STATISTICS IN TRANSITION new series, June 2023
Vol. 24, No. 3, pp. 257–263, https://doi.org/10.59170/stattrans-2023-047
Received – 15.06.2022; accepted – 26.01.2023

Bayesian estimation of a geometric distribution using informative priors based on a Type-I censoring scheme

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Abstract

In this paper, the geometric distribution parameter is estimated under a type-I censoring scheme by means of the Bayesian estimation approach. The Beta and Kumaraswamy informative priors, as well as five loss functions are used for this purpose. Expressions of Bayes estimators and Bayes risks are derived under the Squared Error Loss Function (SELF), the Quadratic Loss Function (QLF), the Precautionary Loss Function (PLF), the Simple Asymmetric Precautionary Loss Function (SAPLF), and the DeGroot Loss Function (DLF) using the two aforementioned priors. The prior densities are obtained through prior predictive distributions. Simulation studies are carried out to make comparisons using Bayes risks. Finally, a real-life data example is used to verify the model's efficiency.

Key words: prior distribution, posterior distribution, geometric distribution, beta distribution, Kumraswamy distribution.

1. Introduction

Type-I censored sampling is helpful for lifetime research. Type-I censored data are used when the last few observations in a series are missing or suppressed due to lack of time or because the experimenter cannot wait for the last observation due to time restriction. Shi and Yan (2010) produced type-I censored empirical Bayes estimates of the two-parameter exponential distribution. Saleem et al. (2010) used type-I censored data to explore the power function mixture distribution. Tahir et al. (2016) study the Bayesian analysis of a three-component mixture of exponential distributions. Khan et al. (2016) designed and compared different loss functions for estimating scale parameter of log-normal distribution under type-I censoring schemes. Yanuar et al. (2019) used Bayesian estimation tool to estimate the scale parameter of Weibull distribution. Kour et al. (2020) employed Bayesian and E-Bayesian techniques to estimate exponential-Lomax distribution's parameters. Abbas et al. (2020) used Bayesian inference to estimate the parameters of Gumbel type-II distribution under censored sample scenario. Long (2021) estimated Rayleigh distribution's parameters

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using double Type-I hybrid censored data. Most of the studies considered continuous life testing models. This study explores the estimation issue for the parameter of a discrete Geometric life testing model in the Bayesian paradigm using different loss functions and priors information under the type-1 censoring sampling scheme. The posterior distributions are derived using Beta and Kumarasawmy priors. In addition, the posterior Bayes estimators (BE) and Bayes risks (BR) are derived using the Squared Error Loss Function (SELF), Quadratic Loss Function (QLF), Precautionary Loss Function (PLF), Simple Asymmetric Precautionary Loss Function (SAPLF), and DeGroot Loss Function (DLF). A simulation study is carried out with different sample sizes and different parametric settings in order to make numerical comparisons. A real data set is also used to validate the simulation findings.

2. Posterior distributions under different priors

The probability density function of the geometric distribution for a random variable X is given below.

$$f(x) = \omega(1-\omega)^{x}, \quad x = 0, 1, \ldots \tag{1}$$

where ω is the parameter of the Geometric distribution. The cumulative distribution function of the geometric distribution for a random variable X is given by:

$$F_X(x) = 1-(1-\omega)^{x+1}, \quad x = 0, 1, \ldots \tag{2}$$

The likelihood function of the Geometric distribution under type-I censoring sampling scheme can be written as:

$$L(\omega) \propto \prod_{i=1}^{r} \omega(1-\omega)^{x_i}\left[(1-\omega)^{t+1}\right]^{n-r} \tag{3}$$

It is assumed that the parameter follows Beta distribution, i.e. $\omega \sim \text{Beta}(f_1, f_2)$ with hyper-parameters f_1 and f_2, using this prior, the posterior distribution of ω given data is:

$$p_\beta(\omega|\underline{x}) = \frac{\omega^{r+f_1-1}(1-\omega)^{\delta_\beta-1}}{B(f_1+r, \delta_\beta)}, 0 < \theta < 1 \tag{4}$$

3. Elicitation of Hyper-Parameters

The Aslam (2003) approach is used to elicit the informative priors. For this purpose, prior distributions derived under Beta and Kumaraswamy priors are used. We consider two intervals for two unknown hyper parameters of Beta prior, i.e. (0, 2) and (6, 8), with the associated probabilities 0.6001 and 0.100 as an expert's belief about these intervals. The hyper-parameters of Beta prior are obtained as follows.

$$f_1 = 0.77930, \; f_2 = 1.4340$$

The resultant value of the hyper-parameter of Kumarswamy prior is $f_3 = 1.241$.

