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Investigation of half-normal model using informative priors under Bayesian structure

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Abstract

This paper considers properties of half-normal distribution using informative priors under the Bayesian criterion. It employs the squared root inverted gamma, Chi-square and Rayleigh distributions as the prior distribution to construct the Posterior distributions of the respective distributional parameters. Hyperparameters are elicited via prior predictive distribution. The properties of posterior distribution are studied, and their graphs are presented using a real data set. A comprehensive simulation scheme is conducted using informative priors. Bayes estimates are obtained using the loss functions (squared error loss function, modified loss function, quadratic loss function and Degroot loss function). Statistical inferences interval estimates and Bayesian hypothesis testing are presented to demonstrate the usefulness of the study.

Key words: informative prior, squared root inverted gamma distribution (SRIG), Bayesian hypothesis testing, loss functions.

1. Introduction

Bayesian Inference is an approach to Statistical Inference, which is distinct from frequentist inference. Bayesian statistics, named for Thomas Bayes, is a set of fields of statistics in which the evidence about the true state of the world is expressed in terms of degrees of belief or, more specifically, Bayesian probabilities. Moravveji etal. (2019) presents a Bayesian approach for the estimation of the parameters of two-piece scale mixtures of normal distributions. This is a rich family of light/heavy-tailed symmetric/asymmetric distributions that includes, as a special case, the heavy-tailed scale mixtures of normal distributions, and is flexible in computations for modelling

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symmetric and asymmetric data. A Bayesian approach is possible from the specification of hierarchical representations of the proposed family[4].

The half-normal distribution (HND) is linked with skewed positive data in describing lifetime process under fatigue. Various studies are done on the characteristics of HND under Bayesian with the choice of various priors. For example, Bland and Altman (1999) studied the half-normal model for dealing with the relationships between measurement and magnitude error whereas Cohen (1992) studied the problem of inference of truncated distributions, including the truncated normal through a classical approach. Classical inference for half-normal model is examined by Pewsey (2002, 2004). Later, Cooray and Ananda (2008) defined the generalized HND derived from a model for static fatigue, which is then followed by Gauss et al. (2012), who study the Kumaraswamy generalized half-normal distribution for modelling skewed positive data. Gupta (2018) estimates the location parameter of a HND is considered. Some unbiased as well as biased estimators are derived. Admissibility and minimaxity of Pitman estimator are proved. A complete class of estimators among multiples of the maximum likelihood estimator is obtained.

Dobler (2015) developed Stein's method for HND and applied it to derive rates of convergence in distributional limit theorems for three statistics of the simple symmetric random walk: the maximum value, the number of returns to the origin and the number of sign changes up to a given time 'n'. Dobler compares the characterizing operator of the limiting HND with suitable characteristics of the discrete approximating distributions. Jeniffer et al. (2014) study the extended generalized half-normal distribution for modelling skewed fatigue life data. The new model contains as special cases the half-normal and generalized half-normal (Cooray and Ananda, 2008) distributions. Several of its structural properties are derived, including the density function, moments, quantile and generating functions, mean deviations and order statistics. They investigate maximum likelihood estimation of the model parameters. Alzaatreh and Knight (2013) propose the gamma-HND. Various structural properties of the gamma-HND are derived. The shape of the distribution may be unimodel or bimodal. Results for moments, limit behaviour, mean deviations and Shannon entropy are provided. To estimate the model parameters, the method of maximum likelihood estimation is proposed. Three real-life data sets are used to illustrate the applicability of the gamma-HND. For the first time, Cordeiro (2012) study the Kumaraswamy generalized HND for modelling skewed positive data. The half-normal and generalized half-normal (Cooray and Ananda, 2008) distributions are special cases of the new model. Several of its structural properties are derived, including explicit expressions for

[4] For recent applications, one can refer to the recent work by Shrivastava et al. (2019), Montagna et al. (2020), Ariyo et al. (2022) and Sindhu and Hussain 2022, among others.

the density function, moments generating and quantile functions, mean deviations, and moments of the order statistics.

Some recent important works related to simulation, the choice of complex priors related to HND, are done by various authors including Van Erp and Brown (2020) and Al Amer et al. (2021), Sindhu and Hussain (2022), Ariyo et al. (2022), Bruch and Felderer (2022), Martin et al. (2022), among others. Here, we would like to summarize their work for the ready reference of the readers. For example, Ariyo et al. (2022) explored the performance of three Bayesian model-selection criteria when vague priors are used for the covariance parameters of the random effects in a linear mixed-effects model using simulation study. They considered five different specifications of inverse-Wishart (IW) prior, five different separation priors and a joint prior. The results show that marginals perform far better over the conditional and the superiority of joint and separation priors over IW in all settings with selection criteria on a practical data set. Second is the work of Bruch and Felderer (2022), who considered prior choice for the variance parameter in multilevel regression and poststratification selective data and their Monte Carlo simulation study was done on the similar way as that of ours. They observed that prior choices are *challenging* when data results from selective inclusion mechanism which may be subject to bias in the estimation of a proportion based on a sample that is subject to a highly selective inclusion mechanism.

Moreover, similar work is done by Martin et al. (2022) using Python instead of SAS. They explored Bayesian modelling and computation in Python with the aim to help beginner Bayesian practitioners to become intermediate modellers. Beside SAS, they used PyMC3, Tensor-flow Probability and Arvi-Z approaches and other libraries focusing on the practice of applied statistics with a summary of references to the package used in, whereas Sindhu and Hussain (2022) derived and performed predictive inference and parameter estimation from the half-normal distribution for the left censored data. They also derive the posterior and predictive distribution in conjunction with informative vis-à-vis uninformative priors. They used SAS and simulated left censored samples from a half-normal distribution are utilized to interpret the results.

