boldsymbol{\varsigma} \partial \lambda} \\ \frac{\partial^2 \ell}{\partial \theta \partial \boldsymbol{\varsigma}} & \frac{\partial^2 \ell}{\partial \theta^2} & \frac{\partial^2 \ell}{\partial \theta \partial \lambda} \\ \frac{\partial^2 \ell}{\partial \lambda \partial \boldsymbol{\varsigma}} & \frac{\partial^2 \ell}{\partial \lambda \partial \theta} & \frac{\partial^2 \ell}{\partial \lambda^2} \end{pmatrix}.$$

Under the usual regularity conditions and that the parameters are in the interior of the parameter space, but not on the boundary and large n, the distribution of $\sqrt{n}\left(\widehat{\boldsymbol{\xi}} - \boldsymbol{\xi}\right)$ can be approximated by $N_{p+2}\left(\mathbf{0}, nJ_n^{-1}(\boldsymbol{\xi})\right)$. This approximation can be used to construct confidence intervals and tests of hypotheses.

6. A simulation study

In this section, we assess the performance of the MLEs of the WGPD distribution as the particular case of LPS^2 distribution with respect to sample size n. Samples of sizes 20, 50, 100, 200 and 500 are generated for different combinations of $\boldsymbol{\xi} = (\alpha, \beta, \theta, \lambda)$ from WGPD distribution by using (5). We repeated the simulation k =1000 times with parameter values I : $\alpha = 2$, $\beta = 1$, $\theta = 0.5$, $\lambda = 1$ and II : $\alpha = 1.5$, $\beta = 0.5$, $\theta = 0.7$, $\lambda = 0.8$, then the MLEs of the parameters are calculated. The standard deviation (SD) of the parameter estimates are computed by inverting the observed information matrices. The bias and mean squared errors (MSE) are given respectively by

$$\text{bias}_{\varepsilon}(n) = \frac{1}{1000}\sum_{i=1}^{1000}(\hat{\varepsilon}_i - \varepsilon)$$

and

$$\text{MSE}_{\varepsilon}(n) = \frac{1}{1000}\sum_{i=1}^{1000}(\hat{\varepsilon}_i - \varepsilon)^2,$$

for $\varepsilon = \alpha, \beta, \theta, \lambda$ where $\hat{\varepsilon}_i$ is ith MLE of ε with standard error $s_{\hat{\varepsilon}_i}$. The empirical results are given in Table 2 indicate that the MLEs perform well for estimating the model parameters. According to the results, it can be concluded that as the sample size n increases, the MSEs decay toward zero. We also observe that for all the parameters, the biases decrease as the sample size n increases.

Table 2: The mean, bias, MSE, standard error of the MLE estimators.

		I					*II*				
n	Parameter	R.value	MLE	Bias	MSE	SD	R.value	MLE	Bias	MSE	SD
$n=20$	α	2	2.0912	0.0912	0.1129	0.4445	1.5	1.5412	0.0412	0.0478	0.2715
	β	1	1.1614	0.1614	0.3196	0.7924	0.5	0.5738	0.0738	0.0827	0.4787
	θ	0.5	0.4743	-0.0257	0.2130	0.9124	0.7	0.6281	-0.0719	0.0671	0.6591
	λ	1	1.2050	0.2050	0.7241	3.2141	0.8	0.7705	-0.0295	0.2252	2.9567
$n=50$	α	2	2.0675	0.0675	0.0621	0.3502	1.5	1.5133	0.0133	0.0319	0.3024
	β	1	1.0758	0.0758	0.2385	0.6444	0.5	0.5641	0.0641	0.1539	0.3816
	θ	0.5	0.4815	-0.0185	0.1861	0.9618	0.7	0.6202	-0.0798	0.0552	0.3252
	λ	1	0.8462	-0.1538	0.6177	2.9721	0.8	0.7888	-0.0112	0.2413	2.6142
$n=100$	α	2	2.0561	0.0561	0.0464	0.2216	1.5	1.5031	0.0310	0.0151	0.1699
	β	1	1.0877	0.0877	0.1445	0.3892	0.5	0.5494	0.0494	0.0483	0.2502
	θ	0.5	0.4921	-0.0079	0.0706	0.7271	0.7	0.6594	-0.0406	0.0387	0.6048
	λ	1	1.0921	0.0921	0.5822	2.3921	0.8	0.7354	-0.0646	0.2166	2.1081
$n=200$	α	2	1.9792	-0.0208	0.0162	0.1648	1.5	1.4985	-0.0015	0.0076	0.1263
	β	1	1.0393	0.0393	0.0485	0.2704	0.5	0.5185	0.0185	0.0176	0.1519
	θ	0.5	0.4216	-0.0787	0.6224	0.7334	0.7	0.6762	-0.0238	0.0198	0.4513
	λ	1	1.0434	0.0434	0.1905	2.1947	0.8	0.7766	-0.0234	0.2347	1.6921
$n=500$	α	2	1.9875	-0.0125	0.0085	0.1147	1.5	1.4955	-0.0045	0.0033	0.0914
	β	1	1.0079	0.0079	0.0277	0.1807	0.5	0.5053	0.0053	0.0084	0.1059
	θ	0.5	0.4888	-0.0112	0.0412	0.4425	0.7	0.6894	-0.0106	0.0177	0.3191
	λ	1	0.9871	-0.0129	0.0927	1.5535	0.8	0.8302	0.0302	0.0962	1.2971