4. Loss functions

Under the assumed priors, BEs and BRs are determined under the SELF, DLF, QLF, PLF and SAPLF. The expressions are provided in the following tables. Under the assumed priors, BEs and BRs are determined under the SELF, DLF, QLF, PLF and SAPLF. The expressions are provided in the following tables.

Table 1. Bayes estimators and Bayes Risks under SELF

Priors	*BE*	*BR*
Beta	$\frac{B(r+f_1+1,\delta_\beta)}{B(r+f_1,\delta_\beta)}$	$\frac{B(r+f_1+2,\delta_\beta)}{B(r+f_1,\delta_\beta)}-(BE)^2$
Kumarswamy	$\frac{B(r+2,\delta_{KS})}{B(r+1,\delta_{KS})}$	$\frac{B(r+3,\delta_{KS})}{B(r+1,\delta_{KS})}-(BE)^2$

Table 2. Bayes estimators and Bayes Risks under DLF

Priors	*BE*	*BR*
Beta	$\frac{B(r+f_1+2,\delta_\beta)}{B(r+f_1+1,\delta_\beta)}$	$1-\frac{(B(r+f_1+1,\delta_\beta))^2}{B(f_1+r,\delta_\beta)B(r+f_1+2,\delta_\beta)}$
Kumarswamy	$\frac{B(r+3,\delta_{KS})}{B(r+2,\delta_{KS})}$	$1-\frac{(B(r+2,\delta_{KS}))^2}{B(r+1,\delta_{KS})B(r+3,\delta_{KS})}$

Table 3. Bayes estimators and Bayes Risks under QLF

Priors	*BE*	*BR*
Beta	$\frac{B(r+f_1-1,\delta_\beta)}{B(r+f_1-2,\delta_\beta)}$	$1-\frac{(B(r+f_1-1,\delta_\beta))^2}{B(f_1+r,\delta_\beta)B(r+f_1-2,\delta_\beta)}$
Kumarswamy	$\frac{B(r,\delta_{KS})}{B(r-1,\delta_{KS})}$	$1-\frac{(B(r,\delta_{KS}))^2}{B(r+1,\delta_{KS})B(r-1,\delta_{KS})}$

Table 4. Bayes estimators and Bayes Risks under PLF

Priors	*BE*	*BR*
Beta	$\sqrt{\frac{B(r+f_1+2,\delta_\beta)}{B(\omega_1+r,\delta_\beta)}}$	$2\left(\sqrt{\frac{B(r+f_1+2,\delta_\beta)}{B(f_1+r,\delta_\beta)}}-\frac{B(r+f_1+1,\delta_\beta)}{B(f_1+r,\delta_\beta)}\right)$
Kumarswamy	$\sqrt{\frac{B(r+3,\delta_{KS})}{B(r+1,\delta_{KS})}}$	$2\left(\sqrt{\frac{B(r+3,\delta_{KS})}{B(r+1,\delta_{KS})}}-\frac{B(r+2,\delta_{KS})}{B(r+1,\delta_{KS})}\right)$

Table 5. Bayes estimators and Bayes Risks under SAPLF

Priors	*BE*	*BR*
Beta	$\sqrt{\frac{B(r+f_1+1,\delta_\beta)}{B(r+f_1-1,\delta_\beta)}}$	$2\left(\frac{1}{B(f_1+r,\delta_\beta)}\times\sqrt{\frac{B(r+f_1+1,\delta_\beta)}{B(r+f_1-1,\delta_\beta)}}-1\right)$
Kumarswamy	$\sqrt{\frac{B(r+2,\delta_{KS})}{B(r,\ \delta_{KS})}}$	$2\left(\frac{1}{B(r+1,\delta_{KS})}\times\sqrt{\frac{B(r+2,\delta_{KS})}{B(r,\delta_{KS})}}-1\right)$

5. Simulations Study

From a lifetime model of geometric distribution, random samples are generated with samples of sizes n =20 and 50 by considering different termination times and different parametric values, simulation process is performed 10,000 times. Based on these samples, BEs and BRs are obtained. The findings of the simulation study are presented in Tables 6 – 15.

