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Under military war weapon support the economic bond level estimation using generalized Petersen graph with imputation

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ABSTRACT

Several countries of the world are involved in mutual and collaborative business of military equipments, weapons in terms of their production, sales, technical maintenance, training and services. As a consequence, manufacturing of boms, rockets, missiles and other ammunitions have taken structured and smooth shape to help others where and when needed. Often the military support among countries remain open for information to the media, but sometime remain secret due to the national security and international political pressure. Such phenomenon (hidden or open support) is a part of military supply chain and could be modeled like a Petersen graph considering vertices as countries and edges as economic bonds. For a large graphical structure, without sampling, it is difficult to find out average economic bonding (open & secret) between any pair of countries involved in the military business or support.

This paper presents a sample based estimation methodology for estimating the mean economic bond value among countries involved in the military support or business. Motivation to the problem is derived from current Russia-Ukraine war situation and a kind of hidden support to war by NATO countries. A node sampling procedure is proposed whose bias, mean-squared error and other properties are derived. Results are supported with empirical studies. Findings are compared with particular cases and confidence intervals are used as a basic tool of comparison. Pattern imputation is used together with a new proposal of CI-Imputation method who has been proved useful for filling the missing value, specially when secret economic support data from involved countries found missing. The current undergoing war between Ukraine and Russia and secret weapon, economic support from NATO countries is an application of the proposed methodology contained in this paper.

Key words: Graph, Petersen Graph, Estimator, Bias, Mean Squared Error (MSE), Optimum Choice, Confidence intervals (CI), Nodes (vertices), Pattern Imputation, CI-Imputation (LL-imputation and UL-imputation), Economic Bonds, Military War, Weapon Support.

1. Introduction

The Russian war in Ukraine is a kind of complecated political event prolonged over time frame. After the pass of many months it is hard to predict about the ultimate date of war end from either side. The Russian invasion was started in Feb, 2022 and by April, 2022 as per United Nations High Commissioner for Humen rights report more than 2800 death of civilians occurred in Ukraine. There is big difference between military capacity of NATO, Russia and Ukraine as per record of 2022.

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Table 1.1 World Nuclear Forces [see link the independent resource on global security]

Country	Deployed warheads*	Other warheads**	Total 2021	Total 2020
USA	1800	3750	5550	5800
Russia	1625	4630	255	6375
UK***	120	105	225	215
France	280	10	290	290
China	–	350	350	320
India	–	156	156	150
Pakistan	–	165	165	160
Israel	–	90	90	90
North Korea****	–	[40-50]	[40-50]	[30-40]
Total	3825	9255	13080	13400

In view of report China support to Ukraine [see Link], the Russia asked China for military assistance of equipments and economic support. It is customary to ask for business deal, financial and military support from either country during the war period from neighbouring coutries, China support to Ukraine and Nine big question answered by Russia [See links].

Assume several countries of the world involved with each other in trading of war-weapons. They are having economic bonding among themselves in terms of export, import, supply and manufacturing of war-weapons etc. As an example, several NATO countries are involved in mutual colllaboration and exchange of weapons during the current war of Ukrain and Russia. All Europian countries can be treated as a group involved in supply of open and secrete war weapons to countries involved in fight to save the own territory. One can visualize the current war scenario as under:

(a) Type I: Between war group countries, the open and accountable war-weapon business.

(b) Type II: Within a country accountable war-weapon business.

(c) Type III: Secret (unaccountable) war-weapon business andsupport between countries.

The table 1.2 reveals such the structure of type I, II, III in terms of numerical values for only five countries A, B, C, D and E (treating a_{ij} as business value, i, j =1,2,3,4).

Table 1.2 Countries and War Period Exchange Economic

Countries	Type I (units)	Type II (units)	Type III (units)
$A \to B$	a_{11}	a_{12}	a_{13}
$B \to C$	a_{21}	a_{22}	a_{23}
$C \to D$	a_{31}	a_{32}	a_{33}
$D \to E$	a_{41}	a_{42}	a_{43}

(d) External Economic Bonding: It is defined as the accountable weapon trade between two countries which is auditable.

(e) Internal Economic Bonding: It is the internal accountable war-weapon trade within country among army, defence, security forces and internal manufacturing companies.

(f) Secret Economic Bonding: The trade of military products between and within countries who are secret (un-accountable) like many NATO countries are supporting Ukraine providing secret war weapons (as per reports).

Remark 1 *The information about type-I, type-II and type-III business (Economic bonds) can be obtained through the National Audit reports, United Nations reports (like IMF reports, Security Council reports, media and spying agencies reports etc.) either immediately or after long time when war is over. For intermediate Economic bonds, within country, the ordinance factories who are producing gun, tanks, arms and ammunitions and supplying those to own army, Paramilitary Forces, Private Security agencies within the country may be considered. For secret Economic bonding, information about only few units in sample is required which may available, at any instant, through authentic media sources.*

1.1. Objective

In view to Ukraine-Russia war, interest of data analyst is to evaluate the average amount of internal economic and secret economic bond together existing between any two countries using sampling techniques and imputation method if secret economic data found missing.

1.2. Motivation

The Europian country organizations (like EU or NATO) have open and free trade policies among them in currency EUROS. During war and military action, the secret economic and infrastructrue exchange is an obvious possible internal factor. A Petersen graph can be used as a model tool to represent such real situation where vertices (inner and outer) be countries and edges (weapon deal) be the trade among them during war period. Outer edges are for accountable weapon business between countries, intermediate edges are for within country and inner edges represent secret business. The current war and hidden weapon supply (with financial support) have motivated to model the real war situation like a Petersen graph.

The generalized Petersen graph G(n,k) was introduced by Coxeter et al. (1950) and named by Watkins (1969) from very interesting family of trivalent graphs that can be described by only two integer parameters. They include Hamiltonian and non-Hamiltonian graph, Bipartite and non-Bipartite graphs, vertex transitive and non-vertex transitive graphs, cayley and non-cayley graphs of girth 3,4,5,6,7 or 8 [Krnc, M. et al. (2018)]. A generalized Petersen graph G(n,k) is a family of cubic graph who is 3-regular graph. Following notations of Watkins et al. (1969) for a given integer n and $k < \frac{n}{2}$ one can define a Petersen graph $G\ (n,k)$ as a graph of vertex set $(\mu_0, \mu_1, ..., \mu_{n-1}, v_0, v_1, ..., v_{n-1})$ and edge set partitioned into three equal parts $(\mu_i \mu_{i+1}, \mu_i v_i, v_i v_{i+k} \mid 0 \le i \le n-1)$ where subscripts are to be read modulo n. The $G\ (3,1)$ and $G\ (4,1)$ are given below as examples (fig 1.1 and 1.2).

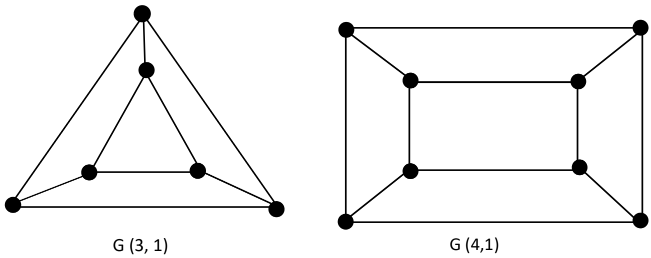

Fig 1.1 Petersen Graph

Let $\mu = (\mu_1, \mu_2, \mu_3, ...)$ denotes a set of vertices and $\varepsilon = (\varepsilon_1, \varepsilon_2, \varepsilon_3, ...)$ is a set of edges. The $G = (\mu, \varepsilon, R)$ consitutes a graph, in general, where R is a set of relations.
Example of five NATO countries linked like a Petersen graph (see fig 1.2) as under:
Vertices (μ_1 *and* ν_1)$\rightarrow$ Poland
Vertices (μ_2 *and* ν_2)$\rightarrow$ Hungary
Vertices (μ_3 *and* ν_3)$\rightarrow$ Bulgaria
Vertices (μ_4 *and* ν_4)$\rightarrow$ Romania
Vertices (μ_5 *and* ν_5)$\rightarrow$ Turkey.
The shape of graph can be extended to 30 or more NATO countries with similar edge-connectivity in inner and outer form (fig 1.2).

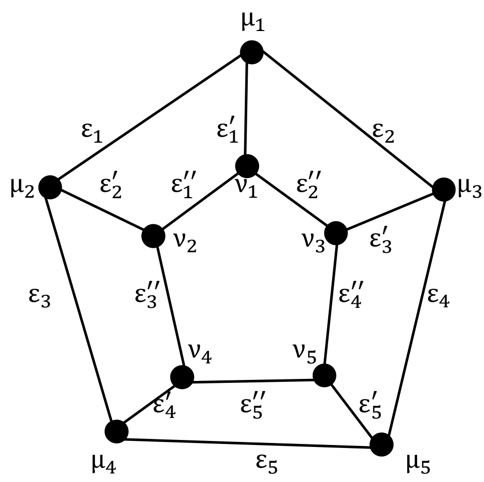

Fig 1.2 Petersen Graph G(5,1)

Table 1.3 Relation of Vertices and Edges in Petersen Graph

S.No.	Set μ	Set ν
1.	$\mu_1 = (\varepsilon_1, \varepsilon_2, \varepsilon_1')$	$\nu_1 = (\varepsilon_1'', \varepsilon_2'', \varepsilon_1')$
2.	$\mu_2 = (\varepsilon_1, \varepsilon_3, \varepsilon_2')$	$\nu_2 = (\varepsilon_1'', \varepsilon_3'', \varepsilon_2')$
3.	$\mu_3 = (\varepsilon_2, \varepsilon_4, \varepsilon_3')$	$\nu_3 = (\varepsilon_2'', \varepsilon_4'', \varepsilon_3')$
4.	$\mu_1 = (\varepsilon_3, \varepsilon_5, \varepsilon_4')$	$\nu_4 = (\varepsilon_3'', \varepsilon_5'', \varepsilon_4')$
5.	$\mu_1 = (\varepsilon_4, \varepsilon_5, \varepsilon_5')$	$\nu_5 = (\varepsilon_4'', \varepsilon_5'', \varepsilon_5')$

Note 1.1 The set of vertices $\mu = (\mu_1, \mu_2, \mu_3, \mu_4, \mu_5)$ denotes countries for external economic level where as set $\nu = (\nu_1, \nu_2, \nu_3, \nu_4, \nu_5)$ denotes some countries for secret economic level. The paired set of vertices $w = \{(\mu_i, \nu_i) : i = 1, 2, 3, 4, 5\}$ represents some countries for internal economic level.

