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Odd log-logistic generalised Lindley distribution with properties and applications

Vahid Ranjbar[1], **Abbas Eftekharian**[2], **Omid Kharazmi**[3], **Morad Alizadeh**[4]

Abstract

In this paper, a new three-parameter lifetime model, called the *odd log-logistic generalised Lindley* distribution, is introduced. Some structural properties of the new distribution including ordinary and incomplete moments, quantile and generating functions and order statistics are obtained. The new density function can be expressed as a linear mixture of exponentiated Lindley densities. Different methods are discussed to estimate the model parameters and a simulation study is carried out to show the performance of the new distribution. The importance and flexibility of the new model are also illustrated empirically by means of two real data sets. Finally, Bayesian analysis and Gibbs sampling are performed based on the two real data sets.

Key words: Lindley distribution, odd log-logistic generalised family, moments, Bayesian analysis, simulation study.

1. Introduction

Modelling and analysing real lifetime data are widely used in many applied fields such as finance, reliability, engineering, medicine. In practice, researchers dealt with different types of survival data and they proposed various lifetime models for modelling such data. The statistical analysis depends on the procedure used by the researcher and the generated family of distributions. Recently, new families of distributions have been introduced in the literature that could considerably help to analyse complex real data. However, it is necessary to find more efficient statistical models; since there are many real data sets in practice that need to be investigated with statistical models that are more flexible. Therefore, the researchers have had many attempts to extend distributions theory by adding new shape parameters to different families of distribution to introduce new families. In particular, some extended distributions demonstrate high flexibility in hazard rate function (hrf) such as increasing, decreasing and bathtub shapes even though the baseline hazard rate function may not have these shapes.

Most of the new generators of G family can be obtained using T-X class, which is proposed by Alzaatreh et al. (2013). For example, Kumaraswamy generated, odd log-logistic-G, Exponentiated-G (Exp-G), gamma generated, proportional odds and generalized

[1]Golestan University, Gorgan, Iran. E-mail: vahidranjbar@gmail.com. ORCID: https://orcid.org/0000-0003-3743-0330.
[2]University of Hormozgan, Bandar Abbas, Iran. ORCID: https://orcid.org/0000-0002-5343-8597.
[3]Vali-e-Asr University of Rafsanjan, Rafsanjan, Iran. ORCID: https://orcid.org/0000-0003-4176-9708.
[4]Persian Gulf University, Bushehr, Iran. ORCID: https://orcid.org/0000-0001-6638-2185.

beta generated. Recently, the extended exponentiated-G (EE-G) family has been defined by Alizadeh et al. (2018a).

Gleaton and Lynch (2010) showed that the extended generalized log-logistic family has appropriate performance for lifetime data. Although, there are several lifetime distributions that we can use, since the proposed family has three parameters, therefore it is better to select a lifetime distribution with only one parameter, for example, exponential or Lindley. It should be noted that hrf of the exponential is constant while the hrf of the Lindley distribution has different shapes as increasing, decreasing, unimodal and bathtub. Moreover, the Lindley distribution is a well-known distribution that is employed widely in different fields such as lifetime and reliability, medical, finance, engineering and insurance. These reasons motivate the use of this distribution for modelling real lifetime data. So, we consider the Lindley distribution as the baseline distribution in this paper.

The Lindley distribution was originally proposed by Lindley(1958) in the Bayesian statistical context. Some properties of this distribution such as moments, failure rate function, characteristic function, mean residual life function, mean deviations, Lorenz curve, stochastic ordering, entropies, asymptotic distribution of the extreme order statistics have been studied by Ghitany et al. (2008). The cdf of the Lindley distribution with scale parameter $\lambda > 0$ is

$$G(x;\lambda) = 1 - \left(1 + \frac{\lambda x}{1+\lambda}\right)\mathrm{e}^{-\lambda x},\ x > 0, \tag{1}$$

and its corresponding probability density function (pdf) is given by

$$g(x;\lambda) = \frac{\lambda^2}{1+\lambda}(1+x)\mathrm{e}^{-\lambda x}. \tag{2}$$

Many authors have published various extensions of the Lindley distribution recently. For example, a three-parameter generalization of the Lindley distribution proposed by Zakerzadeh and Dolati (2009), Nadarajah et al. (2011) defined a generalized Lindley distribution, a new generalized Lindley distribution based on the weighted mixture of two gamma distributions was studied by Abouammoh et al. (2015).

Asgharzadeh et al. (2016) and Asgharzadeh et al. (2018) introduced a weighted Lindley distribution and Weibull Lindley distribution, respectively and Alizadeh et al. (2017a), Alizadeh et al. (2017b), Alizadeh et al. (2018b) proposed several generalizations of the Lindley distribution based on the odd log-logistic model. Given the vast amount of papers published recently, we can only mention a few of the most recent contributions: Gomes-Silva et al. (2017), Afify et al. (2019) and Alizadeh et al. (2019).

The problem here is to construct a new extension of the Lindley distribution that may be useful for complex situations. The suggested distribution provides an acceptable flexibility based on the pdf and hazard rate function and it can be applied in actuarial science, finance, bioscience, telecommunications and lifetime data analysis. Alzaatreh et al. (2013) defined a generalization of odd ratio and it called as transformer (T-X) generator, where $W[G(x)] = \frac{G(x)^\alpha}{[1-G(x)]^\beta} = \frac{G(x)^\alpha}{1-\left\{1-[1-G(x)]^\beta\right\}}$ is an increasing and continuous function of $G(x)$. One can say that $W[G(x)] = \frac{G(x)^{\alpha}}{[1-G(x)]^{\beta}}$ for integer α, β is a relative odd ratio of two systems, the first system with α parallel subcomponents and the second with β series subcomponents, which are useful in reliability theory. Motivated by Alzaatreh et al. (2013), we propose a new lifetime distribution called *odd log-logistic generalized Lindley* (OLLG-L) distribution by integrating the log-logistic density function, which yields the cdf

$$F(x) = \frac{\left[1-(1+\frac{\lambda x}{1+\lambda})\mathrm{e}^{-\lambda x}\right]^{\alpha\beta}}{\left[1-(1+\frac{\lambda x}{1+\lambda})\mathrm{e}^{-\lambda x}\right]^{\alpha\beta} + \left[1-\left[1-(1+\frac{\lambda x}{1+\lambda})\mathrm{e}^{-\lambda x}\right]^{\alpha}\right]^{\beta}} \tag{3}$$

where $\alpha, \beta > 0$ are the extra shape parameters. Then, the corresponding pdf of the OLLGL distribution is given by

$$f(x) = \frac{\alpha\beta\lambda^2(1+x)\mathrm{e}^{-\lambda x}\left[1-(1+\frac{\lambda x}{1+\lambda})\mathrm{e}^{-\lambda x}\right]^{\alpha\beta-1}\left[1-\left[1-(1+\frac{\lambda x}{1+\lambda})\mathrm{e}^{-\lambda x}\right]^{\alpha}\right]^{\beta-1}}{(1+\lambda)\left\{\left[1-(1+\frac{\lambda x}{1+\lambda})\mathrm{e}^{-\lambda x}\right]^{\alpha\beta} + \left[1-\left[1-(1+\frac{\lambda x}{1+\lambda})\mathrm{e}^{-\lambda x}\right]^{\alpha}\right]^{\beta}\right\}^{2}} \tag{4}$$

A random variable X with pdf (4) is denoted by $X \sim \mathrm{OLLGL}(\alpha, \beta, \lambda)$. The OLLGL distribution is more flexible than the Lindley distribution and allows for greater flexibility of the tails.