7. Two application examples

In this section, we present two applications of LPS^2 family of distributions using real-life data sets. In the applications, we use the Adequacy Model package version 1.0.8 available in the R programming language. The fit is compared to other distributions based on the maximized log-likelihood, the Kolmogorov-Smirnov test (K-S), Akaike Information Criterion (AIC), corrected Akaike Information Criterion (AICc) and Bayesian Information Criterion (BIC). Finally, we provide the histograms of the data sets to have a visual comparison of the fitted density functions.

Data set 1

The first set consists of the number of successive failures for the air conditioning system of each member in a fleet of 13 Boeing 720 airplanes. The pooled data, yielding a total of 213 observations, were first analyzed by Proschan (1963) and further discussed in Dahiya and Gurland (1972), Adamidis and Loukas (1998) and Tahmasbi and Rezaei (2008). Table 3 gives some descriptive statistics for the first data set. Figure 5a displays the Gaussian kernel density estimation for this data set.

For this data set, the new distributions, exponential geometric binomial (EGBD) and

Table 3: Descriptive statistics for data set 1.

n	Mean	Q_1	Median	Q_3	Mode	Variance	Skewness	Kurtosis	Min	Max
213	93.14	22	57	118	14	11398.47	2.11	7.92	1	603

Weibull geometric geometric (WGGD) distributions given by the following pdfs were fitted:

$$f_{EGBD}(x;\alpha,\beta,\theta,\lambda) = \frac{m\lambda\beta(1-\theta)e^{-\beta x}\left[1-\theta+(\theta+\lambda)e^{-\beta x}\right]^{m-1}}{\left[(1+\lambda)^m-1\right]\left\{1-\theta\left(1-e^{-\beta x}\right)\right\}^{m+1}},$$

and

$$f_{WGGD}(x;\alpha,\beta,\gamma) = \frac{\alpha\beta(1-\gamma)x^{\alpha-1}e^{-\beta x^\alpha}}{\left\{1-\gamma e^{-\beta x^\alpha}\right\}^2}$$

for $x > 0$, $\alpha,\beta > 0$, $\gamma,\theta < 1$, $\lambda \in \mathbb{R}$. For comparison purposes, we also fit the generalization of Weibull distribution (GWD) (Shanker and Shukla, 2019), the generalization of generalized gamma distribution(GGD) (Shanker and Shukla, 2019), beta exponential (BE) (Nadarajah and Kotz, 2006) and odd Weibull (OW) (Cooray, 2006) distributions. Estimates of the parameters of the distributions, standard errors (in parentheses), log-likelihood function evaluated at the parameter estimates, K-S statistic and its p-value are shown in Table 4. Furthermore, to compare the models, the AIC, AICc, and BIC indices are obtained too. In general, the smaller the values of these criteria, the better the fit. According to these formal tests, the WGGD model has the largest likelihood, the smallest K-S statistic, the largest p-value, and the smallest values for all other indices, among all fitted models.

Figure 6a gives the graph of the estimated pdfs of the WGGD, EGBD, and other competitive models that are used to fit the data after replacing the unknown parameters included in each distribution by their MLEs. The fitted pdf of the WGGD distribution captures the observed histograms better than others for the data sets 1. This real example suggests that the three-parameter WGGD fits data set 1 very well when compared to the other distributions.

Data set 2

The second application takes into account the data related to the breaking stress of carbon fibres of 50 mm in length from Nicholas and Padgett (2006). This data set was used by Cordeiro and Lemonte (2011) which is given in Table 5. Table 6 gives some descriptive statistics for this data set. Figure 5b displays the Gaussian kernel density estimation for the second data set.