In this paper, the posterior distributions of the parameter using informative priors are derived in Section 2. The prior predictive distributions are derived in Section 3. Section 4 presents the elicitation of the hyperparameters via prior predictive distribution. The graphs of posterior distributions using a real data set are drawn in Section 5. In Section 6, the expressions of Bayes estimates under different loss functions are obtained. Section 7 presents Bayes estimates and Posterior risks using real data set. Section 8 contains credible intervals and hypothesis testing using a real data set. A simulation study is conducted using Mathematica and SAS packages[5] in Section 9. Section 10 contains some concluding remarks.

[5] For the use of other software and computing subroutines one can refer the work of Martin et al. (2022)

2. Posterior Distribution of the Parameter Using Informative Priors

A random variable X is said to be half-normal distribution with location parameter zero and unknown scale parameter θ if its p.d.f is:

$$f(x;\theta) = \sqrt{\frac{2}{\pi}}\frac{1}{\theta}exp\{-(\frac{x^2}{2\theta^2})\}, \quad \theta > 0, 0 < x < \infty \qquad 2.1$$

Let $x_{1,}x_{2,} ..., x_n$ be a random sample taken from HND with unknown parameter θ and its likelihood function is:

$$L(\theta\ \mathbf{x}) = \left(\sqrt{\frac{2}{\pi}}\right)^n \frac{1}{\theta^n} exp\left\{-\left(\frac{\sum_{i=1}^n x^2}{2\theta^2}\right)\right\} \qquad 2.2$$

2.1. Posterior Distribution using Informative Priors

The posterior distribution using informative priors, i.e. squared root inverted gamma prior, inverted chi-square prior and inverse Raleigh prior, are presented in the following sections.

2.1.1. Posterior Distribution Using Squared Root Inverted Gamma Prior

The squared root inverted gamma (SRIG) with hyperparameters 'a' and 'b' is defined as:

$$p(\theta) = \frac{2b^a}{\Gamma(a)}\theta^{-(2a+1)}exp\left\{-\left(\frac{b}{2\theta^2}\right)\right\}, \quad a, b, \theta > 0 \qquad 2.3$$

Using equations (2.2) and (2.3), the posterior distribution of the parameter θ given data $\mathbf{x}$ is:

$p(\theta|\mathbf{x}) \propto p(\theta)L(\theta, x)$

$$p(\theta|\mathbf{x}) \propto \frac{2\left(b+\frac{\sum x^2}{2}\right)^{\frac{n}{2}}}{\Gamma\left(a+\frac{n}{2}\right)}\theta^{-\left[2\left(a+\frac{n}{2}\right)+1\right]}exp\{-[\tfrac{1}{\theta^2}(b+\tfrac{\sum x^2}{2})]\}, \quad 0 < \theta < \infty \qquad 2.4$$

which is the density kernel of (SRIG) distribution, so the posterior distribution of $\theta|\mathbf{x}$ is

$$SRIG(\alpha,\beta)\ where\ \alpha = a + \frac{n}{2} \text{ and } \beta = b + \frac{\sum x^2}{2}.$$

2.1.2. Posterior Distribution using Inverted Chi-square Prior

The inverted chi-square (IC) with hyperparameter 'ν' and is defined as:

$$p(\theta) = \frac{\left(\frac{1}{2}\right)^{\frac{\nu}{2}}}{\Gamma\left(\frac{\nu}{2}\right)}\theta^{-\left(\frac{\nu}{2}+1\right)}exp\{-(\tfrac{1}{2\theta^2})\}, \quad \nu, \theta > 0 \qquad 2.5$$

Using equations (2.2) and (2.5), the posterior distribution of the parameter $\theta|\mathbf{x}$ is:

$$p(\theta|\mathbf{x}) \propto \frac{2\left(\frac{\sum x^2+1}{2}\right)^{\frac{\nu}{4}+\frac{n}{2}}}{\Gamma\left(\frac{\nu}{4}+\frac{n}{2}\right)}\theta^{-\left[2\left(\frac{\nu}{4}+\frac{n}{2}\right)+1\right]}exp - \left[\frac{1}{\theta^2}\left(\frac{\sum x^2+1}{2}\right)\right], \quad 0 < \theta < \infty \qquad 2.6$$

which is the density kernel of (SRIG) distribution, so the posterior distribution of $\theta|\mathbf{x}$ is

$$SRIG(\alpha,\beta)\ where\ \alpha = \frac{\nu}{4} + \frac{n}{2} \text{ and } \beta = \frac{\sum x^2+1}{2}.$$

2.1.2. Posterior Distribution using Inverse Rayleigh Prior

The inverse Rayleigh (IR) with Hyperparameter 'c' is defined as:

$$p(\theta) = \frac{2c}{\theta^3} exp - \left(\frac{c}{\theta^2}\right), \quad c, \theta > 0 \qquad 2.7$$