Special cases: Let $X \sim EOLL-L(\alpha, \beta, \lambda)$.

- If $\alpha = 1$, then X reduces to the Odd Log-Logistic Lindley (OLL-L).
- If $\beta = 1$, then X reduces to the Generalized Lindley (GL).
- For $\alpha = \beta = 1$, X is ordinary Lindley.

Plots of the density function for the OLLGL distribution are shown in Figure 1 for several values of parameters. As seen from Figure 1, the density function can take various forms depending on the parameter values. Both unimodal, symmetric, skewed, and monotonically decreasing shapes appear to be possible.

The rest of the paper is organized as follows. In Section 2, main properties of the OLLGL distribution such as moments, parameters estimation and asymptotic properties are obtained. A simulation study is reported in Section 3. In Section 4, the performance and application of the OLLGL distribution are evaluated using a real data set. Bayesian inference and Gibbs sampling procedure for the considered data sets are investigated in Section 5. Finally, some conclusions are stated in Section 6.

2. Main Properties

2.1. Survival and Hazard Rate Functions

The survival function is a function that gives the probability that a patient, device, or other object of interest will survive beyond any given specified time. The survival function is also known as the survivor function or reliability function. We obtain the survival function corresponding to (3) as

$$S(x;\alpha,\beta,\lambda)=1-\frac{\left[1-(1+\frac{\lambda x}{1+\lambda})\mathrm{e}^{-\lambda x}\right]^{\alpha\beta}}{\left[1-(1+\frac{\lambda x}{1+\lambda})\mathrm{e}^{-\lambda x}\right]^{\alpha\beta}+\left[1-\left[1-(1+\frac{\lambda x}{1+\lambda})\mathrm{e}^{-\lambda x}\right]^{\alpha}\right]^{\beta}} \tag{5}$$

In reliability studies, the hazard rate function (hrf) is an important characteristic and fundamental to the design of safe systems in a wide variety of applications. The hrf of the OLLGL distribution takes the form

$$h(x;\alpha,\beta,\lambda)=$$
$$\frac{\alpha\beta\lambda^2(1+x)\mathrm{e}^{-\lambda x}\left[1-(1+\frac{\lambda x}{1+\lambda})\mathrm{e}^{-\lambda x}\right]^{\alpha\beta-1}}{(1+\lambda)\left[1-\left[1-(1+\frac{\lambda x}{1+\lambda})\mathrm{e}^{-\lambda x}\right]^{\alpha}\right]\left\{\left[1-(1+\frac{\lambda x}{1+\lambda})\mathrm{e}^{-\lambda x}\right]^{\alpha\beta}+\left[1-\left[1-(1+\frac{\lambda x}{1+\lambda})\mathrm{e}^{-\lambda x}\right]^{\alpha}\right]^{\beta}\right\}} \tag{6}$$

where $\alpha>0$, $\beta>0$ and $\lambda>0$.

Plots for the hrfs of the OLLGL distribution for several parameter values are displayed in Figure 1. As seen in Figure 1, the hrf of the OLLGL distribution has very flexible shapes such as increasing, decreasing, upside-down, bathtub. It is evident that the OLLGL distribution is more flexible than the Lindley distribution, i.e. the additional parameters $\alpha>0,\beta>0$ allow for a high degree of flexibility of the OLLGL distribution. This attractive flexibility makes the hrf of the OLLGL useful for non-monotone empirical hazard behaviour, which is more likely to be observed in real life situations.

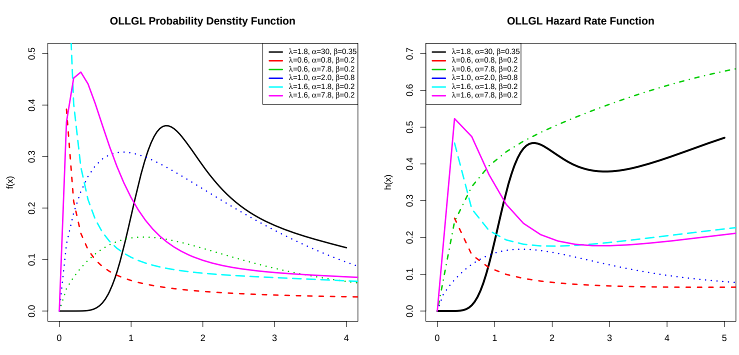

Figure 1: Plots of the density and hazard rate functions for the OLLGL distribution for some selected values.

2.2. Quantile Function

Quantile function is generally used to find representations in terms of lookup tables for key percentiles. Let X be a OLLGL distributed random variable with parameters $\alpha,\beta,\lambda,\gamma$. The quantile function, $Q(p)$, defined by $F[Q(p)] = p$ is the root of the equation as

$$p = \frac{\left[1-\left(1+\frac{\lambda}{1+\lambda}Q(p)\right)\mathrm{e}^{-\lambda Q(p)}\right]^{\alpha\beta}}{\left[1-\left(1+\frac{\lambda}{1+\lambda}Q(p)\right)\mathrm{e}^{-\lambda Q(p)}\right]^{\alpha\beta}+\left[1-\left[1-\left(1+\frac{\lambda}{1+\lambda}Q(p)\right)\mathrm{e}^{-\lambda Q(p)}\right]^{\alpha}\right]^{\beta}}. \tag{7}$$

For $\alpha = \beta$, the closed form for the quantile function can be obtained. Then, we define

$$[1+\lambda+\lambda Q(p)]\,\mathrm{e}^{-\lambda Q(p)} = -(1+\lambda)\left[1-\frac{p^{\frac{1}{\alpha\beta}}}{(p^{\frac{1}{\beta}}+(1-p)^{\frac{1}{\beta}})^{\frac{1}{\alpha}}}\right] \tag{8}$$

for $0 < p < 1$. Substituting $Z(p) = -1-\lambda-\lambda Q(p)$, one can write (8) as

$$Z(p)\,\mathrm{e}^{Z(p)} = -(1+\lambda)\mathrm{e}^{-1-\lambda}\left[1-\frac{\mathrm{p}^{\frac{1}{\alpha\beta}}}{(\mathrm{p}^{\frac{1}{\beta}}+(1-\mathrm{p})^{\frac{1}{\beta}})^{\frac{1}{\alpha}}}\right]. \tag{9}$$