Table 1.4 Node-Edge Matrix of Petersen Graph

	ε_1	ε_2	ε_3	ε_4	ε_5	ε_1'	ε_2'	ε_3'	ε_4'	ε_5'	ε_1''	ε_2''	ε_3''	ε_4''	ε_5''	*row total*
μ_1	1	1	0	0	0	1	0	0	0	0	1	0	0	0	0	3
μ_2	1	0	1	0	0	0	1	0	0	0	0	0	0	0	0	3
μ_3	0	1	0	1	0	0	0	1	0	0	0	0	0	0	0	3
μ_4	0	0	1	0	1	0	0	0	1	0	0	0	0	0	0	3
μ_5	0	0	0	1	1	0	0	0	0	1	0	0	0	0	0	3
ν_1	0	0	0	0	0	1	0	0	0	0	1	1	0	0	0	3
ν_2	0	0	0	0	0	0	1	0	0	0	1	0	1	0	0	3
ν_3	0	0	0	0	0	0	0	1	0	0	0	1	0	1	0	3
ν_4	0	0	0	0	0	0	0	0	1	0	0	0	1	0	1	3
ν_5	0	0	0	0	0	0	0	0	0	1	0	0	0	1	1	3

1.3. Pattern Imputation

In light of fig 1.2 and table 1.4, for large number of outer vertices N and large number of inner vertices N, the general relationship R is
At i=1 $\mu_1 \rightarrow (\varepsilon_1, \varepsilon_2, \varepsilon_1'); \nu_1 \rightarrow (\varepsilon_1'', \varepsilon_2'', \varepsilon_1')$

$\mu_i \rightarrow (\varepsilon_{i-1}, \varepsilon_{i+1}, \varepsilon_i'); \nu_i \rightarrow (\varepsilon_{i-1}'', \varepsilon_{i+1}'', \varepsilon_i'), i = 2, 3...n-1$

At i=N $\mu_N \rightarrow (\varepsilon_{N-1}, \varepsilon_{N+1}, \varepsilon_N'); \nu_N \rightarrow (\varepsilon_{N-1}'', \varepsilon_N'', \varepsilon_N')$.

Under large N, for external set of vertices μ, secret set of vertices ν and internal set ω, the pattern imputation is proposed as under:
Step I At i=2 take $\mu_i \rightarrow (\varepsilon_{i-1}, \varepsilon_{i+1}, \varepsilon_i'); \nu_i \rightarrow (\varepsilon_{i-1}'', \varepsilon_{i+1}'', \varepsilon_i'), i = 2, 3...N-1$

Step II At i=1 impute in step I, ε_0 by ε_1, ε_0'' by ε_1'' and take $\mu_1 \rightarrow (\varepsilon_1, \varepsilon_2, \varepsilon_1'); \nu_1 \rightarrow (\varepsilon_1'', \varepsilon_2'', \varepsilon_1')$

Step III At i=N, impute in step I, ε_{N+1} by ε_N and ε_{N+1}'' by ε_N'' and take $\mu_N \rightarrow (\varepsilon_{N-1}, \varepsilon_N, \varepsilon_N')$;
$\nu_N \rightarrow (\varepsilon_{N-1}'', \varepsilon_N'', \varepsilon_N')$.

To note that imputation of ε_0 *by* ε_1, ε_{N+1} *by* ε_N and ε_0'' by ε_1'', ε_{N+1}'' by ε_N'' is like a specific imputation just to maintaine a pattern so it is called pattern imputation. In general, it may random imputation also like ε_0 to replace by any ε_i, ε_{N+1} by any ε_i, ε_0'' by any ε_i'' ε_{N+1}'' by any ε_i'' randomly chosen. The pattern imputation is closed to the nearest neighbour imputation, but earlier maintains a pattern but later do not.

1.4. Economic Bond Structure Between Countries

Looking at fig 1.2 and assuming large N, the Generalised Petersen Graph G(N,k) can be expressed having edge weights as different economic level bonds between vertices (countries).

(a) **Single Economic Bonding:** The bonding is between any vertex pair (μ_i, μ_{i+1}) at external level, any pair (ν_i, ν_{i+1}) at secret level and any vertex pair (μ_i, ν_i) at internal level. The symbols δ_i, δ_i', δ_i'' represent value of corresponding bonding as shown in fig 1.3.

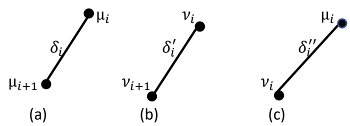

Fig 1.3 Single Economic Bonding

(b) **Double Economic Bonding:** This bonding is between one external and one internal pair of vertices or one internal with one secret pair of vertices. The α_i and α_i' are edge-weights revealed in fig 1.4. Double economic bond may be taken as external+internal as one part (one variable) and (internal+secret) as another part (other variable).

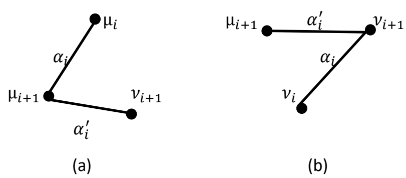

Fig 1.4 Double Economic Bonding

(c) **Triple Economic Bonding:** This consitutes bonding among two vertex pairs at external and secret level and one pair at internal level. The β_i, β_i', β_i'' are edge weights as economic levels shown in fig 1.5.

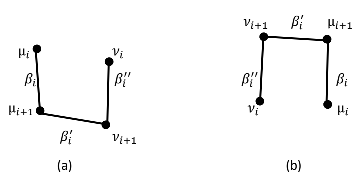

Fig 1.5 Triple Economic Bonding

2. Estimation

In view of Ukraine-Russia war situation and secret military help by NATO countries, authors considered the case of double economic bond estimation only in the content of this paper assuming large N. Define $U_i = \varepsilon_i + \varepsilon_i'$ as external+internal edges and $Z_i = \varepsilon_i'' + \varepsilon_i'$ as secret+internal edges.

$\bar{U} = \frac{\sum_{i=1}^{N} U_i}{N} = \frac{\sum_{i=1}^{N} (\varepsilon_i + \varepsilon_i')}{N}$; $\bar{Z} = \frac{\sum_{i=1}^{N} Z_i}{N} = \frac{\sum_{i=1}^{N} (\varepsilon_i'' + \varepsilon_i')}{N}$ (Population means)

$S_U^2 = \frac{\sum_{i=1}^{N} (U_i - \bar{U})^2}{N-1}$; $S_Z^2 = \frac{\sum_{i=1}^{N} (Z_i - \bar{Z})^2}{N-1}$ (Population mean square)

$C_U = (\frac{S_U}{\bar{U}})$; $C_Z = (\frac{S_Z}{\bar{Z}})$ (Population coefficient of variation)

$S_{UZ} = \frac{\sum_{i=1}^{N} (U_i - \bar{U})(Z_i - \bar{Z})}{N-1}$; $\rho_{UZ} = \rho_{ZU} = \frac{S_{UZ}}{S_U . S_Z}$ (Population correlation coefficient)

Let a simple random sample of large size n ($n < N$) containing vertices like (μ_j, ν_j), j=1,2,3...n is drawn from N vertices using without replacement procedure.

Sample statistic are:

$$\bar{u} = \frac{\sum_{j=1}^{n} u_j}{n} \quad (sample\ mean\ of\ external + internal\ edges) \tag{1}$$

$$\bar{z} = \frac{\sum_{j=1}^{n} z_j}{n} \quad (sample\ mean\ of\ internal + secret\ edges) \tag{2}$$

$$s_u^2 = \frac{\sum_{j=1}^{n} (u_j - \bar{u})^2}{n-1};\ s_z^2 = \frac{\sum_{j=1}^{n} (z_j - \bar{z})^2}{n-1} \quad (sample\ mean\ square\ of\ U\ and\ Z) \tag{3}$$

$$c_u = (\frac{s_u}{\bar{u}});\ c_z = (\frac{s_z}{\bar{z}}) \quad (sample\ coefficient\ of\ variation) \tag{4}$$

$$s_{uz} = \frac{\sum_{j=1}^{n} (u_j - \bar{u})(z_j - \bar{z})}{n-1} \tag{5}$$

$$\rho_{uz} = \frac{s_{uz}}{s_u . s_z} \quad (sample\ correlation\ between\ u\ and\ z) \tag{6}$$

The out mean $\bar{U}$ is assumed known but inner mean $\bar{Z}$ is unknown and the aim of this paper is to estimate $\bar{Z}$ using known ($\bar{u}$,$\bar{z}$, $\bar{U}$) by an appropriate efficient estimation strategy along with imputation for missing.

2.1. Proposed Estimation Strategy

To estimate unknown $\bar{Z}$ in the internal+secret double economic bond and support obtained between any two war involved countries, the proposed estimation strategy $[using\ \bar{z}, \bar{U}, \bar{u}]$ is:

$$E = (\bar{z})[\phi_1(\bar{u}, \bar{U})][\phi_2(\bar{u}, \bar{U})]^{-1}$$

where,

$$\phi_1(\bar{u},\bar{U}) = [(A+C+D)\bar{U}+gB\bar{u}]$$

$$\phi_2(\bar{u},\bar{U}) = [(A+gB+D)\bar{U}+C\bar{u}]$$

$A=(q-1)(q-2); B=(q-1)(q-4); C=(q-2)(q-3)(q-4); D=(q-1)(q-2)(q-3)(q-4)(q-5), g=\frac{n}{N}$, $0<q<\infty$

The proposed is in accordance with shukla et al. (2014) but as a part of new structure, a term is added D which is in power five in q . At q=4, as a special case, the proposed strategy converts to the internal+secret sample mean based economic bond value estimation through a sample.

3. Setting Approximations

For two real numbers h_1 *and* h_2, $|h_1|<1$ *and* $|h_2|<1$, assuming both N, n large, one can express approximations as per Singh et al.(2003), Chochran et al. (2005), Shukla et al. (2014) and Rajoriya et al. (2021).

$$\bar{z} = \bar{Z}(1+h_1) \tag{7}$$

$$\bar{u} = \bar{U}(1+h_2) \tag{8}$$

Let $E^*(.)$ denotes expected value of random variables $\bar{z}$ and $\bar{u}$, then one can get the followings Deo et al. (2001), Shukla et al. (2002), Shukla et al. (2018), Shukla et al. (2020), Donga et al. (2021) and Rajoriya et al.(2021).

$$E^*(h_1) = E^*(h_2) = 0 \tag{9}$$

$$E^*({h_1}^2) = \frac{(N-n)}{Nn}C_Z^2 \tag{10}$$

$$E^*({h_2}^2) = \frac{(N-n)}{Nn}C_U^2 \tag{11}$$

$$E^*(h_1h_2) = \left(\frac{(N-n)}{Nn}\right)(\rho_{ZU}.C_Z.C_U) \tag{12}$$

Theorem 1 *Under large sample approximations, the proposed E could be expressed as:*

$$E = \bar{Z}\left[(1+h_1)+\Delta^*\left\{(h_1+h_1h_2)-\frac{Ch_2^2}{\Delta}\right\}\right]$$

where, $\Delta=(A+gB+C+D)$; $\Delta^*=[\frac{(gB-C)}{\Delta}]$

proof:

$$E = (\bar{z})[\phi_1(\bar{u},\bar{U})][\phi_2(\bar{u},\bar{U})]^{-1}$$

where,

$$\phi_1(\bar{u},\bar{U}) = [(A+C+D)\bar{U} + gB\bar{u}]$$

$$\phi_2(\bar{u},\bar{U}) = [(A+gB+D)\bar{U} + C\bar{u}]$$

Using (7) and (8), $|h_1| < 1$, $|h_2| < 1$

$$\phi_1(\bar{u},\bar{U}) = [(A+C+D)\bar{U} + gB\{\bar{U}(1+h_2)\}] \quad (13)$$

$$\phi_2(\bar{u},\bar{U}) = [(A+gB+D)\bar{U} + C\{\bar{U}(1+h_2)\}] \quad (14)$$

Then $\phi_1(\bar{u},\bar{U})$ could be expressed as:

$$\phi_1(\bar{u},\bar{U}) = [\bar{U}(A+gB+C+D)]\left[1+\frac{(gBh_2)}{(A+gB+C+D)}\right] \quad (15)$$

Since $|h_2| < 1$, therefore $|\frac{gBh_2}{(A+gB+C+D)}| < 1, \ \forall\, g > 0, q > 0$