Using equations (2.2) and (2.7), the posterior distribution of the parameter $\theta|\mathbf{x}$ is:

$$p(\theta|\mathbf{x}) \propto \frac{2\left(\frac{\sum x^2}{2}+c\right)^{\frac{n}{2}+1}}{\Gamma\left(\frac{n}{2}+1\right)} \theta^{-\left[2\left(\frac{n}{2}+1\right)+1\right]} exp - \left[\frac{1}{\theta^2}\left(\frac{\sum x^2}{2}+c\right)\right], \quad 0 < \theta < \infty \qquad 2.8$$

which is the density kernel of (SRIG) distribution, so the posterior distribution of $\theta|\mathbf{x}$ is

$$SRIG(\alpha, \beta) \text{ where } \alpha = \frac{n}{2} + 1 \text{ and } \beta = \frac{\sum x^2}{2} + c.$$

3. Prior Predictive Distribution Using Informative Priors

The prior predictive distribution is the model predicts over the observed variables before any of data are considered. The prior predictive distribution is also known as marginal distribution of an unobserved value which is the prior distribution of θ and single variable p.d.f integrating out this parameter. The derivations of the prior predictive distribution using informative priors are given below. Let Y be the random variable having the HND with unknown parameter θ.

The prior predictive distribution can be obtained by the following equation

$$p(y) = \int_0^\infty p(\theta) f(y, \theta)\, d\theta \qquad 3.1$$

where y represents future random variable.

3.1. Prior Predictive Distribution using Squared root Inverted Gamma Prior

The prior predictive distribution using equation (2.3) and (3.1) is:

$$p(y) = \sqrt{\frac{2}{\pi}} \frac{b^a \Gamma\left(a+\frac{1}{2}\right)}{\Gamma(a)\left(b+\frac{y^2}{2}\right)^{a+\frac{1}{2}}}, y > 0 \qquad 3.2$$

The above equation is used for the elicitation of hyperparameters 'a' and 'b'.

3.2. Prior Predictive Distribution using Inverted Chi-Square Prior

The prior predictive distribution using equation (2.5) and (3.1) is:

$$p(y) = \sqrt{\frac{2}{\pi}} \left(\frac{1}{2}\right)^{\frac{\nu}{2}} \frac{b^a \Gamma\left(\frac{\nu}{4}+\frac{1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\left(\frac{y^2+1}{2}\right)^{\frac{\nu}{4}+\frac{1}{2}}}, y > 0 \qquad 3.3$$

The above equation is used for the elicitation of Hyperparameter 'ν'.

3.3. Prior Predictive Distribution using Inverse Rayleigh Prior

The prior predictive distribution using equation (2.7) and (3.1) is:

$$p(y) = \frac{3c}{2\sqrt{2}\left(c^2+\frac{y^2}{2}\right)^{\frac{3}{2}}}, y > 0 \qquad 3.4$$

The above equation is used for the elicitation of Hyperparameter 'c'.

4. Elicitation of Hyperparameters

The methods of elicitation through prior predictive distribution are defined by Aslam (2003). For the elicitation of the hyperparameters of the informative priors, we use prior predictive distributions given in Ssection 3 and consider the intervals that are used in the elicitation.

4.1. Elicitation of Hyperparameters of Squared root inverted Gamma Prior

Using the prior predictive distribution given in equation (3.2), expert's probabilities are to be 0.15 and 0.10, which are associated with the intervals $0.01 \le y \le 0.5$ and $3 \le y \le 5$ respectively.

$$\int_{0.01}^{0.05} \sqrt{\frac{2}{\pi}} \frac{b^a \Gamma\left(a+\frac{1}{2}\right)}{\Gamma(a)\left(b+\frac{y^2}{2}\right)^{a+\frac{1}{2}}} dy = 0.15$$

$$\int_{3}^{5} \sqrt{\frac{2}{\pi}} \frac{b^a \Gamma\left(a+\frac{1}{2}\right)}{\Gamma(a)\left(b+\frac{y^2}{2}\right)^{a+\frac{1}{2}}} dy = 0.10$$

To elicit the hyperparameters 'a' and 'b', the above equations are simultaneously solved through the program developed in SAS package using 'PROC SYSNLIN' commands and the values of the hyperparameters 'a' and 'b' are found to be 0.7136 and 0.1330 respectively .

4.2. Elicitation of Hyperparameter of Inverted chi-square prior

Using the prior predictive distribution given in equation (3.3).The expert's probability for the interval (0, 0.5) is to be 0.5.

$$\int_{0}^{0.5} \sqrt{\frac{2}{\pi}} \left(\frac{1}{2}\right)^{\frac{\nu}{2}} \frac{b^a \Gamma\left(\frac{\nu}{4}+\frac{1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\left(\frac{y^2+1}{2}\right)^{\frac{\nu}{4}+\frac{1}{2}}} = 0.18$$

The above equation is used to elicit the hyperparameter of inverted chi-square 'ν' by applying 'PROC SYSNLIN' and the value of the hyperparameter 'ν' is found to be 0.8963.

4.3. Elicitation of Hyperparameter of Inverse Rayleigh Prior

Using the prior predictive distribution given in equation (3.4), expert's probability is to be 0.08, which is associated with the interval $4 \le y \le 6$.

$$\int_{4}^{6} \frac{3c}{2\sqrt{2}\left(c^2+\frac{y^2}{2}\right)^{\frac{3}{2}}} = 0.08$$

The above equation is used to elicit the hyperparameter of inverse Rayleigh 'c' by applying 'PROC SYSNLIN' and the value of the hyperparameter 'c' is found to be 0.8531.

5. Graphs of Posterior Distribution Using Real Data Set

This section represents the graphs of the posterior distribution using informative priors. We draw graphs in SAS package.