Hence, the solution $Z(p)$ is given by

$$Z(p) = W_{-1}\left\{-(1+\lambda)\mathrm{e}^{-1-\lambda}\left[1-\frac{\mathrm{p}^{\frac{1}{\alpha\beta}}}{(\mathrm{p}^{\frac{1}{\beta}}+(1-\mathrm{p})^{\frac{1}{\beta}})^{\frac{1}{\alpha}}}\right]\right\}, \tag{10}$$

where $W_{-1}[.]$ is the negative branch of the Lambert function (Corless (1996)). Inserting (10), we obtain

$$Q(p) = -1-\frac{1}{\lambda}-\frac{1}{\lambda}W_{-1}\left\{-(1+\lambda)\mathrm{e}^{-1-\lambda}\left[1-\frac{\mathrm{p}^{\frac{1}{\alpha\beta}}}{(\mathrm{p}^{\frac{1}{\beta}}+(1-\mathrm{p})^{\frac{1}{\beta}})^{\frac{1}{\alpha}}}\right]\right\}. \tag{11}$$

Note that the particular case of (11) for $\alpha = \beta = \gamma = 1$ is derived by Jodr´a (2010).

Now, we propose following two different algorithms for generating random data from the OLLGL distribution for the case $\alpha = \beta$.

(a) The first algorithm is based on generating random data from the Lindley distribution mixturing the exponential and gamma distributions.

Algorithm 1 (Mixture form of the Lindley distribution)

- Generate $U_i \sim \text{Uniform}(0,1), \quad i = 1,\ldots,n$;
- Generate $V_i \sim \text{Exponential}(\lambda), \quad i = 1,\ldots,n$;
- Generate $W_i \sim \text{Gamma}(2,\lambda), \quad i = 1,\ldots,n$;
- If $\frac{U^{\frac{1}{\alpha\beta}}}{(U^{\frac{1}{\beta}}+(1-U)^{\frac{1}{\beta}})^{\frac{1}{\alpha}}} \le \frac{\lambda}{1+\lambda}$ set $X_i = V_i$, otherwise, set $X_i = W_i, \quad i = 1,\ldots,n.$

(b) The second algorithm is based on generating random data from the inverse cdf in (3) of the OLLGL distribution.

Algorithm 2 (Inverse cdf)

- Generate $U_i \sim$ Uniform(0,1), $\quad i = 1,\ldots,n$;
- Set

$$X_i = -1 - \frac{1}{\lambda} - \frac{1}{\lambda} W_{-1}\left\{-(1+\lambda)\mathrm{e}^{-1-\lambda}\left[1 - \frac{\mathrm{U_i}^{\frac{1}{\alpha\beta}}}{(\mathrm{U_i}^{\frac{1}{\beta}} + (1-\mathrm{U_i})^{\frac{1}{\beta}})^{\frac{1}{\alpha}}}\right]\right\}, \quad i = 1,\ldots,n.$$

2.3. Mixture representations for the pdf and cdf

The cdf and pdf can be written as mixture representations and such forms of cdf and pdf can be used to derive some mathematical properties, e.g. moments, moments of residual life and incomplete moments. To this purpose, first let us remind inverse of a power series using the following Remark.

Remark 1 *(Gradshteyn and Ryzhik (2007), page 17)*
Inverse of a power series $\sum_{k=0}^{\infty} b_k x^k$ is

$$\frac{1}{\sum_{k=0}^{\infty} b_k x^k} = \sum_{k=0}^{\infty} c_k x^k,$$

where $c_0 = \frac{1}{b_0}$ and for $k \geq 1$, and $c_k = -\frac{1}{b_0}\sum_{r=1}^{k} c_{k-r} b_r$.

To obtain the mixture representation of the cdf of OLLGL, note that for any $0 < u < 1$,

$$\begin{aligned} u^{\alpha\beta} &= \sum_{i=1}^{\infty}(-1)^i \binom{\alpha\beta}{i}(1-u)^i = \sum_{i=1}^{\infty}\sum_{k=0}^{i}(-1)^{i+k}\binom{\alpha\beta}{i}\binom{i}{k}u^k \\ &= \sum_{k=0}^{\infty}\sum_{i=k}^{\infty}(-1)^{i+k}\binom{\alpha\beta}{i}\binom{i}{k}u^k = \sum_{k=0}^{\infty} a_k u^k, \end{aligned}$$

where $a_k = a_k(\alpha\beta) = \sum_{i=k}^{\infty}(-1)^{i+k}\binom{\alpha\beta}{i}\binom{i}{k}$. By similar argument, we have

$$[1-(1+\frac{\lambda}{1+\lambda}x)e^{-\lambda x}]^{\alpha\beta} + \left[1-[1-(1+\frac{\lambda}{1+\lambda}x)e^{-\lambda x}]^{\alpha}\right]^{\beta} = \sum_{k=0}^{\infty} b_k[1-(1+\tfrac{\lambda}{1+\lambda}x)e^{-\lambda x}]^k,$$

where $b_k = a_k(\alpha\beta) + \sum_{j=0}^{\infty}(-1)^j \binom{\beta}{j} a_k(\alpha j)$. Now, using Remark 1, we get

$$F(x) = \frac{[1-(1+\frac{\lambda}{1+\lambda}x)e^{-\lambda x}]^{\alpha\beta}}{\sum_{k=0}^{\infty} b_k[1-(1+\frac{\lambda}{1+\lambda}x)e^{-\lambda x}]^k} = \sum_{k=0}^{\infty} c_k[1-(1+\frac{\lambda}{1+\lambda}x)e^{-\lambda x}]^{k+\alpha} = \sum_{k=0}^{\infty} c_k G(x;\lambda)^{k+\alpha\beta}, \tag{12}$$

where $c_0 = \frac{1}{b_0}$ and for $k \geq 1$,

$$c_k = \frac{-1}{b_0}\sum_{r=1}^{k} b_r c_{k-r}.$$

The equation (12) can be interpreted as a linear combination of generalized Lindley distribution. Using this equation, the mixture representation of pdf is given by

$$f(x) = \sum_{k=0}^{\infty} (k+\alpha\beta)\, c_k\, g(x;\lambda) G(x;\lambda)^{k+\alpha\beta-1}. \tag{13}$$