Moreover, for $\phi_2(\bar{u},\bar{U})$ using expansion of $(1+x)^{-1}$, one gets

$$[\phi_2(\bar{u},\bar{U})]^{-1}\,[(A+gB+C+D)\bar{U} + \bar{U}Ch_2]^{-1}$$

$$= (\bar{U})^{-1}[A+gB+C+D]^{-1}\left[1+\frac{Ch_2}{(A+gB+C+D)}\right]^{-1}$$

$$= (\bar{U})^{-1}[A+gB+C+D]^{-1}\left[1-\frac{Ch_2}{(A+gB+C+D)}+\frac{C^2h_2^2}{(A+gB+C+D)}\cdots\right]$$

Define $\Delta = (A+gB+C+D)$, then one can express proposed E as

$$E = (\bar{z})[\phi_1(\bar{u},\bar{U})][\phi_2(\bar{u},\bar{U})]^{-1}$$

$$E = \bar{Z}(1+h_1)\left[1+\frac{gBh_2}{\Delta}\right]\left[1-\frac{Ch_2}{\Delta}+\frac{C^2h_2^2}{\Delta^2}\cdots\right]$$

$$E = \bar{Z}(1+h_1)\left[1-\frac{Ch_2}{\Delta}+\frac{C^2h_2^2}{\Delta^2}+\left\{\frac{gBh_2}{\Delta}-\frac{gBCh_2^2}{\Delta^2}+\frac{gBC^2h_2^3}{\Delta^3}\cdots\right\}\right]$$

$$E = \bar{Z}\left[(1+h_1)+\frac{(gB-C)}{\Delta}\left\{(h_2+h_1h_2)-\frac{Ch_2^2}{\Delta}\right\}\right]$$

which is expressed after ignoring terms $(h_1^s.\ h_2^t), (s+t) > 2,\ s,t = 0,1,2,3,4...$ because of having high power on h_1 and h_2. The denominator Δ is high for $g > 0$, therefore, one can narrate that contribution of these terms in estimation will be very low (negligible).

Define $\Delta^* = \frac{(gB-C)}{\Delta}$. Then $E = \bar{Z}\left[(1+h_1)+\Delta^*\left\{(h_1+h_1h_2)-\frac{Ch_2^2}{\Delta}\right\}\right]$

Theorem 2 *The bias of estimator E under (7), (8) using theorem 1 is:*

$B[E] = Bias[E] = \bar{Z}\left[\Delta^*\left\{\frac{N-n}{Nn}\right\}\left\{\rho_{ZU}.C_Z.C_U) - \frac{C}{\Delta}C_U^2\right\}\right]$
where $\rho_{UZ} = \rho_{ZU}$ is correlation coefficient between double economic bond variables U and Z in Petersen graph.

proof: The $E^*(.)$ denotes expected value of the proposed estimator E and $B[E] = [E^*(E) - \bar{Z}]$

Now $E^*(E) = E^*\left[\bar{Z}(1+h_1) + \bar{Z}\Delta^*\left\{h_1 + h_1h_2 - \frac{Ch_2^2}{\Delta}\right\}\right]$

$= \left[\bar{Z} + \bar{Z}E^*(h_1) + \bar{Z}\Delta^*\left\{E^*(h_1) + E^*(h_1h_2) - \frac{CE^*(h_2^2)}{\Delta}\right\}\right]$

$= \left[\bar{Z} + \bar{Z}\Delta^*\left\{E^*(h_1h_2) - \frac{C}{\Delta}E^*(h_2^2)\right\}\right]$ Using (7) and (8) and theorem 1

$B[E] = [E^*(E) - \bar{Z}]$

$= \bar{Z}\left[\Delta^*\left\{E^*(h_1h_2) - \frac{C}{\Delta}E^*(h_2^2)\right\}\right]$

$= \bar{Z}\left[\Delta^*\left(\frac{N-n}{Nn}\right)\left\{(\rho_{ZU}C_zC_U) - (\frac{C}{\Delta})C_U^2\right\}\right]$

Corollary 1 *The estimator E is almost unbiased under condition*

$(\rho_{ZU}C_zC_U) = (\frac{C}{\Delta})C_U^2 \implies \frac{C}{\Delta} = \rho_{ZU}(\frac{C_Z}{C_U}) = M\ (Let)$

$\implies \frac{C}{(A+gB+C+D)} = M$

$$\implies M(A+gB+C+D) + C(M-1) + MD = 0 \tag{16}$$

Note 3.1 The equation (17) is having highest power five in terms of q. Therefore, it may has maximum of five roots satisfying the equation. Best root will be that having lowest mean square error (MSE).

Theorem 3 *The mean squared error (MSE) of the proposed strategy is*

$MSE[E] = \bar{Z}^2\left[(\frac{N-n}{Nn})\left\{C_Z^2 + (\Delta^*)^2C_U^2 + 2\Delta^*\rho_{UZ}C_UC_Z\right\}\right]$
proof: $MSE[E] = E^*[E - \bar{Z}]^2$

$= E^*\left[\bar{Z}(1+h_1) + \Delta^*\left\{h_1 + h_1h_2 - \frac{Ch_2^2}{\Delta} + ...\right\} - \bar{Z}\right]^2$

$= E^*\left[\bar{Z}(h_1 + \Delta^*h_2)\right]^2$ ignoring terms $(h^sh^t), (s+t) > 2, s, t = 1,2,3,4,5$

$= \bar{Z}^2\left[E^*(h_1^2) + (\Delta^*)^2E^*(h_2^2) + 2\Delta^*E^*(h_1h_2)\right]$

$$\implies MSE[E] = \bar{Z}^2\left[(\frac{N-n}{Nn})\left\{C_Z^2 + (\Delta^*)^2C_U^2 + 2\Delta^*\rho_{UZ}C_UC_Z\right\}\right] \tag{17}$$

Theorem 4 *The minimum (optimum) mean squared error is attained when* $\Delta^* = -M$

where $M = \rho_{UZ}(\frac{C_Z}{C_U})$.

proof: Differentiating MSE[E] with respect to the term Δ^* and equate to zero, one gets;

$$\frac{MSE[E]}{\Delta^*} = 0 \implies \Delta^* = -\rho_{UZ}(\frac{C_Z}{C_U}) = -M \tag{18}$$

Corollary 2 *The optimuum MSE expression (19) could be expressed as*

$$\frac{(gB-C)}{(A+gB+C+D)} = -M$$

$$\implies AM + gB(M+1) + C(M-1) + DM = 0 \tag{19}$$

Note 3.2 Equation (20) of optimum MSE is having highest power five on term q, therefore, there will be maximum of five roots of equation (20). The best q will be that containing lowest bias value. The proposed strategy E attains the optimum level of MSE and reduces the bias too. This is a novel feature of proposed estimation procedure E.

4. Numerical Illustration

Remark 2 *It is difficult to get real and reliable data of secret Economic bond immediately during the war (like Russia & Ukraine and support of NATO countries). But data of Internal Economic Bond among various ordinance factories within countries could be obtained when the audit and assessment reports, by Auditors, are available. It takes several years to come and to get published. The current Russia-Ukraine war be treated as an application of the proposed whose data will be published after long time. In absence of that, an artificial data set is used just to test the proposed methodology and to demonstrate the suggested procedure to the article readers.*

Remark 3 *There may uneven economic distribution support (as open & secrete) by various involved countries. But, one can assume nearly homogeneous support by most of NATO countries to the Ukraine, specially at the starting duration of war. Later on, as the war progresses, the open and hidden, both kinds of economic support may convert into heterogeneous distributions. In this paper, the almost homogeneous economic support, as was in beginning period of the war is assumed. It is a restriction also in the content of the paper.*

Remark 4 *The size N, if large, will not affect the properties of the proposed methodology using Petersen graph model. In fact, the Petersen graph is a closed network of vertices which can accommodate any number of additional vertices, as and when requried, without loosing structure and properties.*

Define F= Secret Economic Bond $=\varepsilon_i^{''}$; G= External Economic Bond $=\varepsilon_i$; H= Internal Economic Bond $=\varepsilon_i^{'}$. Consider the generalized Petersen structure with N=150. The assumed economic bond values are considered below:

Table 4.1 Military War Weapon Assumed Data of N=150 Countries as Population

S.No.	F = ε_i''	G = ε_i	H = ε_i'
1.	25 units	43units	86units
2.	53 units	81units	64units
3.	34 units	14units	86units
4.	43 units	61units	74units
5.	37 units	28units	69units
6.	91 units	23units	41units
7.	34 units	48units	72units
8.	92 units	43units	21units
9.	35 units	63units	71units
10.	27 units	83units	34units
11.	51 units	63units	86units
12.	63 units	72units	65units
13.	39 units	84units	42units
14.	52 units	26units	75units
15.	84 units	35units	42units
16.	28 units	39units	67units
17.	56 units	42units	63units
18.	81 units	33units	26units
19.	29 units	57units	76units
20.	85 units	38units	43units
21.	91 units	34units	78units
22.	38 units	49units	65units
23.	57 units	63units	84units
24.	19 units	43units	96units
25.	65 units	36units	73units
26.	48 units	96units	21units
27.	43 units	65units	92units
28.	45 units	39units	17units
29.	83 units	91units	26units
30.	57units	48units	21units
31.	23 units	58units	61units
32.	47 units	82units	53units
33.	27 units	63units	73units
34.	98 units	34units	61units
35.	45 units	23units	54units
36.	81 units	53units	66units
37.	22 units	93units	81units
38.	55 units	42units	76units
39.	29 units	63units	66units
40.	68 units	41units	96units
41.	25 units	93units	46units
42.	63 units	71units	32units
43.	73 units	61units	24units
44.	58 units	83units	46units
45.	48 units	43units	22units
46.	31 units	48units	69units
47.	47 units	33units	26units
48.	35 units	87units	76units
49.	63 units	71units	36units
50.	85 units	53units	46units
51.	76 units	29units	36units
52.	32 units	61units	59units
53.	47 units	93units	73units
54.	93 units	84units	64units
55.	55 units	84units	29units
56.	48 units	19units	36units
57.	71 units	94units	68units
58.	92units	83units	57units
59.	28 units	59units	28units
60.	38 units	47units	71units
61.	93 units	72units	65units
62.	35 units	83units	57units
63.	45 units	84units	91units
64.	46 units	52units	29units
65.	15 units	73units	82units
66.	37 units	87units	62units
67.	93 units	13units	96units
68.	75 units	84units	56units
69.	39 units	83units	92units
70.	72units	65units	86units
71.	47 units	41units	68units
72.	85 units	38units	21units
73.	68 units	91units	26units
74.	45 units	38units	56units
75.	30 units	43units	82units
76.	41 units	87units	34units
77.	75 units	32units	66units
78.	48 units	32units	71units
79.	87 units	92units	56units
80.	49 units	22units	76units
81.	65 units	86units	56units
82.	45 units	33units	31units
83.	49 units	64units	88units
84.	93 units	21units	65units
85.	75 units	83units	89units
86.	46 units	26units	18units
87.	68 units	37units	28units
88.	88 units	63units	29units
89.	28 units	44units	75units
90.	39 units	42units	56units
91.	37units	47units	76units
92.	82 units	56units	96units
93.	17 units	47units	89units
94.	76 units	44units	28units
95.	45 units	63units	60units
96.	77 units	42units	63units
97.	29 units	51units	36units
98.	39 units	53units	56units
99.	78 units	88units	40units
100.	20 units	75units	64units
101.	73units	37units	58units
102.	84 units	73units	36units
103.	95 units	43units	21units
104.	58 units	68units	28units
105.	71 units	39units	50units
106.	47 units	40units	19units
107.	85 units	73units	26units
108.	60 units	53units	44units
109.	28 units	49units	81units
110.	35 units	63units	66units
111.	48 units	28units	39 units
112.	56 units	54units	87 units
113.	41 units	40 units	81 units
114.	45 units	63 units	21 units
115.	35 units	71 units	66 units
116.	88 units	23 units	86 units
117.	35 units	43 units	88 units
118.	69 units	40 units	66 units
119.	38 units	33units	96units
120.	68 units	43units	56units
121.	21 units	84 units	26 units
122.	25 units	49units	77units
123.	48 units	64 units	92 units
124.	20 units	63units	29 units
125.	28 units	33 units	83 units
126.	77 units	62 units	55 units
127.	60 units	43 units	56 units
128.	65 units	74 units	78 units
129.	48 units	66 units	58 units
130.	94 units	47units	76units
131.	59 units	31 units	63 units
132.	76 units	93 units	84 units
133.	95 units	73 units	66units
134.	70 units	83 units	56 units
135.	46 units	29 units	46units
136.	79 units	92 units	36units
137.	54 units	54 units	47 units
138.	80 units	43units	98 units
139.	95 units	46units	19 units
140.	39 units	63 units	93 units
141.	97 units	76 units	34 units
142.	85 units	94 units	33 units
143.	76 units	33 units	57 units
144.	79 units	65 units	88 units
145.	83 units	60 units	59 units
146.	90 units	22 units	86units
147.	79 units	39 units	88 units
148.	46 units	55 units	39 units
149.	98 units	68 units	88 units
150.	29 units	85 units	89 units