For the second data set, the new distributions, generalized half normal geometri Poisson distribution (GHNGPD) and WGPD were fitted:

$$f_{CHNGPD}(x;\alpha,\beta,\theta,\lambda) = \frac{\sqrt{\frac{2}{\pi}}\alpha\beta\lambda(1-\theta)x^{\alpha-1}e^{-\frac{1}{2}(\beta x^\alpha)^2}}{\left[e^\lambda-1\right]\left\{1-2\theta\Phi(-\beta x^\alpha)\right\}^2}\exp\left[\frac{2\lambda(1-\theta)\Phi(-\beta x^\alpha)}{1-2\theta\Phi(-\beta x^\alpha)}\right],$$

Table 4: Estimates and goodness-of-fit measures for the first data set.

Model	MLEs (standard errors)	Log-likelihood	K-S	p-value	AIC	AIC$_c$	BIC
EGBD	0.0067, 0.0403, 0.2201	-1175.871	0.0564	0.633	2357.74	2357.85	2367.82
SE	(0.0022, 0.3957, 0.2170)						
WGGD	1.1982, 0.0017, 0.7651	-1174.180	0.0504	0.765	2354.36	2354.47	2364.44
SE	(0.029, 2.14×10^{-5}, 0.0430)						
GWD	0.9395, 0.9219, 0.0168	-1177.586	0.0662	0.425	2361.17	2361.34	2371.26
SE	(1.0679, 0.0487, 0.0168)						
GGD	3.7958, 0.4419, 0.6943, 0.6911	-1174.514	0.0537	0.685	2357.03	2357.218	2370.47
SE	(2.8461, 0.1786, 0.2453, 0.2701)						
BE	1.0483, 2.2710, 0.0104	-1177.771	0.0637	0.475	2361.54	2361.73	2371.62
SE	(0.5925, 1.5206, 0.0058)						
OW	0.6667, 0.0469, 1.4838	-1176.062	0.0587	0.581	2358.12	2358.24	2368.20
SE	(0.1581, 0.0312, 0.3907)						

Table 5: Breaking stress of carbon fibres data.

0.39	0.85	1.08	1.25	1.47	1.57	1.61	1.61	1.69	1.80	1.84
1.87	1.89	2.03	2.03	2.05	2.12	2.35	2.41	2.43	2.48	2.50
2.53	2.55	2.55	2.56	2.59	2.67	2.73	2.74	2.79	2.81	2.82
2.85	2.87	2.88	2.93	2.95	2.96	2.97	3.09	3.11	3.11	3.15
3.15	3.19	3.22	3.22	3.27	3.28	3.31	3.31	3.33	3.39	3.39
3.56	3.60	3.65	3.68	3.70	3.75	4.20	4.38	4.42	4.70	4.90

and

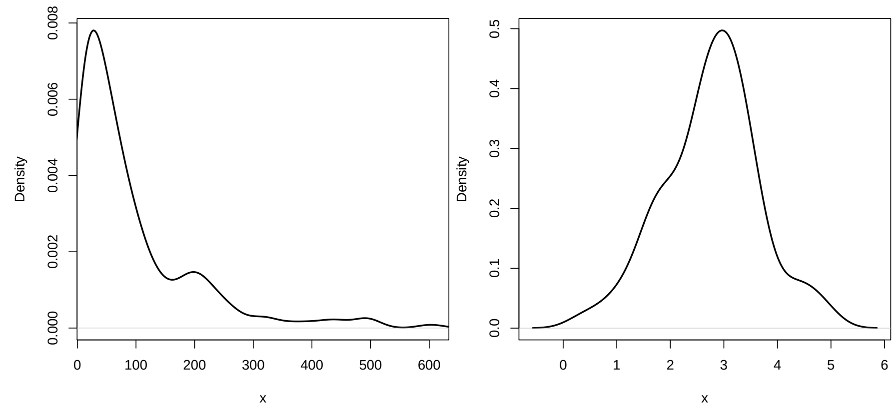
$$f_{WGPD}\left(x;\alpha,\beta,\theta,\lambda\right)=\frac{\alpha\beta\lambda\left(1-\theta\right)x^{\alpha-1}e^{-\beta x^{\alpha}}}{\left[e^{\lambda}-1\right]\left\{1-\theta e^{-\beta x^{\alpha}}\right\}^{2}}\exp\left[\frac{\lambda\left(1-\theta\right)e^{-\beta x^{\alpha}}}{1-\theta e^{-\beta x^{\alpha}}}\right]$$

for $x>0$, $\alpha,\beta>0$, $\theta<1$, $\lambda\in\mathbb{R}$. We also fit the BE, beta Weibull (BW) (Famoye et al., 2005), Cauchy Weibull logistic (CWL) (Almheidat et al., 2015), Gumbel Weibull (GW) (Al-Aqtash et al., 2014) distributions to make a comparison with the new models. The parameter estimates, the log-likelihood values, the Kolmogorov-Smirnov statistics, and respective p-values are given in Table 7. Additionally, a comparison of these proposed distributions using the criteria, explained earlier, is presented.