5.1. Real Data Set

The real data set is used for analysis. From Serge et al. (2010), the data set of maximum flood levels (in millions cubic feet per second) for the Susquehanna River at Harrisburg, Pennsylvania over four-year periods. We have the following 20 observations:

0.654, 0.613, 0.402, 0.379, 0.269, 0.740, 0.416, 0.338, 0.315, 0.449, 0.297, 0.423, 0.379, 0.3235, 0.418, 0.412, 0.494, 0.392, 0.484, 0.268.

The mean, variance and CV of the above data are as follows.

$$\bar{X} = 0.423 \quad \sigma^2 = 0.016 \text{ CV} = 0.295$$

5.1.1. Graphs of Posterior Distributions

The graphs of posterior distribution using SRIG prior with parameters $\alpha_{SRIG} = 10.7136$, $\beta_{SRIG} = 2.07245$, IC prior with parameters $\alpha_{IC} = 10.224075$, $\beta_{IC} = 2.3945$, and IR prior with parameters $\alpha_{IR} = 11, \beta_{IR} = 2.79255$ are presented below in Figures 5.1, 5.2 and 5.3.

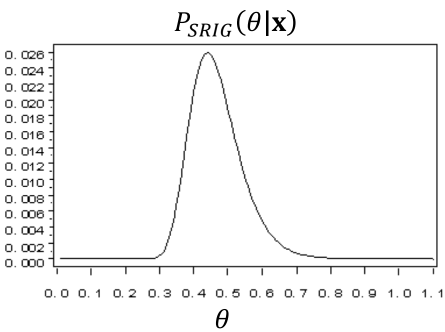

Figure 5.1: Graph using SRIG prior

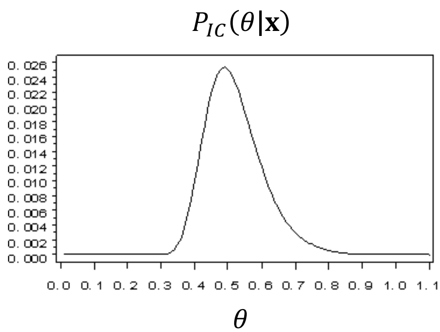

Figure 5.2: Graph using IC prior

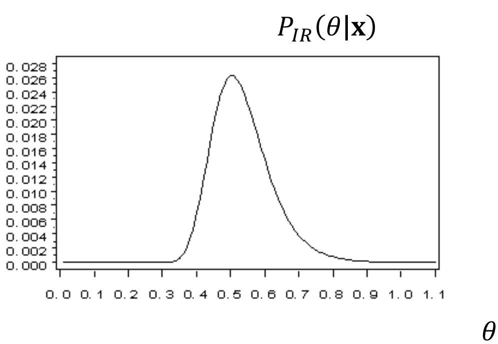

Figure 5.3: Graph using IR prior

The graphs of posterior distributions using informative priors in Figures 5.1, 5.2 and 5.3 are similar and positively skewed.

5.2 Properties of Posterior Distribution Using Real Data Set

The properties of posterior distribution using a real data set mentioned in 5.1 are determined and given below.

Table 5.1: Properties of Posterior Distribution

n=20	Mean	Variance	Mode	C.V
SRIG Prior	1.3724	0.0054	0.4299	5.3638%
IC Prior	1.4854	0.2342	0.4769	32.5835%
IR Prior	1.5950	0.1414	0.4927	23.5793%

From the above Table 5.1, if we compare informative priors, squared root inverted Gamma prior is more efficient than other priors, as variance is minimum using Squared root inverted Gamma prior.

6. Bayes Estimates Under Different Loss Functions

In statistics, typically a loss function is used for parameter estimation, and the event in question is a function of the difference between estimated and true values for an instance of data. In this section, we have used four different loss functions. The details are given below.

6.1. Squared Error Loss Function

The loss function: $L(\theta,\hat{\theta}) = (\theta - \hat{\theta})^2$ is called squared error loss function (SELF), where θ is the parameter and $\hat{\theta}$ is an estimator.

By minimizing the risk function $\rho(\hat{\theta}) = EL(\theta,\hat{\theta})$ with respect to θ, we have the Bayes estimator

$$\hat{\theta} = E(\theta) \qquad 6.1$$

which is the posterior mean under SELF.

The Bayes posterior risk is

$$\rho(\hat{\theta}) = E(\theta^2) - \{E(\theta)\}^2 \qquad 6.2$$

which is the posterior variance, and it is the Bayes posterior risk under SELF.

6.2. Quadratic Loss Function

The loss function: $L(\theta,\hat{\theta}) = \left(1 - \frac{\hat{\theta}}{\theta}\right)^2$ is called quadratic loss function (QLF).

By minimizing the risk function, we have $\hat{\theta} = \frac{E(\theta^{-1})}{E(\theta^{-2})}$, which is the Bayes estimator under QLF.

The Bayes posterior risk is

$\rho(\hat{\theta}) = 1 - \frac{\{E(\theta^{-1})\}^2}{E(\theta^{-2})}$. This is the Bayes posterior risk under quadratic loss function.

6.3. Modified Loss Function

The loss function $L(\theta,\hat{\theta}) = \frac{(\theta-\hat{\theta})^2}{\theta}$ is called modified loss function (MLF).

By minimizing the risk function, we have $\hat{\theta} = \frac{1}{E(\theta^{-1})}$, which is the Bayes estimator under MLF.

The Bayes posterior risk is $\rho(\hat{\theta}) = E(\theta) - \frac{1}{E(\theta^{-1})}$.This is the Bayes posterior risk under modified loss function.