Remark 5 *Define Secret level+ Internal level Economic Bond* $=\varepsilon_i'' + \varepsilon_i' = Z_i$ *; Internal level + External level Economic Bond* = $\varepsilon_i' + \varepsilon_i = U_i$.

Table 4.2 Double Economic Bond (in U and Z) Data of N=150 Countries as Population (from table 4.1)

$Z_i = \varepsilon_i'' + \varepsilon_i' = F+G$	$U_i = \varepsilon_i' + \varepsilon_i = G+H$	$Z_i = \varepsilon_i'' + \varepsilon_i' = F+G$	$U_i = \varepsilon_i' + \varepsilon_i = G+H$
111	129	75	121
117	145	141	98
120	100	119	103
117	135	143	148
106	97	125	98
132	64	121	142
106	120	76	64
113	64	137	152
106	134	158	86
61	117	164	172
137	149	64	44
128	137	96	65
81	126	117	92
127	101	103	119
126	77	95	98
95	106	113	123
119	105	178	152
107	59	106	136
105	133	104	72
128	81	105	123
169	112	140	105
103	114	65	87
141	147	95	109
115	139	118	128
138	109	84	139
69	117	131	95
135	157	105	94
62	56	116	64
109	117	86	96
78	69	121	89
84	119	66	59
100	135	111	99
100	136	104	97
159	95	109	130
99	77	101	129
147	119	87	67
103	174	143	141
131	118	122	121
95	129	66	84
164	137	101	137
71	139	174	109
95	103	123	131
97	85	135	106
104	129	134	129
70	65	124	99
100	117	47	110
73	59	102	126
111	163	140	56
99	107	49	92
131	99	111	116
112	65	132	117
91	120	116	99
120	166	143	152
157	148	106	124
84	113	170	123
84	55	122	94
139	162	160	177
149	140	161	139
56	87	126	139
109	118	92	75
158	137	115	128
92	140	101	101
136	175	178	141
75	81	114	65
97	155	132	156
99	149	131	110
189	109	118	129
131	140	133	90
131	175	167	153
158	151	142	119
115	109	176	108
106	59	167	127
94	117	85	94
101	94	186	156
112	125	118	174

Table 4.3 Petersen Graph Population Parameters (table 4.1)

S.No.	Parameters	Value	Description/(Section 2.0)
1.	N	150	Population size
2.	n	40	Sample size
3.	$\bar{Z}$	116	Population Mean
4.	$\bar{U}$	115	Population Mean
5.	S_Z	29.4903	Populatin Mean Square
6.	S_U	30.2076	Population Mean Square
7.	C_Z	0.2542	Population Coefficient of Variation
8.	C_U	0.2626	Population Coefficient of Variation
9.	ρ_{UZ}	0.4217	Population Correlation Coefficient
10.	M	0.4082	Using Corollary 1

Table 4.4 Almost Unbiased Choice of q for given (M, g) [from eq (17)]

S.No.	M	g	Choice of q	Bias	MSE
1.	0.4082	0.2666	$q_1 = 1.0756$	0.1925	28.4753
2.	0.4082	0.2666	$q_2 = 1.9709$	0.0420	56.1204
3.	0.4082	0.2666	$q_3 = 2.9073$	-0.0323	14.7597
4.	0.4082	0.2666	$q_4 = --$	–	–
5.	0.4082	0.2666	$q_5 = --$	–	–

Table 4.5 Choice of q for Optimum MSE for given (M, g) [from eq (20)]

S.No.	M	g	Choice of q	MSE	Bias
1.	0.4082	0.2666	$q_{1(opt)} = 0.6335$	13.1075	-0.0007
2.	0.4082	0.2666	$q_{2(opt)} = 1.8270$	13.1075	-0.3053
3.	0.4082	0.2666	$q_{3(opt)} = 2.9830$	13.1075	-0.0236
4.	0.4082	0.2666	$q_{4(opt)} = --$	–	–
5.	0.4082	0.2666	$q_{5(opt)} = --$	–	–

Table 4.6 Special Cases At q = 1,2,3,4,5 for (g= 0.2666, M= 0.0773)

S.No.	q	A	B	C	D	Bias(theorem 2)	MSE (theorem 3)
1.	1	0	0	-6	0	0.0868	19.0684
2.	2	0	-2	0	0	0.0599	48.8626
3.	3	2	-2	0	0	-0.0217	13.1413
4.	4	6	0	0	0	0.0000	15.9440
5.	5	12	4	6	0	-0.0035	13.4878

Tables 4.7 Ready Reckoner for Choice of q Providing almost Unbiasedness for given (M,g) (Using corollary 1, eq. (17))[$Range\ 0.05 \leq M \leq 0.95;\ Range\ 0.3 \leq g \leq 0.9$]

S.No.	M	g	Choice of q	Bias	MSE	S.No.	M	g	Choice of q	Bias	MSE
1.	0.05	0.3	$q_1 = 1.0220$	0.0977	19.9902	31.	0.65	0.3	$q_1 = 1.1050$	0.2707	35.7252
2.	0.05	0.3	$q_2 = 1.8939$	0.0578	47.2832	32.	0.65	0.3	$q_2 = 1.9599$	0.0357	51.0935
3.	0.05	0.3	$q_3 = 2.9837$	-0.0277	13.1919	33.	0.65	0.3	$q_3 = 2.8847$	-0.0383	20.5160
4.	0.05	0.3	$q_4 = --$	–	–	34.	0.65	0.3	$q_4 = --$	–	–
5.	0.05	0.3	$q_5 = --$	–	–	35.	0.65	0.3	$q_5 = --$	–	–
6.	0.05	0.6	$q_1 = 1.0220$	0.1097	19.8885	36.	0.65	0.6	$q_1 = 1.1071$	0.2536	33.4299
7.	0.05	0.6	$q_2 = 1.9903$	0.5489	278.9120	37.	0.65	0.6	$q_2 = 1.9124$	0.0285	51.6161
8.	0.05	0.6	$q_3 = 2.9980$	-0.90664	62.8639	38.	0.65	0.6	$q_3 = 2.9365$	-0.1204	159.8629
9.	0.05	0.6	$q_4 = --$	–	–	39.	0.65	0.6	$q_4 = --$	–	–
10.	0.05	0.6	$q_5 = --$	–	–	40.	0.65	0.6	$q_5 = --$	–	–
11.	0.05	0.9	$q_1 = 1.0121$	0.0964	19.8288	41.	0.65	0.9	$q_1 = 1.1090$	0.2370	31.2449
12.	0.05	0.9	$q_2 = 1.9856$	0.0578	47.2592	42	0.65	0.9	$q_2 = 1.8745$	0.0271	51.1854
13.	0.05	0.9	$q_3 = 2.9886$	-0.5703	1526.2520	43.	0.65	0.9	$q_3 = 2.9742$	-0.6875	6756.3080
14.	0.05	0.9	$q_4 = --$	–	–	44.	0.65	0.9	$q_4 = --$	–	–
15.	0.05	0.9	$q_5 = --$	–	–	45.	0.65	0.9	$q_5 = --$	–	–
16.	0.35	0.3	$q_1 = 1.0773$	0.1743	26.7770	46.	0.95	0.3	$q_1 = 1.1050$	0.2707	35.7252
17.	0.35	0.3	$q_2 = 1.9714$	0.0448	49.6382	47.	0.95	0.3	$q_2 = 1.9599$	0.0357	51.0935
18.	0.35	0.3	$q_3 = 2.9198$	-0.0364	15.4016	48.	0.95	0.3	$q_3 = 2.8847$	-0.0383	20.5160
19.	0.35	0.3	$q_4 = --$	–	–	49.	0.95	0.3	$q_4 = --$	–	–
20.	0.35	0.3	$q_5 = --$	–	–	50.	0.95	0.3	$q_5 = --$	–	–
21.	0.35	0.6	$q_1 = 1.0682$	0.1676	25.8888	51.	0.95	0.6	$q_1 = 1.2370$	0.3544	42.4224
22.	0.35	0.6	$q_2 = 1.9445$	0.0444	49.5347	52.	0.95	0.6	$q_2 = 1.9000$	0.0106	53.5250
23.	0.35	0.6	$q_3 = 2.9553$	-0.1174	87.4308	53.	0.95	0.6	$q_3 = 2.9172$	-0.1041	258.4500
24.	0.35	0.6	$q_4 = --$	–	–	54.	0.95	0.6	$q_4 = --$	–	–
25.	0.35	0.6	$q_5 = --$	–	–	55.	0.95	0.6	$q_5 = --$	–	–
26.	0.35	0.9	$q_1 = 1.0690$	0.1609	25.0245	56.	0.95	0.9	$q_1 = 1.1397$	0.3251	38.4883
27.	0.35	0.9	$q_2 = 1.9950$	0.0439	49.4185	57.	0.95	0.9	$q_2 = 1.8420$	0.0080	52.5609
28.	0.35	0.0.9	$q_3 = 2.9970$	-0.6871	3462.7230	58.	095	0.9	$q_3 = 2.9819$	-0.6017	9656.5140
29.	0.35	0.9	$q_4 = --$	–	–	59.	0.95	0.9	$q_4 = --$	–	–
30.	0.35	0.9	$q_5 = --$	–	–	60.	0.95	0.9	$q_5 = --$	–	–

Tables 4.8 Ready Reckoner for Choice of q Providing Optimum MSE for given (M,g) (Using corollary 2, eq. (20)) [$Range\ 0.05 \leq M \leq 0.95;\ Range\ 0.3 \leq g \leq 0.9$]