2.4. Moments and Moment Generating Function

Some of the most important features and characteristics of a distribution can be studied through moments (e.g., central tendency, dispersion, skewness and kurtosis). Now, we obtain ordinary moments and the moment generating function (mgf) of the OLLGL distribution. Nadarajah et al. (2011) defined the following equation for the ordinary moments as

$$A(a,b,c,\delta) = \int_0^{\infty} x^c (1+x) \left[1 - \left(1 + \frac{bx}{b+1}\right) \mathrm{e}^{-bx}\right]^{a-1} \mathrm{e}^{-\delta x} dx \tag{14}$$

which can be used to produce ordinary moments (μ_r'). Then, we have

$$A(a,b,c,\delta) = \sum_{l=0}^{\infty} \sum_{r=0}^{l} \sum_{s=0}^{r+1} \binom{a-1}{l} \binom{l}{r} \binom{r+1}{s} \frac{(-1)^l b^r \Gamma(s+c+1)}{(1+b)^l (bl+\delta)^{c+s+1}}. \tag{15}$$

From equations (12) and (13), we obtain the ordinary moments of the OLLGL distribution as

$$\mu_r' = E\left[X^r\right] = \frac{\lambda^2}{1+\lambda} \sum_{k=0}^{\infty} (k+\alpha\beta)\, c_k A(k+\alpha\beta, \lambda, r, \lambda). \tag{16}$$

We now provide a formula for the conditional moments of the OLLGL distribution.
Nadarajah et al. (2011) defined the following equation for the conditional moments

$$L(a,b,c,\delta,t) = \int_t^{\infty} x^c (1+x) \left[1 - \left(1 + \frac{bx}{b+1}\right) \mathrm{e}^{-bx}\right] \mathrm{e}^{-\delta x} dx. \tag{17}$$

Using the generalized binomial expansion, we have

$$L(a,b,c,\delta,t) = \sum_{l=0}^{\infty} \sum_{r=0}^{l} \sum_{s=0}^{r+1} \binom{a-1}{l} \binom{l}{r} \binom{r+1}{s} \frac{(-1)^l b^r \Gamma(s+c+1, (bl+\delta)t)}{(1+b)^l (bl+\delta)^{c+s+1}} \tag{18}$$

where

$$\Gamma(a,x) = \int_x^{\infty} t^{a-1} \mathrm{e}^{-t}\, dt \tag{19}$$

denotes the incomplete gamma function. From equations (13) and (18), we obtain the conditional moments of the OLLGL distribution as

$$\mu_r'(t) = E\left[X^r | X > t\right] = \frac{\lambda^2}{1+\lambda} \sum_{k=0}^{\infty} (k+\alpha\beta)\, c_k L(k+\alpha\beta, \lambda, r, \lambda, t). \tag{20}$$

The incomplete moments of the OLLGL distribution can be calculated directly from (20).

The mgf of a random variable provides the basis of an alternative route to analytical results compared with working directly with its pdf and cdf. Using (13) and (15), we obtain

$$M_X(t) = E\left[\mathrm{e}^{tX}\right] = \frac{\lambda^2}{1+\lambda}\sum_{k=0}^{\infty}(k+\alpha)\,c_k\,A(k+\alpha\beta,\lambda,0,\lambda-t).$$

Remark 2 *The central moments (μ_n) and cumulants (κ_n) of X are easily obtained from (16) as*

$$\mu_n = \sum_{k=0}^{n}(-1)^k\binom{n}{k}\mu_1'^k\,\mu_{n-k}' \qquad \text{and} \qquad \kappa_n = \mu_n' - \sum_{k=1}^{n-1}\binom{n-1}{k-1}\kappa_k\,\mu_{n-k}',$$

respectively, where $\kappa_1 = \mu_1'$. Thus, $\kappa_2 = \mu_2' - \mu_1'^2$, $\kappa_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3$, etc.

Figure 2 shows skewness and kurtosis measures of the OLLGL distribution. The skewness and kurtosis are calculated from the ordinary moments given in (16) for $\lambda = 2$. Figure 2 shows that skewness and kurtosis are very sensitive for the shape parameters and it indicates the importance of the proposed distribution.

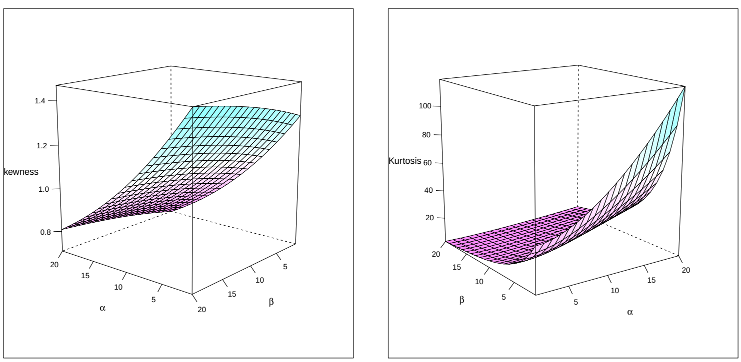

Figure 2: The skewness (left) and kurtosis (right) plots of OLLGL distribution for selected α, β for $\lambda = 2$.

Theorem 1 *If the baseline distribution $G(x)$ has a mgf, then $F(x)$ has also a mgf.*

The proof of this theorem was done by Gleaton and Lynch (2010). Since any moment of the Lindley distribution exists, all moments of the OLLGL distribution can be obtained.

2.5. Asymptotic properties

The asymptotic of cdf, pdf and hrf of the OLLGL distribution as $x \to 0$ are, respectively, given by

$$F(x) \sim (\lambda x)^{\alpha\beta} \quad as \quad x \to 0,$$
$$f(x) \sim \alpha\beta\,\lambda^{\alpha\beta}\,x^{\alpha\beta-1} \quad as \quad x \to 0,$$
$$h(x) \sim \alpha\beta\,\lambda^{\alpha\beta}\,x^{\alpha\beta-1} \quad as \quad x \to 0.$$

The asymptotic of cdf, pdf and hrf of the OLLGL distribution as $x \to \infty$ are, respectively, as follows

$$
\begin{aligned}
&1-F(x) \sim (\frac{\alpha\lambda}{1+\lambda})^{\beta}\, x^{\beta}\, \mathrm{e}^{-\lambda\beta \mathrm{x}} \quad \text{as} \quad \mathrm{x} \to \infty,\\
&f(x) \sim \beta\lambda(\frac{\alpha\lambda}{1+\lambda})^{\beta}\, x^{\beta}\, \mathrm{e}^{-\lambda\beta \mathrm{x}} \quad \text{as} \quad \mathrm{x} \to \infty,\\
&h(x) \sim \beta\lambda \quad as \quad x \to \infty.
\end{aligned}
$$

These equations show the effect of parameters on the tails of the OLLGL distribution.