6.4. Degroot Loss Function

The loss function $L(\theta,\hat{\theta}) = \left(\frac{\theta-\hat{\theta}}{\hat{\theta}}\right)^2$ is called Degroot loss function (DLF).

By minimizing the risk function, we have $\hat{\theta} = \frac{E(\theta^2)}{E(\theta)}$, which is the Bayes estimator under DLF.

The Bayes posterior risk is

$$\rho(\hat{\theta}) = \frac{Var(\theta)}{E(\theta)} \qquad 6.3$$

This is the Bayes posterior risk under Degroot loss function. The expressions of Bayes estimators and posterior risks using SRIG, IC and IR priors are given in Tables 6.1, 6.2 and 6.3 respectively.

Table 6.1: Bayes Estimators and Posterior Risks Assuming SRIG Prior

Loss Functions	Bayes Estimators	Posterior Risks
SELF	$\hat{\theta} = \sqrt{b+\frac{\sum x^2}{2}}\frac{\Gamma\left(a+\frac{n-1}{2}\right)}{\Gamma\left(a+\frac{n}{2}\right)}$	$\rho(\hat{\theta}) = \frac{2b+\sum x^2}{2a+n-2} - \left(\sqrt{b+\frac{\sum x^2}{2}}\frac{\Gamma\left(a+\frac{n-1}{2}\right)}{\Gamma\left(a+\frac{n}{2}\right)}\right)^2$
QLF	$\hat{\theta} = \sqrt{b+\frac{\sum x^2}{2}}\frac{\Gamma\left(a+\frac{n+1}{2}\right)}{\Gamma\left(a+\frac{n+2}{2}\right)}$	$\rho(\hat{\theta}) = 1 - \left[\left(\frac{1}{a+\frac{n}{2}}\right)\left(\frac{\Gamma\left(a+\frac{n+1}{2}\right)}{\Gamma\left(a+\frac{n+2}{2}\right)}\right)^2\right]$
MLF	$\hat{\theta} = \sqrt{b+\frac{\sum x^2}{2}}\frac{\Gamma\left(a+\frac{n}{2}\right)}{\Gamma\left(a+\frac{n+1}{2}\right)}$	$\rho(\hat{\theta}) = \sqrt{b+\frac{\sum x^2}{2}}\left[\left(\frac{\Gamma\left(a+\frac{n-1}{2}\right)}{\Gamma\left(a+\frac{n}{2}\right)} - \frac{\Gamma\left(a+\frac{n}{2}\right)}{\Gamma\left(a+\frac{n+1}{2}\right)}\right)\right]$
DLF	$\hat{\theta} = \sqrt{b+\frac{\sum x^2}{2}}\frac{\Gamma\left(a+\frac{n-2}{2}\right)}{\Gamma\left(a+\frac{n-1}{2}\right)}$	$\rho(\hat{\theta}) = \frac{\frac{2b+\sum x^2}{2a+n-2} - \left[\sqrt{b+\frac{\sum x^2}{2}}\frac{\Gamma\left(a+\frac{n-1}{2}\right)}{\Gamma\left(a+\frac{n}{2}\right)}\right]^2}{\sqrt{b+\frac{\sum x^2}{2}}\frac{\Gamma\left(a+\frac{n-2}{2}\right)}{\Gamma\left(a+\frac{n}{2}\right)}}$

Table 6.2: Bayes Estimators and Posterior Risks Assuming IC Prior

Loss Functions	Bayes Estimators	Posterior Risks
SELF	$\hat{\theta}=\sqrt{\frac{\sum x^2+1}{2}}\frac{\Gamma\left(\frac{v}{4}+\frac{n-1}{2}\right)}{\Gamma\left(\frac{v}{4}+\frac{n}{2}\right)}$	$\rho(\hat{\theta})=\frac{\sum x^2+1}{v+2n-2}-\left(\sqrt{\frac{\sum x^2+1}{2}}\frac{\Gamma\left(\frac{v}{4}+\frac{n-1}{2}\right)}{\Gamma\left(\frac{v}{4}+\frac{n}{2}\right)}\right)^2$
QLF	$\hat{\theta}=\sqrt{\frac{\sum x^2+1}{2}}\frac{\Gamma\left(\frac{v}{4}+\frac{n+1}{2}\right)}{\Gamma\left(\frac{v}{4}+\frac{n+2}{2}\right)}$	$\rho(\hat{\theta})=1-\left[\left(\frac{1}{\frac{v}{4}+\frac{n}{2}}\right)\left(\frac{\Gamma\left(\frac{v}{4}+\frac{n+1}{2}\right)}{\Gamma\left(\frac{v}{4}+\frac{n}{2}\right)}\right)^2\right]$
MLF	$\hat{\theta}=\sqrt{\frac{\sum x^2+1}{2}}\frac{\Gamma\left(\frac{v}{4}+\frac{n}{2}\right)}{\Gamma\left(\frac{v}{4}+\frac{n+1}{2}\right)}$	$\rho(\hat{\theta})=\sqrt{\frac{\sum x^2+1}{2}}\left[\left(\frac{\Gamma\left(\frac{v}{4}+\frac{n-1}{2}\right)}{\Gamma\left(\frac{v}{4}+\frac{n}{2}\right)}-\frac{\Gamma\left(\frac{v}{4}+\frac{n}{2}\right)}{\Gamma\left(\frac{v}{4}+\frac{n+1}{2}\right)}\right)\right]$
DLF	$\hat{\theta}=\sqrt{\frac{\sum x^2+1}{2}}\frac{\Gamma\left(\frac{v}{4}+\frac{n-2}{2}\right)}{\Gamma\left(\frac{v}{4}+\frac{n-1}{2}\right)}$	$\rho(\hat{\theta})=\frac{\frac{\sum x^2+1}{v+2n-4}-\left[\sqrt{\frac{\sum x^2+1}{2}}\frac{\Gamma\left(\frac{v}{4}+\frac{n-1}{2}\right)}{\Gamma\left(\frac{v}{4}+\frac{n}{2}\right)}\right]^2}{\sqrt{\frac{\sum x^2+1}{2}}\frac{\Gamma\left(\frac{v}{4}+\frac{n-2}{2}\right)}{\Gamma\left(\frac{v}{4}+\frac{n}{2}\right)}}$