S.No.	M	g	Choice of q	Bias	MSE	S.No.	M	g	Choice of q	Bias	MSE
1.	0.05	0.3	$q_1 = 1.8017$	-0.0320	15.3031	31.	0.65	0.3	$q_1 = 0.8497$	0.0219	14.1011
2.	0.05	0.3	$q_2 = 3.0989$	-0.0036	15.3001	32.	0.65	0.3	$q_2 = 1.8150$	-0.5915	14.1012
3.	0.05	0.3	$q_3 = 4.1930$	-0.0024	15.3008	33.	0.65	0.3	$q_3 = 2.9500$	-0.0336	14.1060
4.	0.05	0.3	$q_4 = --$	–	–	34.	0.65	0.3	$q_4 = --$	–	–
5.	0.05	0.3	$q_5 = --$	–	–	35.	0.65	0.3	$q_5 = --$	–	–
6.	0.05	0.6	$q_1 = 1.6810$	-0.2043	15.3018	36.	0.65	0.6	$q_1 = 0.8501$	0.0207	14.1020
7.	0.05	0.6	$q_2 = 3.6084$	-0.0036	15.3010	37.	0.65	0.6	$q_2 = 1.6905$	-2.7535	14.1026
8.	0.05	0.6	$q_3 = 4.3269$	-0.0019	15.3011	38.	0.65	0.6	$q_3 = 3.1127$	-0.0457	14.1052
9.	0.05	0.6	$q_4 = --$	–	–	39.	0.65	0.6	$q_4 = --$	–	–
10.	0.05	0.6	$q_5 = --$	–	–	40.	0.65	0.6	$q_5 = --$	–	–
11.	0.05	0.9	$q_1 = 1.5999$	0.0598	15.3012	41.	0.65	0.9	$q_1 = 0.2017$	0.0194	14.1020
12.	0.05	0.9	$q_2 = 3.7734$	-0.0034	15.3010	42.	0.65	0.9	$q_2 = 1.5951$	1.01378	14.1058
13.	0.05	0.9	$q_3 = 4.5800$	-0.0012	15.3010	43.	0.65	0.9	$q_3 = 3.2500$	-0.0502	14.1053
14.	0.05	0.9	$q_4 = --$	–	–	44.	0.65	0.9	$q_4 = --$	–	–
15.	0.05	0.9	$q_5 = --$	–	–	45.	0.65	0.9	$q_5 = --$	–	–
16.	0.35	0.3	$q_1 = 0.5400$	-0.0036	13.1652	46.	0.95	0.3	$q_1 = 0.9851$	0.0790	18.0995
17.	0.35	0.3	$q_2 = 1.8105$	-0.2723	13.1612	47.	0.95	0.3	$q_2 = 1.8176$	-0.9932	18.0961
18.	0.35	0.3	$q_3 = 3.0311$	-0.0219	13.1648	48.	0.95	0.3	$q_3 = 2.8670$	-0.0383	18.0973
19.	0.35	0.3	$q_4 = --$	–	–	49.	0.95	0.3	$q_4 = --$	–	–
20.	0.35	0.3	$q_5 = --$	–	–	50.	0.95	0.3	$q_5 = --$	–	–
21.	0.35	0.6	$q_1 = 0.5060$	-0.0043	13.1651	51.	0.95	0.6	$q_1 = 0.9840$	0.0748	18.0941
22.	0.35	0.6	$q_2 = 1.6875$	-1.2295	13.1600	52.	0.95	0.6	$q_2 = 1.6818$	-4.4638	18.0933
23.	0.35	0.6	$q_3 = 3.2101$	-0.0257	13.1651	53.	0.95	0.6	$q_3 = 3.0461$	-0.0635	18.0941
24.	0.35	0.6	$q_4 = --$	–	–	54.	0.95	0.6	$q_4 = --$	–	–
25.	0.35	0.6	$q_5 = --$	–	–	55.	0.95	0.6	$q_5 = --$	–	–
26.	0.35	0.9	$q_1 = 0.4718$	-0.0049	13.1653	56.	0.95	0.9	$q_1 = 0.9732$	0.0744	18.0995
27.	0.35	0.9	$q_2 = 1.5101$	0.4831	13.1601	57.	0.95	0.9	$q_2 = 1.5816$	1.6632	18.0925
28.	0.35	0.0.9	$q_3 = 3.3773$	-0.0267	13.1649	58.	095	0.9	$q_3 = 3.1741$	-0.0733	18.09847
29.	0.35	0.9	$q_4 = --$	–	–	59.	0.95	0.9	$q_4 = --$	–	–
30.	0.35	0.9	$q_5 = --$	–	–	60.	0.95	0.9	$q_5 = --$	–	–

5. Confidence Interval Estimation and Imputation

Consider the 10 random samples $A_1, A_2, A_3, ..., A_{10}$ each of size n=40 from population N=150 (from table 4.2). Description of samples is in table 5.1 given below:

Table 5.1 Ten Random Sample Selection

Sample No.	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9	A_{10}
(z_1, u_1)	(117,145)	(156,156)	(120,100)	(111,129)	(118,174)	(85,94)	(117,135)	(176,108)	(91,239)	(167,127)
(z_2, u_2)	(106,97)	(107,127)	(132,64)	(120,,100)	(85,94)	(142,119)	(106,97)	(142,119)	(114,63)	(133,100)
(z_3, u_3)	(132,64)	(142,119)	(113,64)	(113,64)	(101,101)	(133,90)	(113,64)	(131,110)	(167,127)	(178,141)
(z_4, u_4)	(61,117)	(131,110)	(61,117)	(128,137)	(170,123)	(131,110)	(128,137)	(178,141)	(167,153)	(126,139)
(z_5, u_5)	(81,126)	(178,141)	(137,149)	(127,101)	(116,99)	(178,141)	(128,81)	(92,75)	(132,156)	(160,177)
(z_6, u_6)	(95,106)	(115,128)	(81,126)	(119,105)	(111,116)	(115,128)	(103,114)	(161,139)	(178,141)	(143,152)
(z_7, u_7)	(128,81)	(160,117)	(126,77)	(141,147)	(140,156)	(134,129)	(138,109)	(122,94)	(92,75)	(111,116)
(z_8, u_8)	(103,114)	(143,152)	(119,105)	(159,95)	(47,110)	(174,109)	(84,119)	(111,116)	(170,123)	(101,137)
(z_9, u_9)	(138,109)	(49,92)	(128,81)	(99,77)	(134,129)	(66,84)	(159,95)	(102,126)	(111,116)	(122,121)
(z_{10}, u_{10})	(62,56)	(134,129)	(141,147)	(131,118)	(135,106)	(109,130)	(147,119)	(135,166)	(47,110)	(109,130)
(z_{11}, u_{11})	(78,69)	(123,131)	(138,109)	(97,85)	(123,131)	(121,89)	(164,137)	(143,141)	(134,129)	(111,99)
(z_{12}, u_{12})	(100,136)	(66,84)	(135,157)	(104,129)	(122,121)	(116,64)	(985)	(109,136)	(143,141)	(121,89)
(z_{13}, u_{13})	(99,77)	(122,121)	(78,69)	(112,65)	(101,129)	(131,95)	(100,117)	(116,64)	(109,130)	(116,64)
(z_{14}, u_{14})	(103,174)	(87,67)	(100,135)	(120,160)	(104,97)	(118,128)	(131,99)	(95,109)	(121,89)	(131,95)
(z_{15}, u_{15})	(131,118)	(111,99)	(84,119)	(139,162)	(140,105)	(140,105)	(97,85)	(106,136)	(95,109)	(118,128)
(z_{16}, u_{16})	(164,137)	(105,123)	(71,139)	(109,118)	(178,152)	(95,98)	(100,117)	(95,98)	(140,105)	(65,87)
(z_{17}, u_{17})	(111,163)	(178,152)	(97,85)	(99,149)	(117,92)	(117,92)	(131,99)	(164,172)	(104,72)	(140,105)
(z_{18}, u_{18})	(112,65)	(113,123)	(100,117)	(131,140)	(113,123)	(96,65)	(97,85)	(125,98)	(95,98)	(104,72)
(z_{19}, u_{19})	(120,166)	(125,98)	(111,136)	(158,151)	(106,136)	(137,152)	(104,129)	(115,109)	(117,92)	(178,152)
(z_{20}, u_{20})	(139,162)	(75,121)	(11,99)	(106,59)	(140,105)	(125,98)	(73,59)	(189,109)	(158,86)	(103,119)
(z_{21}, u_{21})	(109,118)	(112,125)	(84,113)	(94,117)	(118,128)	(112,125)	(99,107)	(75,81)	(121,142)	(96,65)
(z_{22}, u_{22})	(92,140)	(101,94)	(149,140)	(143,148)	(116,64)	(106,59)	(112,65)	(139,162)	(119,103)	(158,86)
(z_{23}, u_{23})	(75,81)	106,59)	(136,175)	(137,152)	(109,130)	(158,151)	(139,162)	(157,148)	(112,125)	(121,142)
(z_{24}, u_{24})	(99,149)	(158,151)	(97,155)	(164,172)	(101,129)	(131,140)	(158,137)	(100,117)	(131,140)	(103,119)
(z_{25}, u_{25})	(131,140)	(131,140)	(131,175)	(103,119)	(143,141)	(136,17)	(97,155)	(95,129)	(99,149)	(96,65)
(z_{26}, u_{26})	(158,151)	(99,149)	(106,59)	(113,123)	(122,121)	(140,140)	(132,117)	(147,119)	(92,140)	(158,86)
(z_{27}, u_{27})	(94,117)	(75,81)	(101,94)	(106,136)	(101,137)	(84,113)	(121,142)	(99,77)	(139,162)	(121,142)
(z_{28}, u_{28})	(141,98)	(92,140)	(141,98)	(105,94)	(123,131)	(91,120)	(164,172)	(100,136)	(157,148)	(119,103)
(z_{29}, u_{29})	(164,172)	(109,118)	(143,148)	(66,84)	(134,129)	(131,99)	(103,119)	(78,69)	(131,99)	(101,94)
(z_{30}, u_{30})	(96,65)	(56,87)	(121,142)	(174,109)	(47,110)	(70,65)	(140,105)	(135,157)	(73,59)	(131,157)
(z_{31}, u_{31})	(103,119)	(84,55)	(137,152)	(135,106)	(49,92)	(95,103)	(118,128)	(103,114)	(104,129)	(131,140)
(z_{32}, u_{32})	(178,152)	(157,148)	(103,119)	(124,99)	(132,117)	(164,137)	(111,99)	(128,81)	(95,103)	(92,140)
(z_{33}, u_{33})	(140,105)	(91,120)	(104,72)	(140,156)	(161,139)	(159,95)	(140,105)	(95,106)	(147,119)	(56,87)
(z_{34}, u_{34})	(118,128)	(73,59)	(140,105)	(132,117)	(92,75)	(78,69)	(118,128)	(127,101)	(109,117)	(139,162)
(z_{35}, u_{35})	(121,89)	(104,129)	(118,128)	(106,124)	(131,110)	(69.117)	(111,99)	(61,117)	(107,59)	(84,113)
(z_{36}, u_{36})	(123,131)	(95,103)	(122,121)	(161,139)	(167,153)	(141,147)	(66,84)	(106,117)	(127,101)	(120,166)
(z_{37}, u_{37})	(102,126)	(71,139)	(174,109)	(92,75)	(176,108)	(169,112)	(174,109)	(106,120)	(128,137)	(128,81)
(z_{38}, u_{38})	(106,124)	(103,174)	(135,106)	(178,141)	(75,81)	(105,133)	(135,106)	(117,135)	(113,64)	(104,129)
(z_{39}, u_{39})	(170,123)	(100,136)	(47,110)	(131,110)	(122,121)	(126,77)	(124,99)	(120,100)	(106,97)	(164,137)
(z_{40}, u_{40})	(85,94)	(115,139)	(170,123)	(176,108)	(160,117)	(61,117)	(102,126)	(81,126)	(120,100)	(115,139)

Table 5.2 Ten Sample Descriptive Statistic [eq. (1) to eq. (6)]

Sample No.	Mean ($\bar{z}$)	Mean($\bar{u}$)	s_z	s_u	c_z	c_u	ρ_{zu}
A_1	114.6250	119.6750	28.2822	33.2145	0.2467	0.2775	0.0165
A_2	112.6250	119.1500	33.2663	30.2329	0.2953	0.2537	0.0479
A_3	116.5500	116.6500	25.8069	31.0306	0.2214	0.2660	0.0274
A_4	125.0750	118.0000	25.4995	29.1196	0.2038	0.24167	0.0732
A_5	121.1500	119.8250	32.2566	22.2767	0.2662	0.1859	0.0437
A_6	119.7256	110.3500	30.3273	27.5249	0.2533	0.2494	0.0363
A_7	120.4000	113.6000	25.8852	28.5359	0.2149	0.2512	0.0303
A_8	118.3000	115.4250	30.7531	27.8815	0.2599	0.2415	0.1967
A_9	124.6250	113.6750	33.5954	28.4906	0.2695	0.2506	0.1885
A_{10}	118.7000	118.6500	28.5401	29.2991	0.2404	0.2469	0.1476

The table 5.2 presents the descriptive statistics of mean, variability and correlation of ten samples using eq. (1) to eq. (6).