2.6. Extreme Value

If $\bar{X} = (X_1 + ... + X_n)/n$ denotes the sample mean, then by the usual central limit theorem, $\sqrt{n}(\bar{X} - E(X))/\sqrt{\mathrm{Var}(X)}$ approaches the standard normal distribution as $n \to \infty$. One may be interested in the asymptotic of the extreme values $M_n = \max(X_1, ..., X_n)$ and $m_n = \min(X_1, ..., X_n)$. Let $\tau(x) = \frac{1}{\lambda}$, we obtain following equations for the cdf in (3) as

$$
\begin{aligned}
\lim_{t\to 0}\frac{F(t\,x)}{F(t)} &= \lim_{t\to 0}\frac{G(t\,x)^{\alpha}}{G(t)^{\alpha}} = \lim_{t\to 0}\frac{\left[1-\left(1+\frac{\lambda\,t\,x}{1+\lambda}\right)\mathrm{e}^{-\lambda\,t\,x}\right]^{\alpha\beta}}{\left[1-\left(1+\frac{\lambda\,t}{1+\lambda}\right)\mathrm{e}^{-\lambda\,t}\right]^{\alpha\beta}} = \lim_{t\to 0}\frac{\left[1-\mathrm{e}^{-\lambda\,t\,x}\right]^{\alpha\beta}}{\left[1-\mathrm{e}^{-\lambda\,t}\right]^{\alpha\beta}}\\
&= \lim_{t\to 0}\frac{(\lambda\,t\,x)^{\alpha\beta}}{(\lambda\,t)^{\alpha\beta}} = x^{\alpha\beta} \qquad (21)
\end{aligned}
$$

and

$$
\lim_{t\to\infty}\frac{1-F(t+x\,\tau(t))}{1-F(t)} = \lim_{t\to\infty}(\frac{1-G(t+x\,\tau(t))^{\alpha}}{1-G(t)^{\alpha}})^{\beta} = \mathrm{e}^{-\beta\,x}. \qquad (22)
$$

Thus, from Leadbetter et al. (2012), there must be norming constants $a_n > 0$, b_n, $c_n > 0$ and d_n such that

$$
Pr\left[a_n(M_n - b_n) \le x\right] \to \mathrm{e}^{-\mathrm{e}^{-\beta\,\mathrm{x}}}
$$

and

$$
Pr\left[c_n(m_n - d_n) \le x\right] \to 1 - \mathrm{e}^{-\mathrm{x}^{\alpha\beta}}
$$

as $n \to \infty$. The form of the norming constants can also be determined. For instance, using Corollary 1.6.3 in Leadbetter et al. (2012), one can see that $b_n = F^{-1}(1 - \frac{1}{n})$ and $a_n = \lambda$, where $F^{-1}(\cdot)$ denotes the inverse function of $F(\cdot)$.

2.7. Maximum likelihood estimation

We determine the maximum likelihood estimates (MLEs) of the parameters of the OLLGL distribution from complete samples. Let $x_1, \ldots, x_n$ be a random sample of size n from the OLLGL(α, β, λ) distribution. The log-likelihood function for the vector of parameters $\theta = (\alpha, \beta, \lambda)^T$ can be written as

$$\begin{aligned} l(\theta) &= n\log\left(\frac{\alpha\beta\lambda^2}{1+\lambda}\right)+\sum_{i=1}^{n}\log(1+x_i)+(\alpha\beta-1)\sum_{i=1}^{n}\log(q_i) \\ &+ (\beta-1)\sum_{i=1}^{n}\log(1-q_i^{\alpha})-2\sum_{i=1}^{n}\log\left[q_i^{\alpha\beta}+(1-q_i^{\alpha})^{\beta}\right] \end{aligned} \tag{23}$$

where $q_i = 1-(1+\frac{\lambda}{1+\lambda}x_i)\mathrm{e}^{-\lambda \mathrm{x_i}}$ is a transformed observation.

The log-likelihood can be maximized either directly by using the SAS (Procedure NLMixed) or the MaxBFGS routine in the matrix programming language Ox (Doomik (2007)) or by solving the nonlinear likelihood equations obtained by differentiating (23). The components of the score vector $U(\theta)$ are given by

$$\begin{aligned} U_\lambda(\theta) &= \frac{2n}{\lambda}-\frac{n}{1+\lambda}-\sum_{i=1}^{n}x_i+(\alpha\beta-1)\sum_{i=1}^{n}\frac{q_i^{(\lambda)}}{q_i}+\alpha(1-\beta)\sum_{i=1}^{n}\frac{q_i^{(\lambda)}q_i^{\alpha-1}}{1-q_i^{\alpha}} \\ &\quad -2\alpha\beta\sum_{i=1}^{n}q_i^{(\lambda)}\ \frac{q_i^{\alpha\beta-1}-q_i^{\alpha-1}\left[1-q_i^{\alpha}\right]^{\beta-1}}{q_i^{\alpha\beta}+(1-q_i^{\alpha})^{\beta}}, \\ U_\alpha(\theta) &= \frac{n}{\alpha}+\beta\sum_{i=1}^{n}\log(q_i)+(1-\beta)\sum_{i=1}^{n}\frac{q_i^{\alpha}\log(q_i)}{1-q_i^{\alpha}} \\ &- 2\beta\sum_{i=1}^{n}\frac{q_i^{\alpha\beta}\log(q_i)-q_i^{\alpha}\left[1-q_i^{\alpha}\right]^{\beta-1}\log(q_i)}{q_i^{\alpha\beta}+(1-q_i^{\alpha})^{\beta}} \end{aligned}$$

and

$$U_\beta(\theta) = \frac{n}{\beta}+\alpha\sum_{i=1}^{n}\log(q_i)+\sum_{i=1}^{n}\log(1-q_i^{\alpha})-2\sum_{i=1}^{n}\frac{\alpha q_i^{\alpha\beta}\log(q_i)+\left[1-q_i^{\alpha}\right]^{\beta}\log\left[1-q_i^{\alpha}\right]}{q_i^{\alpha\beta}+(1-q_i^{\alpha})^{\beta}}.$$

For interval estimation and hypothesis tests on the model parameters, the 2×2 observed information matrix $J = J(\theta)$ is required.

Under conditions that are fulfilled for parameters in the interior of the parameter space but not on the boundary, the asymptotic distribution of $\sqrt{n}(\widehat{\theta}-\theta)$ is $N_3(0, I(\theta)^{-1})$, where $I(\theta)$ is the expected information matrix. In practice, we can replace $I(\theta)$ by the observed information matrix evaluated at $\widehat{\theta}$ (say $J(\widehat{\theta})$). We can construct approximate confidence intervals and confidence regions for the individual parameters and for the hazard and survival functions based on the multivariate normal $N_3(0, J(\widehat{\theta})^{-1})$ distribution.