Table 6.3: Bayes Estimators and Posterior Risks Assuming IR Prior

Loss Functions	Bayes Estimators	Posterior Risks
SELF	$\hat{\theta}=\sqrt{\frac{\sum x^2+c}{2}}\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)}$	$\rho(\hat{\theta})=\frac{\sum x^2+2c}{n}-\left(\sqrt{\frac{\sum x^2+c}{2}}\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)}\right)^2$
QLF	$\hat{\theta}=\sqrt{\frac{\sum x^2}{2}+c}\frac{\Gamma\left(\frac{n+3}{2}\right)}{\Gamma\left(\frac{n}{2}+2\right)}$	$\rho(\hat{\theta})=1-\left[\left(\frac{1}{\frac{n}{2}+1}\right)\left(\frac{\Gamma\left(\frac{n+3}{2}\right)}{\Gamma\left(\frac{n}{2}+2\right)}\right)^2\right]$
MLF	$\hat{\theta}=\sqrt{\frac{\sum x^2}{2}+c}\frac{\Gamma\left(\frac{n}{2}+1\right)}{\Gamma\left(\frac{n+3}{2}\right)}$	$\rho(\hat{\theta})=\sqrt{\frac{\sum x^2}{2}+c}\left[\left(\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)}-\frac{\Gamma\left(\frac{n}{2}+1\right)}{\Gamma\left(\frac{n+3}{2}\right)}\right)\right]$
DLF	$\hat{\theta}=\sqrt{\frac{\sum x^2}{2}+c}\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)}$	$\rho(\hat{\theta})=\frac{\frac{\sum x^2+2c}{n}-\left[\left(\sqrt{\frac{\sum x^2}{2}+c}\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)}\right)^2\right]}{\sqrt{\frac{\sum x^2}{2}+c}\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)}}$

We have simulated the values of the Bayes estimators and posterior risks given in Appendix under different loss functions. If we compare the results of the Bayes estimates and posterior risks, we can see the simulated values are closer to the true parameter as we increase our sample sizes under different loss functions. Bayes estimates and posterior risks have minimum values for SRIG prior, which shows SRIG prior has more efficient results than other priors. While comparing the loss functions, the SELF has more precise results than all other loss functions. We can conclude that among all prior distributions SRIG has better results.

7. Bayes Estimation and Posterior Risks Using Real Data Set

By using the above loss functions, the Bayes estimates and posterior risks of the parameter through informative priors, i.e. SRIG, IC and IR priors, are as follows, with posterior risks in parentheses.

Table 7.1: Bayes Estimates and Posterior Risk under Different loss Functions

n=20	Prior Distributions		
Loss Functions	SRIG	IC	IR
SELF	1.3724(0.0067)	1.4854(0.0070)	1.5951(0.0068)
QLF	1.3781(0.0231)	1.4922(0.0242)	1.7470(0.0225)
MLF	1.5067(0.0110)	1.6382(0.0128)	1.7470(0.0122)
DLF	1.3691(0.0028)	1.4815(0.0053)	1.5915(0.0041)

If we compare informative priors, we observe that posterior risk using Squared root inverted gamma prior is less than other priors hence SRIG prior gives more efficient results. We observe that MLF performance in terms of posterior risk is better than other loss functions.

8. Bayesian Point and Interval Estimates Using Real Data Sets

In this section, we obtained Bayesian point and interval estimates. The Bayesian analog of a classical confidence is called a credible set. For details about credible sets, see Saleem and Aslam (2009), Lynn et al. (2003) and Saleem and Raza (2011), among others. The Bayesian credible intervals are obtained by using the posterior distribution of the respective parameter of interest.

8.1. Credible Intervals

A credible interval or Bayesian confidence interval is an interval in which domain of a posterior probability distribution is used for interval estimation. Credible intervals are not unique on a posterior distribution.

The credible intervals are constructed as:

$$1-\alpha = p\left\{\chi^2{}_{(1-\frac{\alpha}{2}, 2p)} < \frac{2(\beta)}{A} < \chi^2{}_{(\frac{\alpha}{2}, 2p)}\right\}$$

We have,

$$\left[C_L^{(\theta)}, C_U^{(\theta)}\right] = \left[\sqrt{\frac{2(\beta)}{\chi^2_{\left(1-\frac{\alpha}{2},2(\alpha)\right)}}}, \sqrt{\frac{2(\beta)}{\chi^2_{\left(\frac{\alpha}{2},2(\alpha)\right)}}}.\right] \quad 8.1$$

Thus $\left(C_L^{(\theta)} < \theta < C_U^{(\theta)}\right)$ is the $(1-\alpha)$ 100% credible interval where ‘α’ and ‘β’ are the respective parameters of posterior distribution.