5.1. Definition of Confidence Interval (CI)

Suppose $\bar{\theta_n}$ be an unbiased estimator of unknown θ based on random sample n from normal population $N[\theta, \sigma^2]$. Then 95% confidence interval is defined as:

$$P[\bar{\theta} - 1.96\sqrt{var(\bar{\theta})} < \theta < \bar{\theta} + 1.96\sqrt{var(\bar{\theta})}] = 0.95$$

where P[.] denotes the probability of event. The lower limit of confidence interval (CI) is LL= $[\bar{\theta} - 1.96\sqrt{var(\bar{\theta})}]$ and upper limit is UL=$[\bar{\theta} + 1.96\sqrt{var(\bar{\theta})}]$. As interpretation, there exists 95% chance that true but unknwon θ lies between lower limit and upper limit of confidence interval (CI). Deriving motivation from this, for biased estimator, two proposed limits are:

$$(LL)_{opt} = Lower\ Limit\ = [estimated\ mean\ - 1.96\sqrt{est(MSE)_{(q_{opt})}}] \tag{20}$$

$$(UL)_{opt} = Upper\ Limit\ = [estimated\ mean\ + 1.96\sqrt{est(MSE)_{(q_{opt})}}] \tag{21}$$

Table 5.3 Estimated Confidence Intervals Over 10 Samples at the q_{opt} Values [using (21) and (22)]

Sample No.	q_{opt}	E	est(MSE)	C.I. $[(LL)_{opt}, (UL)_{opt}]$	Length
A_1	$q_{1(opt)} = 0.6335$	112.7530	17.5324	[78.38,147.11]	68.7270
A_2	$q_{1(opt)} = 0.6335$	110.9900	22.1005	[67.67,154.30]	86.6338
A_3	$q_{1(opt)} = 0.6335$	115.8710	14.8170	[86.82,144.91]	58.0828
A_4	$q_{1(opt)} = 0.6335$	123.7570	13.9681	[96.37,151.13]	54.7549
A_5	$q_{1(opt)} = 0.6335$	119.1090	20.1490	[79.61,158.60]	78.9839
A_6	$q_{1(opt)} = 0.6335$	121.7330	19.0932	[84.31,159.15]	74.8452
A_7	$q_{1(opt)} = 0.6335$	121.0010	14.7219	[92.14,149.85]	57.7099
A_8	$q_{1(opt)} = 0.6335$	118.1220	17.2456	[84.32,151.92]	67.6072
A_9	$q_{1(opt)} = 0.6335$	125.2140	20.7113	[84.61,165.80]	81.1885
A_{10}	$q_{1(opt)} = 0.6335$	117.1810	15.7098	[86.39,147.97]	61.5823
Avg Length of CI				[84.06,153.07]	69.0115
A_1	$q_{2(opt)} = 1.8270$	112.2777	17.5233	[77.93,146.62]	68.6913
A_2	$q_{2(opt)} = 1.8270$	110.6306	22.0938	[67.32,153.93]	86.6080
A_3	$q_{2(opt)} = 1.8270$	115.8193	14.8085	[86.79,144.84]	58.0496
A_4	$q_{2(opt)} = 1.8270$	123.5596	13.9605	[96.19,150.92]	54.7252
A_5	$q_{2(opt)} = 1.8270$	118.5700	20.449	[79.08,158.05]	78.9682
A_6	$q_{2(opt)} = 1.8270$	121.3830	19.0856	[83.97,158.79]	74.8156
A_7	$q_{2(opt)} = 1.8270$	120.9650	14.7139	[92.19,149.80]	57.6786
A_8	$q_{2(opt)} = 1.8270$	118.1187	17.2418	[84.32,151.91]	67.5882
A_9	$q_{2(opt)} = 1.8270$	125.1801	20.7067	[84.59,165.76]	81.1705
A_{10}	$q_{2(opt)} = 1.8270$	116.8956	15.7045	[86.11,147.67]	61.5619
Avg Length of CI				[83.84,15282]	68.9857
A_1	$q_{3(opt)} = 2.9830$	112.7620	17.5268	[78.37,147.07]	68.0752
A_2	$q_{3(opt)} = 2.9830$	110.9680	22.0964	[67.65,154.27]	86.6180
A_3	$q_{3(opt)} = 2.9830$	115.8680	14.8119	[86.83,144.89]	58.0625
A_4	$q_{3(opt)} = 2.9830$	123.7450	13.9635	[96.37,151.11]	54.7300
A_5	$q_{3(opt)} = 2.9830$	119.0780	20.1465	[79.59,158.56]	78.9743
A_6	$q_{3(opt)} = 2.9830$	121.7000	19.0886	[84.28,159.11]	74.8271
A_7	$q_{3(opt)} = 2.9830$	120.9980	14.7170	[92.15,149.84]	57.6908
A_8	$q_{3(opt)} = 2.9830$	118.1220	17.2433	[84.32,151.91]	67.5938
A_9	$q_{3(opt)} = 2.9830$	125.2110	20.7085	[84.62,165.79]	81.1775
A_{10}	$q_{3(opt)} = 2.9830$	117.1640	15.7066	[86.37,147.94]	61.5698
Avg. Length of CI				[84.05,152.95]	68.9319

6. Application of Confidence Interval For Missing Value Imputation

6.1. Proposed CI- Imputation Procedure

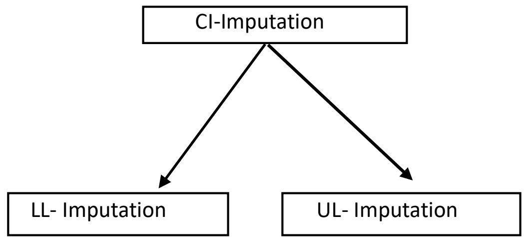

Fig 6.1 Type of CI-Imputation

Let random sample of size n drawn from N $(n < N)$ has only one missing value. The value imputed through pattern procedure is assumed available and missing one in sample is other than that. The proposed CI- imputation procedure is as under:

Step I: Find mean of sample of size (n-1) eliminating missing observation.

Step II: Calculate mean and MSE of sample data by suggested estimation method (eliminating missing value).

Step III: Calculate lower limit $(LL)_{opt} = [estimated\ mean - 1.96\sqrt{MSE(estimated\ mean)q_{opt}}]$ which is termed as LL-Imputation.

Moreover, calculate upper limit $(UL)_{opt} = [estimated\ mean + 1.96\sqrt{MSE(estimated\ mean)q_{opt}}]$ which is termed as UL-Imputation.

Step IV: Use $(LL)_{opt}$ or $(UL)_{opt}$ for imputing the missing value in sample.

Step V: Repeat the procedure over multiple random samples and average out the estimated mean value with imputation.

Note 6.1 CI-Imputaion seems logically better since it incorporates both mean and MSE in $(LL)_{opt}$ or $(UL)_{opt}$ while sample mean imputation of (n-1) observations does not incorporate variability information.