Further, the likelihood ratio (LR) statistic can be used for comparing this distribution with some of its special sub-models. We can compute the maximum values of the unrestricted and restricted log-likelihoods to construct the LR statistics for testing some sub-models of the OLLGL distribution. For example, the test of $H_0: \alpha = \beta = 1$ versus $H_1: H_0$ *is not true* is equivalent to comparing the OLLGL and Lindley distributions and the LR statistic reduces to

$$w = 2\{\ell(\hat{\alpha}, \hat{\beta}, \hat{\lambda}) - \ell(1, 1, \tilde{\lambda})\},$$

where $\hat{\alpha}$, $\hat{\beta}$ and $\hat{\lambda}$ are the MLEs under H and $\tilde{\lambda}$ is the estimate under H_0.

2.8. Least-Square Estimator

Let $x_{(1)}, x_{(2)}, x_{(3)}, \cdots, x_{(n)}$ denote the ordered sample of the random sample of size n from the OLLGL distribution function in (3). The least square estimators (LSEs) of the OLLGL distribution can be obtained by minimizing the following equation

$$\ell(\theta) = \sum_{i=1}^{n} \left\{ \frac{\left[1-(1+\frac{\lambda x}{1+\lambda})e^{-\lambda x_{(i)}}\right]^{\alpha\beta}}{\left[1-(1+\frac{\lambda x_{(i)}}{1+\lambda})e^{-\lambda x_{(i)}}\right]^{\alpha\beta} + \left[1-\left[1-(1+\frac{\lambda x_{(i)}}{1+\lambda})e^{-\lambda x_{(i)}}\right]^{\alpha}\right]^{\beta}} - \frac{i}{n+1} \right\}^2. \quad (24)$$

The **optim** function of R software can be used to minimize the (24). The partial derivatives of (24) with respect to α, λ and β can be obtained from authors upon request.

3. Simulation

In this section, a simulation study on the model parameters is investigated. We consider MLE and LSE methods for estimating unknown parameters of the OLLGL distribution and compare the efficiency of parameters using these methods. The simulation procedure is as follows:

1. Set the sample size n and the vector of parameters $\theta = (\lambda, \alpha, \beta)$,
2. Generate random observations from the $OLLGL(\lambda, \alpha, \beta)$ distribution with size n,
3. Estimate $\hat{\theta}$ by means of MLE and LSE methods using the generated random observations in Step 2,
4. Repeat Steps 2 and 3 for N times,
5. Compute the mean relative estimates (MREs) and mean square errors (MSEs) using $\hat{\theta}$ and θ with the following equations:

$$MRE = \sum_{j=1}^{N} \frac{\hat{\theta}_{i,j}/\theta_i}{N},$$
$$MSE = \sum_{j=1}^{N} \frac{\left(\hat{\theta}_{i,j}-\theta_i\right)^2}{N}, i = 1,2,3.$$

where $\hat{\theta}_{i,j}$ for $i = 1,2,3$ and $j = 1,...,N$, is the estimation of ith element of parameter vector in jth iteration. The simulation results are obtained with software R. The chosen parameters of the simulation study are $\theta = (\lambda = 1.2, \alpha = 2, \beta = 0.2)$, $N = 1000$ and $n = (50,55,60,...,500)$. We expect that MREs are closer to one when the MSEs are near zero. Figures 3 represents estimated MSEs and MREs from MLE and LSE methods. Based on Figures 3, the MSE of all estimates tends to zero for large n and also as expected, the values of MREs tend to one. It is clear that the estimates of parameters are asymptotically unbiased. In estimation of β and λ, the MLE method approach to nominal values of the MSEs and MREs faster than the LSE method. The LSE method exhibits better performance than the MLE method for the large sample size in estimating α. Therefore, the MLE is a more suitable method than other for estimating parameters of the OLLGL distribution for small a sample size.

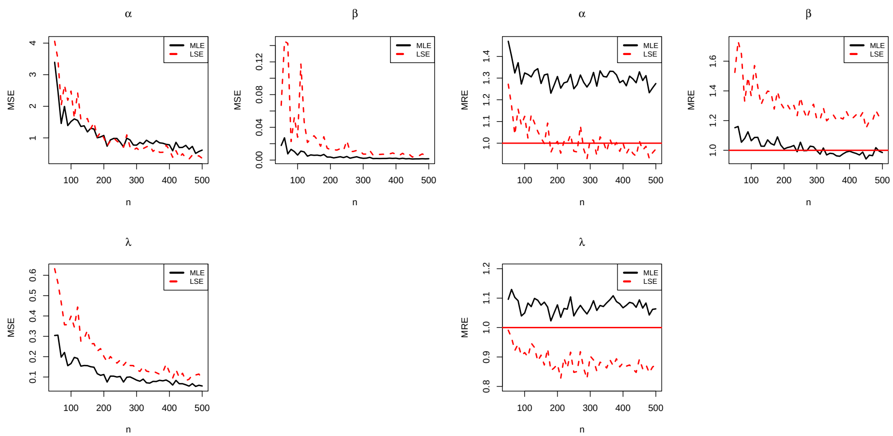

Figure 3: Estimated MREs and MSEs for the selected parameter values.

4. Applications

In this section, we illustrate the fitting performance of the OLLGL distribution using a real data set. For the purpose of comparison, we fitted the following models to show the fitting performance of the OLLGL distribution by means of real data set:

- Lindley Distribution, $L(\lambda)$.
- Power Lindley distribution, $PL(\beta,\lambda)$.
- Generalized Lindley, $GL(\alpha,\lambda)$, (Nadarajah et al. (2011)), with distribution function given by

$$F(x) = \left(1-(1+\frac{\lambda x}{1+\lambda})\mathrm{e}^{-\lambda x}\right)^{\alpha}.$$

- Beta Lindley, $BL(\alpha,\beta,\lambda)$, Merovci and Sharma (2014), with distribution function given by

$$F(x) = \int_0^{L(x,\lambda)} t^{\alpha-1}(1-t)^{\beta-1}dt.$$

- Exponentiated power Lindley distribution, Ashour and Eltehiwy (2015), $EPL(\alpha,\beta,\lambda)$, with distribution function given by

$$F(x) = \left(1-(1+\frac{\lambda x^{\beta}}{1+\lambda})\mathrm{e}^{-\lambda x^{\beta}}\right)^{\alpha}.$$

- Odd log-logistic Lindley distribution $OLL-L(\alpha,\lambda)$, (Ozel et al. (2017)), with distribution function given by

$$F(x) = \frac{L(x,\lambda)^{\alpha}}{L(x,\lambda)^{\alpha}+(1-L(x,\lambda))^{\alpha}}.$$

- Kumaraswamy Power Lindley, $KPL(\alpha,\beta,\gamma,\lambda)$ (Oluyede et al. (2016)

$$F(x) = 1-[1-PL(x,\beta,\lambda)^{\alpha}]^{\gamma}.$$

- Extended generalized Lindley, $EGL(\alpha,\gamma,\lambda)$, (Ranjbar et al. (2018)),

$$F(x)=\frac{L(x,\lambda)^{\alpha}}{L(x,\lambda)^{\alpha}+1-(1-L(x,\lambda))^{\gamma}}.$$