The Credible intervals for real data set by using equation (8.1) are given in Table 8.1.

Table 8.1: Credible Intervals using Informative Priors

Prior Distributions	90% Credible Interval	95% Credible Interval	99% Credible Interval
SRIG	(0.3452,0.4788)	(0.3358,0.4960)	(0.3185,0.5326)
IC	(0.3349,0.5054)	(0.3532,0.5238)	(0.3349,0.5630)
IR	(0.3724,0.5155)	(0.3622,0.5339)	(0.3437,0.5731)

In comparison, we can observe that 90% credible intervals are narrower than 99% and 95%.When we compare informative priors’ credible intervals under squared root inverted gamma prior are shorter than all other priors.

8.2 Bayesian Hypothesis Testing

Hypothesis testing has been subject to polemic since its early formulation by the Neyman and Pearson in the 1930s. It is more difficult to carry out a point null hypothesis test in a Bayesian analysis. Bayesian model comparison is a method of selection based on the Bayes factors. Bayes Factor is ratio of probabilities for null and alternative hypotheses.

Jeffreys (1961) gives the following typology for comparing H_a *vs* H_b where H_a is used for null hypothesis and H_b is used for alternative hypothesis. (i) $B > 1$ H_a is supported, (ii) $10^{-\frac{1}{2}} \le B \le 1$ Minimal evidence against H_a (iii) $10^{-1} \le B \le 10^{-\frac{1}{2}}$ Substantial evidence against H_a.

(iv) $10^{-2} \le B \le 10^{-1}$ Strong evidence against H_a (v) $B < 10^{-2}$ Decisive evidence againstH_a.

Table 8.2: Hypothesis testing using Real Data Set

H_a *vs* H_b	Using SRIG Prior	Using IC Prior	Using IR Prior
	B.F	B.F	B.F
$H_a: \theta \le 0.34$ $H_b: \theta > 0.34$	0.0268	0.0031	0.0009
$H_a: \theta \le 0.43$ $H_b: \theta > 0.43$	0.6696	0.2049	0.1280
$H_a: \theta \le 0.55$ $H_b: \theta > 0.55$	8.5737	2.7522	2.1075
$H_a: \theta \le 0.68$ $H_b: \theta > 0.68$	115.713	26.6151	21.592

The above Table 8.2 shows:

- While considering the hypothesis

$$H_a: \theta \leq 0.34 \text{ Versus } H_b: \theta > 0.34$$

Bayes factor using squared root inverted gamma priors lies between $10^{-2} \leq B \leq 10^{-\frac{1}{2}}$. So we conclude that there is substantial evidence against the posterior distribution under H_a, and $B \leq 10^{-2}$ so we conclude decisive evidence against the posterior distribution under H_a.

- While considering the hypothesis

$$H_a: \theta \leq 0.43 \text{ Versus } H_b: \theta > 0.43$$

As $10^{-\frac{1}{2}} \leq B \leq 1$ we have minimal evidence against H_a for all priors.

- While considering the hypothesis

$$H_a: \theta \leq 0.55 \text{ Versus } H_b: \theta > 0.55$$

As $B > 1$, so we strongly supported H_a using all informative priors.

- While considering the hypothesis

$$H_a: \theta \leq 0.68 \text{ Versus } H_b: \theta > 0.68$$

As $B > 1$, so we strongly supported H_a using all informative priors.

9. Properties of Posterior Distribution using Simulation Study

Simulation is the process of imitating a real phenomenon with a set of mathematical formulas. Here, we discuss some properties of posterior distribution through a simulation study of parameter θ. We have done all simulations in Mathematica package.

Table 9.1: Properties of Posterior Distribution under SRIG Prior

n	$\theta = 2$			$\theta = 4$		
	Mean	Variance	Mode	Mean	Variance	Mode
50	1.9907	1.2314	1.9980	3.9425	1.2268	3.9792
100	1.9964	1.2088	1.9996	3.9945	1.2096	3.9987
500	2.0050	1.1984	2.0048	4.0510	1.1914	4.0070
1000	2.0007	1.1871	2.0006	4.0014	1.1804	4.0003

Table 9.2: Properties of Posterior Distribution under IC Prior

n	$\theta = 2$			$\theta = 4$		
	Mean	Variance	Mode	Mean	Variance	Mode
50	1.9867	1.2173	1.9825	3.9839	1.2040	3.9971
100	1.9969	1.2054	1.9963	3.9914	1.2039	3.9997
500	2.0968	1.1978	2.0963	4.0775	1.1907	4.0553
1000	2.0012	1.1882	2.0010	4.0003	1.1847	4.0072

Table 9.3: Properties of Posterior Distribution under IR Prior

n	$\theta = 2$			$\theta = 4$		
	Mean	Variance	Mode	Mean	Variance	Mode
50	1.9867	1.2173	1.9825	3.9839	1.2040	3.9971
100	1.9969	1.2054	1.9963	3.9914	1.2039	3.9997
500	2.0968	1.1978	2.0963	4.0775	1.1907	4.0553
1000	2.0012	1.1882	2.0010	4.0003	1.1847	4.0072

From the Tables 9.1, 9.2 and 9.3, it is observed that as we increase our sample sizes, our simulated values through mean tend to true values of parameter. Similarly, mode is closely to the true parameter as we increase sample sizes. Squared root inverted gamma prior is more precise than all other priors in the case of comparing informative priors. We have also simulated values of variances, which can show as we increase the sample sizes it becomes less.