It is observed that the WGPD distribution provides the best fit. In particular, we can see that the largest log-likelihood value, the largest p-value, the smallest AIC value, the smallest

Table 6: Descriptive statistics for data set 2.

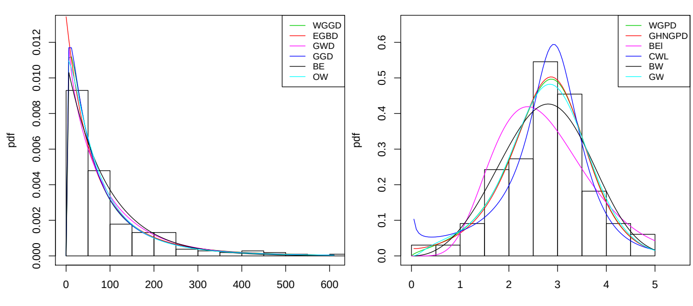
n	Mean	Q_1	Median	Q_3	Mode	Variance	Skewness	Kurtosis	Min	Max
66	2.178	2.178	2.853	3.278	1.61	0.795	-0.131	3.223	0.390	4.90



(a) Data set 1 (b) Data set 2

Figure 5: The Gaussian kernel density estimation for: (a) data set 1 (b) data set 2.

AICc value, and the smallest BIC value are obtained for the WGPD distribution. The fitted densities (with the respective histogram) are shown in Figure 6b. These indicate a good fit for the WGPD distribution for the second data set. It is clear from Tables 4 and 7 and also Figures 6a and 6b that the WGGD and WGPD models provide the best fits to these two real data sets.



(a) Data set 1 (b) Data set 2

Figure 6: Estimates of the density functions for the: (a) data set 1 (b) data set 2.

More on estimated hazard functions

The failure rate of a system usually depends on time, with the rate varying over the life cycle of the system. the hazard rate refers to the rate of failure for a system of a given age

Table 7: Estimates and goodness-of-fit measures for the second data set.

Model	MLEs (standard errors)	Log-likelihood	K-S	p-value	AIC	AIC$_c$	BIC
GHNGPD	0.8290, 1.0219, -37.8800, -0.8760	-84.849	0.0744	0.882	177.70	178.35	186.46
SE	(0.4745, 0.9104, 73.3213, 4.1871)						
WGPD	1.6669, 0.5589, -11.9420, -1.0240	-84.705	0.0723	0.902	177.41	178.07	186.17
SE	(0.8913, 0.9326, 29.3124, 2.8723)						
BE	0.1131, 7.5072, 20.9967	-91.223	0.1393	0.181	188.47	188.85	195.01
SE	(0.0170, 0.7642, 1.4865)						
CWL	2.1437, 7.9321, 2.9530	-86.989	0.1040	0.515	179.98	180.37	186.55
SE	(0.7221, 1.8887, 0.1083)						
BW	3.6790, 0.0136, 0.8820, 1.0594	-85.971	0.0830	0.786	179.94	180.60	188.70
SE	(0.7915, 0.0133, 0.3241, 1.2743)						
GW	3.4359, 5.5673, 2.4231, 1.1324	-84.834	0.0733	0.893	177.67	187.32	186.43
SE	(1.1494, 2.8064, 0.5078, 0.4524)						

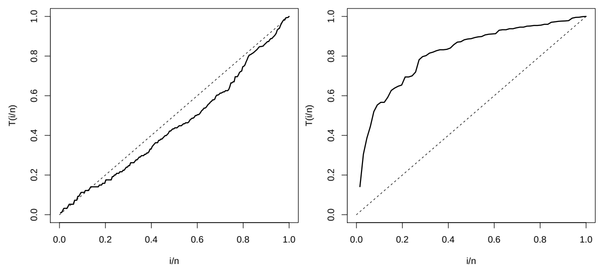
x and is defined as $h(x) = f(x)/S(x)$. Hazard rate provides an alternative characterization for the distribution of a random variable, especially when dealing with lifetime data and it is quite useful in defining and formulating a model. In this section, we focus on estimated hrfs as for the previous sections.