Table 6. BEs and BRs under SELF for $T = 5$

Priors	n	$\omega = 0.4$		$\omega = 0.5$		$\omega = 0.6$	
		BEs	BRs	Bes	BRs	BEs	BRs
Beta	20	0.398549	0.0048122	0.497784	0.0058672	0.5928	0.0065180
	50	0.391299	0.001964	0.494964	0.0024617	0.595256	0.0027738
Kumaraswamy	20	0.402898	0.0048229	0.502888	0.0058569	0.598677	0.0064739
	50	0.393052	0.0019670	0.497044	0.0024608	0.597672	0.0027669

Table 7. BEs and BRs under SELF for $T = 7$

Priors	n	$\omega = 0.4$		$\omega = 0.5$		$\omega = 0.6$	
		BEs	BRs	Bes	BRs	BEs	BRs
Beta	20	0.405604	0.0046816	0.501316	0.0057882	0.594287	0.0064853
	50	0.400007	0.0019197	0.500538	0.0024332	0.59743	0.0027578
Kumaraswamy	20	0.4098	0.0046908	0.506334	0.0057780	0.600126	0.0064418
	50	0.401704	0.0019217	0.502591	0.0024320	0.599834	0.0027508

Table 8. BEs and BRs under DLF for $T = 5$

Priors	n	$\omega = 0.4$		$\omega = 0.5$		$\omega = 0.6$	
		BEs	BRs	Bes	BRs	BEs	BRs
Beta	20	0.399297	0.004819	0.497697	0.0058707	0.591013	0.0065107
	50	0.392174	0.001969	0.494613	0.0024616	0.594912	0.0027743
Kumaraswamy	20	0.403651	0.004829	0.502801	0.0058606	0.596875	0.0064675
	50	0.39393	0.001972	0.496694	0.0024607	0.597328	0.0027674

Table 9. BEs and BRs under DLF for $T = 7$

Priors	n	$\omega = 0.4$		$\omega = 0.5$		$\omega = 0.6$	
		BEs	BRs	Bes	BRs	BEs	BRs
Beta	20	0.405099	0.0046789	0.501034	0.0057820	0.593663	0.0064793
	50	0.399603	0.0019172	0.500154	0.0024312	0.597708	0.0027578
Kumaraswamy	20	0.409296	0.0046880	0.50605	0.0057718	0.599497	0.0064360
	50	0.401299	0.0019193	0.502206	0.0024301	0.600114	0.0027508

Table 10. BEs and BRs under QLF for $T = 5$

Priors	n	$\omega = 0.4$		$\omega = 0.5$		$\omega = 0.6$	
		BEs	BRs	Bes	BRs	BEs	BRs
Beta	20	0.38389	0.032751	0.478039	0.026819	0.570583	0.0217898
	50	0.39330	0.012797	0.4907470	0.0103919	0.588859	0.0082929
Kumaraswamy	20	0.38823	0.032165	0.48322	0.0262729	0.576646	0.0212587
	50	0.39501	0.012705	0.4927920	0.0103066	0.591267	0.0082101

Table 11. BEs and BRs under QLF for $T = 7$

Priors	n	$\omega = 0.4$		$\omega = 0.5$		$\omega = 0.6$	
		BEs	BRs	Bes	BRs	BEs	BRs
Beta	20	0.384534	0.0316685	0.478954	0.0264339	0.571199	0.0216775
	50	0.394012	0.0123857	0.491567	0.0102510	0.588864	0.0082620
Kumaraswamy	20	0.388734	0.031120	0.484075	0.025902	0.577245	0.0211505
	50	0.395762	0.0122944	0.49359	0.0101678	0.591263	0.0081797

Table 12. BEs and BRs under PLF for $T = 5$

Priors	n	$\omega = 0.4$		$\omega = 0.5$		$\omega = 0.6$	
		BEs	BRs	Bes	BRs	BEs	BRs
Beta	20	0.414857	0.0116282	0.509512	0.0115902	0.600445	0.0109775
	50	0.406027	0.0048758	0.504217	0.0048912	0.599809	0.0046379
Kumaraswamy	20	0.418962	0.0115406	0.514367	0.0114678	0.606064	0.0108136
	50	0.407697	0.0048613	0.506212	0.0048706	0.602139	0.0046097