Table 6.1 Sample With one Missing Value

Sample No.	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9	A_{10}	Remark
(z_1,u_1)	(117,145)	(156,156)	(120,100)	(111,129)	(118,174)	(85,94)	(117,135)	(176,108)	(91,239)	(167,127)	
(z_2,u_2)	(106,97)	(107,127)	(132,64)	(120,,100)	(85,94)	(142,119)	(106,97)	(142,119)	(114,63)	(133,100)	
(z_3,u_3)	(132,64)	(142,119)	(113,64)	(113,64)	(101,101)	(133,90)	(113,64)	(131,110)	(167,127)	(178,141)	
(z_4,u_4)	(61,117)	(131,110)	(61,117)	(128,137)	(170,123)	(131,110)	(128,137)	(178,141)	(167,153)	(126,139)	
(z_5,u_5)	(81,126)	(178,141)	(137,149)	(127,101)	(116,99)	(178,141)	(128,81)	(92,75)	(132,156)	(160,177)	
(z_6,u_6)	(95,106)	(115,128)	(81,126)	(119,105)	(111,116)	(115,128)	(103,114)	(161,139)	(178,141)	(143,152)	
(z_7,u_7)	(128,81)	(160,117)	(126,77)	(141,147)	(140,156)	(134,129)	(138,109)	(122,94)	(92,75)	(111,116)	
(z_8,u_8)	(103,114)	(143,152)	(119,105)	(159,95)	(47,110)	(174,109)	(84,119)	(111,116)	(170,123)	(101,137)	
(z_9,u_9)	(138,109)	(49,92)	(128,81)	(99,77)	(134,129)	(66,84)	(159,95)	(102,126)	(111,116)	(122,121)	
(z_{10},u_{10})	(62,56)	(134,129)	(141,147)	(131,118)	(135,106)	(109,130)	(147,119)	(135,166)	(47,110)	(109,130)	
(z_{11},u_{11})	(78,69)	(123,131)	(138,109)	(97,85)	(123,131)	(121,89)	(164,137)	(143,141)	(134,129)	(111,99)	
(z_{12},u_{12})	(100,136)	(66,84)	(135,157)	(104,129)	(122,121)	(116,64)	(985)	(109,136)	(143,141)	(121,89)	
(z_{13},u_{13})	(99,77)	(122,121)	(78,69)	(112,65)	(101,129)	(131,95)	(100,117)	(116,64)	(109,130)	(116,64)	
(z_{14},u_{14})	(103,174)	(87,67)	(100,135)	(120,160)	(104,97)	(118,128)	(131,99)	(95,109)	(121,89)	(131,95)	
(z_{15},u_{15})	(131,118)	(111,99)	(84,119)	(139,162)	(140,105)	(140,105)	(97,85)	(106,136)	(95,109)	(118,128)	
$\rightarrow (z_{16},u_{16})$	(–,137)	(–,123)	(–,139)	(–,118)	(–,152)	(–,98)	(–,117)	(–,98)	(–,105)	(–,87))	Missing Values
(z_{17},u_{17})	(111,163)	(178,152)	(97,85)	(99,149)	(117,92)	(117,92)	(131,99)	(164,172)	(104,72)	(140,105)	
(z_{18},u_{18})	(112,65)	(113,123)	(100,117)	(131,140)	(113,123)	(96,65)	(97,85)	(125,98)	(95,98)	(104,72)	
(z_{19},u_{19})	(120,166)	(125,98)	(111,136)	(158,151)	(106,136)	(137,152)	(104,129)	(115,109)	(117,92)	(178,152)	
(z_{20},u_{20})	(139,162)	(75,121)	(11,99)	(106,59)	(140,105)	(125,98)	(73,59)	(189,109)	(158,86)	(103,119)	
(z_{21},u_{21})	(109,118)	(112,125)	(84,113)	(94,117)	(118,128)	(112,125)	(99,107)	(75,81)	(121,142)	(96,65)	
(z_{22},u_{22})	(92,140)	(101,94)	(149,140)	(143,148)	(116,64)	(106,59)	(112,65)	(139,162)	(119,103)	(158,86)	
(z_{23},u_{23})	(75,81)	106,59)	(136,175)	(137,152)	(109,130)	(158,151)	(139,162)	(157,148)	(112,125)	(121,142)	
(z_{24},u_{24})	(99,149)	(158,151)	(97,155)	(164,172)	(101,129)	(131,140)	(158,137)	(100,117)	(131,140)	(103,119)	
(z_{25},u_{25})	(131,140)	(131,140)	(131,175)	(103,119)	(143,141)	(136,17)	(97,155)	(95,129)	(99,149)	(96,65)	
(z_{26},u_{26})	(158,151)	(99,149)	(106,59)	(113,123)	(122,121)	(140,140)	(132,117)	(147,119)	(92,140)	(158,86)	
(z_{27},u_{27})	(94,117)	(75,81)	(101,94)	(106,136)	(101,137)	(84,113)	(121,142)	(99,77)	(139,162)	(121,142)	
(z_{28},u_{28})	(141,98)	(92,140)	(141,98)	(105,94)	(123,131)	(91,120)	(164,172)	(100,136)	(157,148)	(119,103)	
(z_{29},u_{29})	(164,172)	(109,118)	(143,148)	(66,84)	(134,129)	(131,99)	(103,119)	(78,69)	(131,99)	(101,94)	
(z_{30},u_{30})	(96,65)	(56,87)	(121,142)	(174,109)	(47,110)	(70,65)	(140,105)	(135,157)	(73,59)	(131,157)	
(z_{31},u_{31})	(103,119)	(84,55)	(137,152)	(135,106)	(49,92)	(95,103)	(118,128)	(103,114)	(104,129)	(131,140)	
(z_{32},u_{32})	(178,152)	(157,148)	(103,119)	(124,99)	(132,117)	(164,137)	(111,99)	(128,81)	(95,103)	(92,140)	
(z_{33},u_{33})	(140,105)	(91,120)	(104,72)	(140,156)	(161,139)	(159,95)	(140,105)	(95,106)	(147,119)	(56,87)	
(z_{34},u_{34})	(118,128)	(73,59)	(140,105)	(132,117)	(92,75)	(78,69)	(118,128)	(127,101)	(109,117)	(139,162)	
(z_{35},u_{35})	(121,89)	(104,129)	(118,128)	(106,124)	(131,110)	(69.117)	(111,99)	(61,117)	(107,59)	(84,113)	
(z_{36},u_{36})	(123,131)	(95,103)	(122,121)	(161,139)	(167,153)	(141,147)	(66,84)	(106,117)	(127,101)	(120,166)	
(z_{37},u_{37})	(102,126)	(71,139)	(174,109)	(92,75)	(176,108)	(169,112)	(174,109)	(106,120)	(128,137)	(128,81)	
(z_{38},u_{38})	(106,124)	(103,174)	(135,106)	(178,141)	(75,81)	(105,133)	(135,106)	(117,135)	(113,64)	(104,129)	
(z_{39},u_{39})	(170,123)	(100,136)	(47,110)	(131,110)	(122,121)	(126,77)	(124,99)	(120,100)	(106,97)	(164,137)	
(z_{40},u_{40})	(85,94)	(115,139)	(170,123)	(176,108)	(160,117)	(61,117)	(102,126)	(81,126)	(120,100)	(115,139)	

Table 6.1 represents the ten samples as in table 5.1 but 16^{th} value is assumed missing in each sample.

Table 6.2 Sample Statistic Excluding Missing Value (table 6.1) for (n-1) observations

Sample No.	Mean ($\bar{z}$)	Mean($\bar{u}$)	s_z	s_u	c_z	c_u	ρ_{zu}
A_1	113.3590	119.6750	32.6380	33.2145	0.2879	0.2775	0.1992
A_2	112.0510	119.1500	36.5645	30.2329	0.3263	0.2537	0.5064
A_3	116.0150	116.6500	32.5492	31.0306	0.2805	0.2660	0.0971
A_4	125.0020	118.0000	32.3158	29.1196	0.2585	0.24167	0.3446
A_5	119.1500	119.8250	36.6034	22.2767	0.3072	0.1859	0.1487
A_6	120.3590	110.3500	35.7087	27.5249	0.2966	0.2494	0.3820
A_7	118.2500	113.6000	32.0423	28.5359	0.2709	0.2512	0.3115
A_8	118.1030	115.4250	35.9131	27.8815	0.3040	0.2415	0.4420
A_9	124.2310	113.6750	39.9801	28.4906	0.3218	0.2506	0.3954
A_{10}	117.3590	118.6500	28.8045	29.2991	0.2198	0.2469	0.0474

The tabele 6.2 reveals decriptive statistic of ten samples in terms of mean, variability and sample correlation when one value is missing.

Table 6.3 Estimated Confidence Intervals Over 10 Samples at the q_{opt} Excluding Missing Value (for (n-1) observations)

Sample No.	q_{opt}	E	est(MSE)	C.I.	Length
A_1	$q_{1(opt)} = 0.6335$	111.508	19.4906	[78.30,149.40]	76.403
A_2	$q_{1(opt)} = 0.6335$	110.4240	19.0999	[72.98,147.86]	74.8718
A_3	$q_{1(opt)} = 0.6335$	115.34000	25.9601	[74.42,156.25]	81.8199
A_4	$q_{1(opt)} = 0.6335$	123.6840	16.9108	[90.53,156.82]	66.2905
A_5	$q_{1(opt)} = 0.6335$	117.1430	24.2574	[69.59,164.68]	95.0888
A_6	$q_{1(opt)} = 0.6335$	122.3780	20.0005	[83.17,161.57]	78.4021
A_7	$q_{1(opt)} = 0.6335$	118.8400	17.0798	[85.36,152.31]	66.9500
A_8	$q_{1(opt)} = 0.6335$	117.9250	19.3529	[79.99,155.85]	75.8633
A_9	$q_{1(opt)} = 0.6335$	124.8180	24.8983	[76.01,173.61]	97.6012
A_{10}	$q_{1(opt)} = 0.6335$	115.8930	14.2437	[87.97,143.81]	55.8355
			Average	[79.83,156.21]	76.816
A_1	$q_{2(opt)} = 1.8270$	111.0377	19.4860	[72.84,149.23]	76.3852
A_2	$q_{2(opt)} = 1.8270$	110.0667	19.1044	[72.62,147.51]	74.8894
A_3	$q_{2(opt)} = 1.8270$	115.2877	20.8657	[74.39,156.18]	81.7938
A_4	$q_{2(opt)} = 1.8270$	123.4870	16.9098	[90.34,156.63]	66.2865
A_5	$q_{2(opt)} = 1.8270$	116.6126	24.2555	[69.07,164.15]	95.0817
A_6	$q_{2(opt)} = 1.8270$	122.0257	20.0014	[82.82,161.22]	78.4058
A_7	$q_{2(opt)} = 1.8270$	118.8049	17.0783	[85.33,152.27]	66.9471
A_8	$q_{2(opt)} = 1.8270$	117.9215	19.3556	[79.98,155.85]	75.8741
A_9	$q_{2(opt)} = 1.8270$	124.7844	24.9004	[75.97,173.58]	97.6098
A_{10}	$q_{2(opt)} = 1.8270$	115.6105	14.2367	[87.70,143.51]	55.8079
			Average	[79.10,156.03]	76.8681
A_1	$q_{3(opt)} = 2.9830$	111.4810	19.4878	[73.28,149.67]	76.3922
A_2	$q_{3(opt)} = 2.9830$	110.4030	19.1027	[72.96,147.84]	74.8820
A_3	$q_{3(opt)} = 2.9830$	115.3360	20.8684	[74.43,156.23]	81.8040
A_4	$q_{3(opt)} = 2.9830$	123.6720	16.9102	[90.52,156.81]	66.2880
A_5	$q_{3(opt)} = 2.9830$	117.1130	24.2562	[69.57,164.65]	95.0845
A_6	$q_{3(opt)} = 2.9830$	122.3450	20.0011	[83.14,161.54]	78.4044
A_7	$q_{3(opt)} = 2.9830$	118.8370	17.0789	[85.36,152.31]	66.9494
A_8	$q_{3(opt)} = 2.9830$	117.9250	19.3546	[79.98,155.85]	75.8699
A_9	$q_{3(opt)} = 2.9830$	124.8150	24.8996	[76.01,173.79]	97.6065
A_{10}	$q_{3(opt)} = 2.9830$	115.8760	14.2394	[87.96,143.78]	55.8186
			Average	[79.31,156.24]	76.9099

Table 6.3 displays the three optimum choices of q over 10 samples along with sample estimates (at opt q) opt MSE and optimum length of CI. Different q_{opt} showing the similar length of CI.

6.2. CI-Imputation Using Lower Limit (LL-Imputation)

In tables 6.4, 6.5 and 6.6, the CI-Imputation with $(LL)_{opt}$ is attempted against missing value as sample in table 6.1

Table 6.4 Sample Where Missing Value Replaced by LL-Imptation (table 6.1)

Sample No.	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9	A_{10}	Remark
$\rightarrow (z_{16}, u_{16})$	(–,137)	(–,123)	(–,139)	(–,118)	(–,152)	(–,98)	(–,117)	(–,98)	(–,105)	(–,87))	16^{th} Missing values in table 6.1
$\rightarrow (z_{16}, u_{16})$	(78.30,137)	(72.98,123)	(74.42,139)	(90.53,118)	(69.59,152)	(83.17,98)	(85.36,117)	(79.99,98)	(76.01,105)	(87.97,87)	imputed 16^{th} values by LL in table 6.1

Table 6.4 shows value under LL-Imputation against 16^{th} missing value of table 6.1.

Table 6.5 Sample Statistic When Missing Value replaced by LL-imputation in 10 Samples of table 6.4

Sample No.	Mean ($\bar{z}$)	Mean($\bar{u}$)	s_z	s_u	c_z	c_u	ρ_{zu}
A_1	112.3575	119.6750	27.5043	33.2145	0.2447	0.2775	0.2653
A_2	111.0510	119.1500	32.0446	30.2329	0.2884	0.2537	0.5707
A_3	116.0100	116.6500	27.1908	31.0306	0.2331	0.2660	0.1626
A_4	124.6133	118.0000	25.6345	29.1196	0.2057	0.24167	0.4235
A_5	118.4398	119.8250	31.5086	22.2767	0.2660	0.1859	0.2481487
A_6	119.4293	110.3500	30.2454	27.5249	0.2532	0.2494	0.4089
A_7	118.2400	113.6000	26.0275	28.5359	0.2181	0.2512	0.3837
A_8	117.9248	115.4250	30.7422	27.8815	0.2606	0.2415	0.4628
A_9	123.0253	113.6750	33.9271	28.4906	0.2757	0.2506	0.4273
A_{10}	117.3180	118.6500	26.0493	29.2991	0.2098	0.2469	0.0991

The table 6.5 is obtained by using data of table 6.4 after LL-imputation.