First, we provide the total time on test (TTT) transform procedure proposed by Aarset (1987) as a tool to identify the hazard behaviour of the distribution. The TTT-transform can illustrate the variety of the hazard rate curves for a lifetime distribution. If the empirical TTT-transform is convex and concave, the shape of the corresponding hrf is decreasing and increasing, respectively. If the TTT-transform is convex and concave, the hrf will have a bathtub shape. Finally, if the TTT-transform is concave and convex, a unimodal hrf will be more appropriate. Figure 7a shows that the TTT-plot for the first data set has a concave and convex shape. It indicates that the hrf has a unimodal shape. Figure 7b shows that the TTT-plot for the second data set has a concave shape. It indicates that the hrf has an increasing shape.

Graphs of the estimated hrfs are displayed in Figures 8 for data sets 1 and 2. Hence, the WGGD and WGPD distributions could be the appropriate models for the fitting of such data sets.

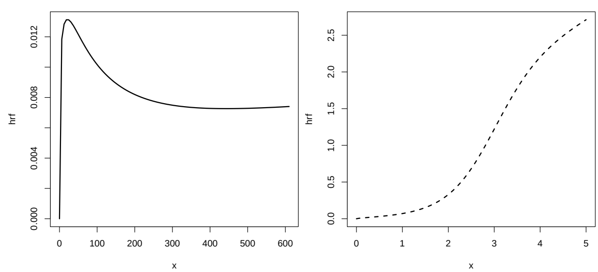
8. Conclusions

We introduce a new generalized class of lifetime distributions, called the LPS^2 family of distributions, by compounding a lifetime and twice power series distributions in a serial and parallel structure. The new models extend several distributions widely used in the lifetime



(a) Data set 1 (b) Data set 2

Figure 7: TTT-plot on the data sets 1 and 2.



(a) Data set 1 (b) Data set 2

Figure 8: Graphs of estimated hrf for the data sets 1 and 2.

literature such as the exponential power series, Weibull power series, and complementary of exponential power series distributions. The pdf of the new distributions can be expressed as a linear combination of baseline distributions and they have a hazard function that displays flexible behaviour. We provide a mathematical treatment of this family, including moments, quantiles, reliability functions, and moment generating function as well as the mean residual lifetime. The method of maximum likelihood was used to estimate the model parameters. We perform a Monte Carlo simulation study to assess the finite sample behavior of the maximum likelihood estimators. Some members of the LPS^2 family are fitted to two real data sets to illustrate the usefulness of the new distributions. They provide better fits than other competing models consistently.

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Appendix

Proof of (3) and (4)

Using the binomial expansion, we have

$$\begin{aligned} F(x;\boldsymbol{\xi}) &= 1-[A(\lambda)]^{-1}\sum_{u=1}^{\infty}a_u\lambda^u\left\{1-[B(\theta)]^{-1}B(\theta\Pi(x,\boldsymbol{\varsigma}))\right\}^u \\ &= [A(\lambda)]^{-1}\sum_{u=1}^{\infty}a_u\lambda^u\left(1-\left\{1-[B(\theta)]^{-1}B(\theta\Pi(x,\boldsymbol{\varsigma}))\right\}^u\right) \\ &= [A(\lambda)]^{-1}\sum_{u=1}^{\infty}a_u\lambda^u\sum_{k=1}^{u}\binom{u}{k}(-1)^{k+1}[B(\theta)]^{-k}\left\{\sum_{z=1}^{\infty}b_z\theta^z[\Pi(x,\boldsymbol{\varsigma})]^z\right\}^k \\ &= [A(\lambda)]^{-1}\sum_{u=1}^{\infty}a_u\lambda^u\sum_{k=1}^{u}\binom{u}{k}(-1)^{k+1}[B(\theta)]^{-k}\sum_{j=0}^{\infty}l_{k,j}\theta^{k+j}[\Pi(x,\boldsymbol{\varsigma})]^{k+j}, \end{aligned}$$

where $j=z-1$ and $l_{k,j}=(jb_1)^{-1}\sum_{m=1}^{j}[m(j+1)-j]\,b_{m+1}l_{k,j-m}$. Then

$$F(x;\boldsymbol{\xi}) = \sum_{k=1}^{\infty}\sum_{j=0}^{\infty}\phi_{k,j}\Pi(x,\boldsymbol{\varsigma})^{k+j},$$

where

$$\phi_{k,j}=\phi_{k,j}(\theta,\lambda)=[A(\lambda)]^{-1}\sum_{u=k}^{\infty}\binom{u}{k}(-1)^{k+1}l_{k,j}a_u\lambda^u\theta^{k+j}[B(\theta)]^{-k}.$$

Finally, Equation (4) is obtained by using the direct differentiation of (3).