Table 13. BEs and BRs under PLF for $T = 7$

Priors	n	$\omega = 0.4$		$\omega = 0.5$		$\omega = 0.6$	
		BEs	BRs	Bes	BRs	BEs	BRs
Beta	20	0.413877	0.0112833	0.508801	0.0114575	0.600232	0.0109478
	50	0.405998	0.0047290	0.503476	0.0048350	0.600198	0.0046205
Kumaraswamy	20	0.417853	0.0112013	0.513593	0.0113383	0.605831	0.0107851
	50	0.407617	0.0047154	0.505446	0.0048149	0.602523	0.0045925

Table 14. BEs and BRs under SAPLF for $T = 5$

Priors	n	$\omega = 0.4$		$\omega = 0.5$		$\omega = 0.6$	
		BEs	BRs	Bes	BRs	BEs	BRs
Beta		BEs	BRs	Bes	BRs	BEs	BRs
	20	0.40045	0.031393	0.496323	0.0254192	0.588079	0.0205064
	50	0.402002	0.0125155	0.49928	0.0101448	0.59496	0.0081071
Kumaraswamy	20	0.4048	0.030806	0.501463	0.0248808	0.594022	0.0199894
	50	0.403758	0.0124211	0.501366	0.0100575	0.597385	0.0080231

Table 15. BEs and BRs under SAPLF for $T = 7$

Priors	n	$\omega = 0.4$		$\omega = 0.5$		$\omega = 0.6$	
		BEs	BRs	Bes	BRs	BEs	BRs
Beta	20	0.40299	0.0301996	0.496625	0.0250879	0.589734	0.0203528
	50	0.400943	0.0121620	0.499331	0.0100232	0.596173	0.0080540
Kumaraswamy	20	0.407222	0.0296524	0.501705	0.0245629	0.595672	0.0198396
	50	0.402642	0.0120733	0.501392	0.0099378	0.598594	0.0079705

The above numerical results indicate that as the sample size increases, the efficiency of the BE increases as the BR decreases. Also, the increasing test termination time results in smaller BRs and enhanced estimating efficiency. The overall results show that Beta prior is a suitable prior and SELF is an efficient loss function in estimating the unknown parameter of the Geometric life testing model under type-I censoring scheme.

6. Applications

In this section, a data set of 137 lung cancer patients' remission duration, given by Krishna and Neha (2017), is analysed with the help of the proposed estimation strategy. The numerical results for this data set are presented in Table 16.

Table 16. BEs and BRs for Lung Cancer data

Prior		SELF		DLF	
		T=14	T=19	T=14	T=19
Beta	BEs	0.15662500	0.0934033	0.119137	0.103447
	BRs	0.000178453	0.0000638503	0.00755068	0.00719107
Kumaraswamy	BEs	0.15691800	0.0935679	0.119358	0.103628
	BRs	0.000178718	0.0000639499	0.00753454	0.00717693
Priors		QLF		PLF	
		T=14	T=19	T=14	T=19
Beta	BEs	0.10662500	0.09204240	0.10871200	0.093753800
	BRs	0.00778249	0.00739421	0.000831595	0.000682338
Kumaraswamy	BEs	0.10681700	0.09219790	0.10890400	0.093909000
	BRs	0.00776680	0.00738051	0.000831408	0.000682216
		SAPLF			
		T=14	T=19		
Beta	BEs	0.107866000	0.093060400		
	BRs	0.007753930	0.007370370		
Kumarasawmy	BEs	0.108070000	0.093225000		
	BRs	0.007737310	0.007355840		

7. Conclusion

In this study, a Bayesian estimation methodology is derived for estimating the parameter of discrete Geometric life testing model under type-I censoring scheme. Two informative priors, namely Beta and Kumaraswamy, and five loss functions (SELF, QLF, SAPLF, DLF and PLF) are used for this purpose. An extensive simulation study and a real-life data analysis is employed to validate the importance of the proposed strategy. The numerical results reveal that Beta is an appropriate prior and SLEF is a better loss function while analysing discrete Geometric life testing model under type-I censoring scheme. The real-life data analysis cements these findings.

Acknowledgements

Thanks to anyone for support, the anonymous referee and the associate editor for their constructive comments and suggestions which had greatly improved the earlier version of this manuscript.

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