Table 6.6 Estimated Confidence Intervals Over 10 Samples at the q_{opt} When Missing Value Replaced by LL-Imputation (at optimum q=0.6335)

Sample No.	q_{opt}	E	est(MSE)	C.I.	Length
A_1	$q_{1(opt)} = 0.6335$	110.5280	13.4334	[84.19,136.85]	52.6591
A_2	$q_{1(opt)} = 0.6335$	109.4615	13.5378	[82.92,135.99]	53.0687
A_3	$q_{1(opt)} = 0.6335$	115.9363	14.4416	[87.65,144.26]	56.6113
A_4	$q_{1(opt)} = 0.6335$	123.30010	9.9386	[103.82,142.77]	38.9593
A_5	$q_{1(opt)} = 0.6335$	116.4447	17.1051	[82.91,149.97]	67.0522
A_6	$q_{1(opt)} = 0.6335$	121.4324	13.9675	[94.05,148.80]	54.7527
A_7	$q_{1(opt)} = 0.6335$	118.8400	17.0798	[85.36,152.31]	66.9500
A_8	$q_{1(opt)} = 0.6335$	117.7472	13.7387	[90.81,144.67]	53.8570
A_9	$q_{1(opt)} = 0.6335$	123.6065	17.3161	[89.66,157.54]	67.8793
A_{10}	$q_{1(opt)} = 0.6335$	116.3164	13.9007	[89.07,143.56]	54.4908
			Average	[88.94,144.87]	56.6280

Table 6.6 provides optimum length of confidence intervals under LL-Imputation.

6.3. CI-Imputation by Upper Limit (UL-Imputation)

In tables 6.7, 6.8 and 6.9 the CI-Imputation with $(UL)_{opt}$ is taken into account against missing values related to table 6.1

Table 6.7 Sample in Which Missing Value Replaced by UL-Imputation (table 6.1)

Sample No.	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9	A_{10}	Remark
$\rightarrow (z_{16}, u_{16})$	(–,137)	(–,123)	(–,139)	(–,118)	(–,152)	(–,98)	(–,117)	(–,98)	(–,105)	(–,87))	16^{th} Missing values in table 6.1
$\rightarrow (z_{16}, u_{16})$	(149.40,137)	(147.86,123)	(156.25,139)	(156.82,118)	(164.68,152)	(161.57,98)	(152.31,117)	(155.85,98)	(173.61,105)	(143.81,87)	imputed 16^{th} values by UL in table 6.1

Table 6.7 displays the value under UL-Imputation against 16^{th} missing value of table 6.1. The fifth column of table 6.3 showing upper limit is used to replace the missing.

Table 6.8 Sample Statistic After When Missing Value is Replaced by UL-Imputation in 10 Samples (obtained by table 6.7)

Sample No.	Mean ($\bar{z}$)	Mean($\bar{u}$)	s_z	s_u	c_z	c_u	ρ_{zu}
A_1	114.26000	119.6750	27.3685	33.2145	0.2395	0.2775	0.3028
A_2	112.9470	119.1500	31.9515	30.2329	0.2828	0.2537	0.5798
A_3	118.6810	116.6500	27.0155	31.0306	0.2276	0.2660	0.02182
A_4	126.2710	118.0000	25.5201	29.1196	0.2021	0.24167	0.4254
A_5	120.8170	119.8250	31.3200	22.2767	0.2592	0.1859	0.3593
A_6	121.3890	110.3500	30.3722	27.5249	0.2502	0.2494	0.3782
A_7	120.9580	113.6000	25.9448	28.5359	0.2114	0.2512	0.3926
A_8	119.8210	115.4250	30.6834	27.8815	0.2560	0.2415	0.4251
A_9	125.4650	113.6750	33.9678	28.4906	0.2707	0.2506	0.4049
A_{10}	120.3590	118.6500	25.7112	29.2991	0.2135	0.2469	0.0864

The table 6.8 is obtained by using data of table 6.7 after UL-Imputation against 16^{th} value of table 6.1.

Table 6.9 Estimated Confidence Intervals Over 10 Samples at the q_{opt} When Missing Values Replaced by UL-Imputation (at optimum q=0.6335)

Sample No.	q_{opt}	E	est(MSE)	C.I.	Length
A_1	$q_{1(opt)} = 0.6335$	112.3940	12.8696	[87.16,137.61]	50.4488
A_2	$q_{1(opt)} = 0.6335$	111.3070	13.2788	[85.28,137.33]	52.0530
A_3	$q_{1(opt)} = 0.6335$	117.9900	13.6392	[91.25,144.72]	53.4657
A_4	$q_{1(opt)} = 0.6335$	124.9400	9.8421	[105.65,144.23]	38.5811
A_5	$q_{1(opt)} = 0.6335$	118.7820	15.7420	[87.92,149.63]	61.7085
A_6	$q_{1(opt)} = 0.6335$	123.4250	14.5061	[94.99,151.85]	56.8641
A_7	$q_{1(opt)} = 0.6335$	121.5620	10.5279	[100.92,142.19]	41.2695
A_8	$q_{1(opt)} = 0.6335$	119.6410	14.1687	[91.86,147.41]	55.5414
A_9	$q_{1(opt)} = 0.6335$	126.0580	17.7000	[91.36,160.75]	69.3841
A_{10}	$q_{1(opt)} = 0.6335$	118.8550	13.8306	[91.74,145.96]	54.2161
			Average	[92.78,146.16]	53.3532

Table 6.9 reveals optimum length of confidence intervals after UL-Imputation for 16^{th} values of table 6.1.

7. Comparison

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LL-Imputation and UL-Impuation mutually using following formula over 10 samples.

$$Percentage\ Relative\ Gain(PRG) = \frac{[Length\ of\ CI\ Under\ (LL)_{opt} - Imputation]}{[Length\ of\ CI\ Under\ (UL)_{opt} - Imputation]} X100$$

Table 7.1 Percentage Relative Gain (PRG)

Sample No.	PRG(%)
A_1	104.3012%
A_2	101.9512%
A_3	105.8833%
A_4	100.9802%
A_5	108.6595%
A_6	106.2809%
A_7	102.2263%
A_8	106.9673%
A_9	104.2353%
A_{10}	100.2826%
Average	104.1767%

$$Percentage\ Relative\ Efficiency(PRE) = \left[\frac{MSE(E)_q - MSE(E)q_{opt}}{MSE(E)_q}\right] X100$$

where q=1, 2, 3, 4, 5

Table 7.2 Percentage Relative Efficiency (PRE) of Proposed Strategy E

S.No.	q	PRE(%)
1	q=1	31.6200%
2	q=2	73.1700%
3	q=3	00.2500%
4	q=4	17.7900%
5	q=5	2.8100%

8. Discussion

The proposed estimation strategy E has constants A, B, C, D who are linked with another single constant $q > 0$. For given data in table 4.2 and population parameters in table 4.3, the most suitable choices of q are in table 4.4 and table 4.5. For given population (M=0.4082, g=0.2666), the proposed estimation strategy is almost unbiased when $q_1 = 1.0756, q_2 = 1.9709$ and $q_3 = 2.9073$. The best is $q = q_3 = 2.9073$ because it reduces MSE also as shown in table 4.4. Likewise, in table 4.5 the choices of q are $q_1 = 0.6335, q_2 = 1.8270, q_3 = 2.9830$ on which the MSE is optimum (minimum). Best option is $q = q_1 = 0.6335$ having the least bias. Overall, for given data in table 4.2, the most suitable q is $q \in (0.6, 3.0)$ producing optimum MSE with least bias.

The general Ready-Reckoner table 4.7 and table 4.8 reveal for any given data where M ranging $M \in (0.05, 0.95)$, g ranging $g \in (0.3, 0.9)$, the best q ranging $q \in (0.6, 4.55)$ for which MSE and bias both are at the lowest level, whatever be positive $M < 1$ and $g < 1$ using the proposed estimation strategy. The simulation results of confidence interval (CI) over 10 samples, each of size n = 40, are in table 5.1 who estimate sample statistic of the proposed over 10 samples as in table 5.2. The calculation of 95% confidence intervals are in table 5.3. All the CI are catching the true mean value of internal+secret economic bond $\bar{Z} = 116$. The length of confidence intervals have extremely minor variations among them.

The proposed strategy E is efficient at q_{opt} compare to other q-values (table 8.2). The table 5.1 presents ten samples each of size fourty and using equations (21) and (22) the confidence intervals limits $(LL)_{opt}$ and $(UL)_{opt}$ are calculated in table 5.3. The critical observation is that confidence intervals are catching the true value $\bar{Z} = 116$ and they are robust for different values of q_{opt}. Predicting confidence intervals using (21) and (22) makes the result independent to the selection of best q_{opt}. Table 6.1 has 16^{th} value missing in terms of Z but U is available. Even after eliminating the missing, remaining (n-1) observations produce confidence intervals containing unknown mean ($\bar{Z} = 116$) but their lengths have variations. A new CI-Imputation is proposed in section 6 having two types strategies as LL-Impuation and UL-Imputation. Both are compared in the convert text. Overall in the ten samples, the UL-Imputation found better (more efficient) than LL-Imputation. Confidence intervals after imputation are close to the before imputation (table 6.6 and table 6.9).

9. Conclusion

On recapitulation, the problem opted is to estimate the average economic inner+secret bond existing between supporting countries involved in the war. Their connectivity is modeled like a generalised Petersen graph, sampling and imputation technique are used as methodological tools. As new methods of imputation, named after "Pattern Imputation and CI- Imputation" are proposed in the content in order to maintain the completeness in the symmetry in view of the sampling strategy implementation. Pattern imputation is found efficient and useful for filling the missing data. An estimation strategy is proposed whose expressions of bias and mean squared error are derived. It has four constants A, B, C and D who are linked with another single constant q having expressions in terms of power five. This has led to the best selection of q for making the proposed estimation strategy optimum with least bias. The most plausible selection of q is $q \in (0.6, 3.0)$ for given M = 0.4082, g = 0.2666. Two Ready-Reckoner tables provide general range of most suitable q as $q \in (0.6, 4.55)$ whatsoever be the positive most frequent value of M and g characterizing the population. As a part of secondary verification of performance of proposed estimation strategy, which is sample based with CI-Imputation, the method of confidence interval (CI) is used as a tool. It is found that all the estimated confidence intervals are catching the true unknown mean value of the internal+secret economic bond levele of interest which is strength of the proposed. The proposed estimation strategy E found robust in terms of different q_{opt} values as the predicted range of confidence intervals are almost same over varying q_{opt}. The UL-Imputation method is better than the LL-Imputation in comparision. The content of this paper has use of double imputations use like Pattern imputation and CI-Imputation together and both are effectively implemented.

In the war-weapon current situation of Ukraine-Russia this methodology can be used to evaluate the average amount of Economic bond (specially secret support) existing among countries assisting or involved indirectly to the war of Ukraine and Russia on either side.

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