- New Odd-log logistic Lindley, $NOLLL(\alpha,\beta,\lambda)$, Alizadeh et al. (2018b)

$$F(x)=\frac{L(x,\lambda)^{\alpha}}{L(x,\lambda)\alpha+(1-L(x,\lambda))^{\beta}}.$$

Estimates of the parameters of OLLGL distribution, Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), Cramer Von Mises and Anderson-Darling statistics (W^* and A^*) are presented for each data set. We have also considered the Kolmogorov-Smirnov (K-S) statistic and its corresponding p-value and the minimum value of the minus log-likelihood function (-Log(L)) for the sake of comparison. Generally speaking, the smaller values of AIC, BIC, W^* and A^*, the better fit to a data set. Furthermore, the likelihood ratio (LR) tests apply for evaluating the OLLGL distribution with its sub-models. For example, the test of $H_0:\beta=1$ against $H_1:\beta\neq 1$ is equivalent to comparing the OLLGL with GL and the LR test statistic is given by

$$LR=2\left[l(\hat{\alpha},\hat{\beta},\hat{\lambda})-l(\hat{\alpha^*},1,\hat{\lambda^*})\right],$$

where $\hat{\alpha^*}$ and $\hat{\lambda^*}$ are the ML estimators under H_0 of α and λ, respectively. All the computations were carried out using the software R.

The data set is given from Murthy (2004). The ML estimates of the parameters and the goodness-of-fit test statistics for the first data set is presented in Table 3 and 4 respectively. As we can see, the smallest values of AIC, BIC, A^*, W^* and $-l$ statistics and the largest p-values belong to the OLLGL distribution. Therefore, the OLLGL distribution outperforms the other competitive considered distribution in the sense of this criteria.

Table 1: Data set.

0.032	0.035	0.104	0.169	0.196	0.260	0.326	0.445	0.449	0.496
0.543	0.544	0.577	0.648	0.666	0.742	0.757	0.808	0.857	0.858
0.882	1.138	1.163	1.256	1.283	1.484	1.897	1.944	2.201	2.365
2.531	2.994	3.118	3.424	4.097	4.100	4.744	5.346	5.479	5.716
5.825	5.847	6.084	6.127	7.241	7.560	8.901	9.000	10.482	11.133

In addition, the profile log-likelihood functions of the OLLGL distribution are plotted in Figure 4. These plots reveal that the likelihood equations of the OLLGL distribution have solutions that are maximizers.

The values of LR test statistics and their corresponding p-values are exhibited in Table 5. From Table 5, we observe that the computed p-values are too small so we reject all the null hypotheses and conclude that the OLLGL fits the data set better than the considered sub-models according to the LR criterion.

We also plotted the fitted pdfs and TTT plots of the considered models for the sake of visual comparison, in figures 5 and 6, respectively. Therefore, the OLLGL distribution can be considered as an appropriate model for fitting the data set.

5. Bayesian estimation

The Bayesian inference procedure has been taken into consideration by many statistical researchers, especially researchers in the field of survival analysis and reliability engineering. In this section, the complete sample data are analysed through a Bayesian point of view. We assume that the parameters α, β and λ of the *OLLGL* distribution have independent prior distributions as

$$\alpha \sim Gamma(a,b), \lambda \sim Gamma(e,f), \beta \sim Gamma(g,h)$$

where a, b, e, f, g and h are positive. Hence, the joint prior density function is formulated as follows:

$$\pi(\alpha,\beta,\lambda) \quad = \quad \frac{b^a f^e h^g}{\Gamma(a)\Gamma(e)\Gamma(g)} \alpha^{a-1}\beta^{h-1}\lambda^{e-1}e^{-(b\alpha+h\beta+f\lambda)}. \tag{25}$$

In the Bayesian estimation, we do not know the actual value of the parameter, which may be adversely affected by loss when we choose an estimator. This loss can be measured by a function of the parameter and corresponding estimator. For the Bayesian discussion, we consider different types of symmetric and asymmetric loss functions such as squared error loss function (*SELF*), weighted squared error loss function (*WSELF*), modified squared error loss function (*MSELF*), precautionary loss function (*PLF*) and *K*-loss function (*KLF*). These loss functions, associated Bayesian estimators and posterior risks are presented in Table 2. For more details see Calabria and Pulcini (1996). Next,

Table 2: Bayes estimator and posterior risk under different loss functions

Loss function	Bayes estimator	Posterior risk
$SELF = (\theta-d)^2$	$E(\theta\|x)$	$Var(\theta\|x)$
$WSELF = \frac{(\theta-d)^2}{\theta}$	$(E(\theta^{-1}\|x))^{-1}$	$E(\theta\|x)-(E(\theta^{-1}\|x))^{-1}$
$MSELF = \left(1-\frac{d}{\theta}\right)^2$	$\frac{E(\theta^{-1}\|x)}{E(\theta^{-2}\|x)}$	$1-\frac{E(\theta^{-1}\|x)^2}{E(\theta^{-2}\|x)}$
$PLF = \frac{(\theta-d)^2}{d}$	$\sqrt{E(\theta^2\|x)}$	$2\left(\sqrt{E(\theta^2\|x)}-E(\theta\|x)\right)$
$KLF = \left(\sqrt{\frac{d}{\theta}}-\sqrt{\frac{\theta}{d}}\right)$	$\sqrt{\frac{E(\theta\|x)}{E(\theta^{-1}\|x)}}$	$2\left(\sqrt{E(\theta\|x)E(\theta^{-1}\|x)}-1\right)$

we provide the posterior probability distributions for a complete data set. Let us define the function φ as

$$\varphi(\alpha,\beta,\lambda) = \alpha^{a-1}\beta^{h-1}\lambda^{e-1}e^{-(b\alpha+h\beta+f\lambda)},\ \alpha>0,\ \beta>0,\ \lambda>0.$$

The joint posterior distribution in terms of a given likelihood function $L(data)$ and joint prior distribution $\pi(\alpha,\beta,\lambda)$ is defined as

$$\pi^*(\alpha,\beta,\lambda|data) \propto \pi(\alpha,\beta,\lambda)L(data). \tag{26}$$