10. Concluding Remarks

We have presented the Bayesian analysis of half-normal model using informative (squared root inverted gamma, inverted chi-square and inverse Rayleigh) priors. Initially, we derive posterior distributions using informative priors. The SAS package is used to draw graphs of posterior distributions. The properties of posterior distribution (mean, median, mode, variance and coefficient of variation) are discussed through simulation as well as real data set. The credible intervals for 90%, 95%, and 99% using informative priors are constructed and the Bayes factors of different hypothesis are computed. By the comparison of results, with increasing the sample size the Bayes estimates converge to the parametric values and their risks tend to be smaller. As under informative priors the Bayes risks for the estimates under SRIG are smaller than the Bayes risks assuming IC and IR priors, thus SRIG is more suitable prior. If we compare the Bayes risk under different loss functions, namely SELF, QLF, MLF and DLF, then the MLF is a better loss function for estimating the parameter θ.

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Appendix

Bayes Estimates and posterior risks under Different Loss Functions

Table 1: Bayes Estimates of Informative Priors using SELF

Priors	SRIG Prior		IC Prior		IR Prior	
N	$\theta = 2$	$\theta = 4$	$\theta = 2$	$\theta = 4$	$\theta = 2$	$\theta = 4$
50	2.0083	3.9706	2.0506	3.9758	2.0167	3.9662
100	2.0044	3.9901	2.0224	3.9964	2.0426	3.9912
500	1.9923	4.0637	1.9849	4.0396	1.9429	4.0826

Table 2: Bayes Estimates of Informative Priors using QLF

Priors	SRIG Prior		IC Prior		IR Prior	
n	$\theta = 2$	$\theta = 4$	$\theta = 2$	$\theta = 4$	$\theta = 2$	$\theta = 4$
50	1.9698	3.9856	2.0720	3.9937	1.9897	3.9927
100	1.9904	3.9942	2.0892	4.0463	1.9983	3.9987
500	2.0097	4.0468	2.0048	4.0930	2.0562	4.0963

Table 3: Bayes Estimates of Informative Priors using MLF

Priors	SRIG Prior		IC Prior		IR Prior	
n	$\theta = 2$	$\theta = 4$	$\theta = 2$	$\theta = 4$	$\theta = 2$	$\theta = 4$
50	2.0387	4.0204	1.9835	3.9913	2.0214	4.0443
100	2.0932	4.0439	1.9991	3.9978	2.0610	4.0728
500	2.0083	4.0992	2.0437	4.0933	1.9886	4.0027
1000	2.0037	4.0070	2.0679	4.0013	1.9989	4.0002

Table 4: Bayes Estimates of Informative Priors using DLF

Priors	SRIG Prior		IC Prior		IR Prior	
n	$\theta = 2$	$\theta = 4$	$\theta = 2$	$\theta = 4$	$\theta = 2$	$\theta = 4$
50	1.9862	3.9999	2.0160	3.9915	1.9925	4.0202
100	1.9999	4.0537	2.0028	3.9996	1.9957	4.0210
500	2.0193	4.0860	2.0921	4.0928	2.0868	4.0114
1000	2.0641	4.0025	2.0944	4.0047	2.0029	3.9993

Table 5: Posterior Risks of Informative Priors using SELF

Priors	SRIG Prior		IC Prior		IR Prior	
n	$\theta = 2$	$\theta = 4$	$\theta = 2$	$\theta = 4$	$\theta = 2$	$\theta = 4$
50	1.2116	1.2376	1.2070	1.2112	1.2296	1.2314
100	1.2048	1.2092	1.1927	1.2042	1.2059	1.2094
500	1.1881	1.1880	1.1885	1.1905	1.1914	1.1902
1000	1.1809	1.1847	1.1864	1.1886	1.1880	1.1888

Table 6: Posterior Risks of Informative Priors using QLF

Priors	SRIG Prior		IC Prior		IR Prior	
n	$\theta = 2$	$\theta = 4$	$\theta = 2$	$\theta = 4$	$\theta = 2$	$\theta = 4$
50	1.2225	1.2427	1.2143	1.2189	1.2281	1.2412
100	1.2153	1.2338	1.2095	1.2161	1.2109	1.2218
500	1.1967	1.2113	1.1918	1.1905	1.1912	1.1912
1000	1.1871	1.1883	1.1882	1.1879	1.1871	1.1863

Table 7: Posterior Risks of Informative Priors using MLF

Priors	SRIG Prior		IC Prior		IR Prior	
n	$\theta = 2$	$\theta = 4$	$\theta = 2$	$\theta = 4$	$\theta = 2$	$\theta = 4$
50	1.2151	1.2363	1.2556	1.2690	1.2352	1.2252
100	1.2058	1.2287	1.2340	1.2566	1.2253	1.2007
500	1.1979	1.2052	1.2075	1.2116	1.1943	1.1946
1000	1.1873	1.1964	1.1991	1.1923	1.1838	1.1877

Table 8: Posterior Risks of Informative Priors using DLF

Priors	SRIG Prior		IC Prior		IR Prior	
n	$\theta = 2$	$\theta = 4$	$\theta = 2$	$\theta = 4$	$\theta = 2$	$\theta = 4$
50	1.2431	1.2492	1.2151	1.2158	1.2226	1.2456
100	1.2364	1.2146	1.2007	1.2097	1.2193	1.2134
500	1.2079	1.1983	1.1878	1.1912	1.1992	1.1886
1000	1.1875	1.1876	1.1822	1.1884	1.1954	1.1805