Hence, we get joint posterior density of parameters α, β and λ for complete sample data by combining the likelihood function and joint prior density (25). Therefore, the joint posterior density function is given by

$$\pi^*(\alpha,\beta,\lambda|\underline{x}) \quad = \quad K\varphi(\alpha,\beta,\lambda)L(\underline{x},\xi) \tag{27}$$

where

$$L(\underline{x};\xi)=\prod_{i=1}^{n}\frac{\alpha\beta\lambda^{2}(1+x)\mathrm{e}^{-\lambda x}\left[1-(1+\frac{\lambda x}{1+\lambda})\mathrm{e}^{-\lambda x}\right]^{\alpha\beta-1}\left[1-\left[1-(1+\frac{\lambda x}{1+\lambda})\mathrm{e}^{-\lambda x}\right]^{\alpha}\right]^{\beta-1}}{(1+\lambda)\left\{\left[1-(1+\frac{\lambda x}{1+\lambda})\mathrm{e}^{-\lambda x}\right]^{\alpha\beta}+\left[1-\left[1-(1+\frac{\lambda x}{1+\lambda})\mathrm{e}^{-\lambda x}\right]^{\alpha}\right]^{\beta}\right\}^{2}}. \tag{28}$$

and K is given as

$$K^{-1} \quad = \quad \int_0^\infty \int_0^\infty \int_0^\infty \varphi(\alpha,\beta,\lambda)L(\underline{x},\xi)d\alpha d\beta d\lambda.$$

Moreover, the marginal posterior pdf of α, γ and β, assuming that $\Theta=(\alpha,\gamma,\beta)$, can be given

$$\pi(\Theta_i|\underline{x})=\int_0^\infty \int_0^\infty \pi^*(\Theta|\underline{x})\Theta_j\Theta_k, \tag{29}$$

where $i,j,k=1,2,3, i\neq j\neq k$ and also Θ_i is ith member of a vector Θ. It is clear from the equations (27) and (29) that there are no closed-form expressions for the Bayesian estimators under the five loss functions described in Table 2. Because of intractable integrals associated with joint posterior and marginal posterior distributions, we need to use numerical software to solve integral equations numerically via MCMC method. The two most popular MCMC methods are: the Metropolis-Hastings algorithm (Metropolis et al. (1953), Hastings (1970)) and the Gibbs sampling (Geman and Geman (1984)). The Gibbs sampling is a special case of the Metropolis-Hastings algorithm which generates a Markov chain by sampling from the full set of conditional distributions. The Gibbs sampling algorithm can be described generically as follows.

Suppose that the general model $f(\underline{x}|\theta)$ is associated with parameter vector $\theta=(\theta_1,\theta_2,...,\theta_p)$ and observed data $\underline{x}$. Thus, the joint posterior distribution is $\pi(\theta_1,\theta_2,...,\theta_p|\underline{x})$. We also assume that $\theta_0=(\theta_1^{(0)},\theta_2^{(0)},...,\theta_p^{(0)})$ is the initial values vector to start the Gibbs sampler. The Gibbs sampler draws the values for each iteration in p steps by drawing a new value for each parameter from its full conditional given the most recently drawn values of all other parameters. In symbols, the steps for any iteration, say iteration k, are as follows:

- starting with an initial estimate $(\theta_1^{(0)},\theta_2^{(0)},...,\theta_p^{(0)})$
- Draw θ_1^k from $\pi\big(\theta_1|\theta_2^{k-1},\theta_3^{k-1},...,\theta_p^{k-1}\big)$
- Draw θ_2^k from $\pi\big(\theta_2|\theta_1^{k},\theta_3^{k-1},...,\theta_p^{k-1}\big)$; and so on down to
- Draw θ_p^k from $\pi\big(\theta_p|\theta_1^{k},\theta_2^{k},...,\theta_{p-1}^{k}\big)$

As mentioned above, often Bayesian inference requires computing intractable integrals to generate posterior samples. Using Gibbs sampling, one can obtain samples from the joint posterior distribution. In practice, simulations related to Gibbs sampling are conducted through a special software WinBUGS. WinBUGS software was developed in 1997 to simulate data of complex posterior distributions, where analytical or numerical integration techniques cannot be applied. Also, we can use OpenBUGS software, which is an open-source version of WinBUGS. Since there is not any prior information about hyper parameters in (25), one can implement the idea of Congdon (2001) and these parameters can be chosen as $a = b = c = d = e = f = 0.0001$. Hence, we can use the *MCMC* procedure to extract posterior samples of (27) by means of the Gibbs sampling process in OpenBUGS software.

Bayesian estimators associated with the parameters of the *OLLGL* distribution are computed based on the single chain of 10000 cycles of the Gibbs sampler with a conservative burn-in period of the first 1000 iterations. The corresponding Bayesian point and interval estimation and posterior risk are provided in Tables 6 and 7 for the data set. Table 7 provides 95% credible and *HPD* intervals for each parameter of the *OLLGL* distribution. The convergence of the Gibbs sampler process is verified through graphical inspection (Trace, Autocorrelation and Histogram plots) of the posterior sampled values. It is observed that the Gibbs samples of all the parameter estimates achieved a good stationary phase for both considered data sets. We provide the posterior summary plots in Figures 7, 8 and 9. These plots confirm that the convergence of the Gibbs sampling process is occurred.

6. Conclusion

In this paper, a new distribution which is called odd log-logistic generalized-Lindley (OLLGL) distribution was introduced. The statistical properties of the OLLGL distribution including the hazard function, quantile function, moments, incomplete moments, generating functions, mean deviations and maximum likelihood estimation for the model parameters were given. Simulation studies were conducted to examine the performance of this distribution. We also presented applications of this new distribution for a real-life data set in order to illustrate the usefulness of the distribution. Finally, the Bayesian estimation and the Gibbs sampling procedure for the considered data sets were discussed.

References

Abouammoh, A., Alshangiti, A. M., and Ragab, I., (2015). A new generalized lindley distribution. *Journal of Statistical Computation and Simulation*, 85(18), pp. 3662–3678.

Afify, A. Z., Cordeiro, G. M., Maed, M. E., Alizadeh, M., Al-Mofleh, H., and Nofal, Z. M., (2019). The generalized odd lindley-g family: properties and applications. *Anais da Academia Brasileira de Ci^encias*, 91(3).

Alizadeh, M., Afify, A. Z., Eliwa, M., and Ali, S., (2019). The odd log- logistic lindley-g family of distributions: properties, bayesian and non-bayesian estimation with applications. *Computational Statistics*, 35, pp. 281–308.

Alizadeh, M., Afshari, M., Hosseini, B., , and Ramires, T. G., (2020). Extended exp-g family of distributions: Properties and applications. *Communication in Statistics-Simulation and Computation*, 49 (7), pp. 1730–1745.

Alizadeh, M., Altun, E., Ozel, G., Afshari, M., and Eftekharian, A., (2018b). A new odd log-logistic lindley distribution with properties and applications. *Sankhya*, 81(2), pp. 323–346.

Alizadeh, M., K MirMostafaee, S., Altun, E., Ozel, G., and Khan Ah- madi, M., (2017). The odd log-logistic marshall-olkin power lindley distribution: Properties and applications. *Journal of Statistics and Management Systems*, 20(6), pp. 1065–1093.

Alizadeh, M., Ozel, G., Altun, E., Abdi, M., and Hamedani, G., (2017). The odd log-logistic marshall-olkin lindley model for lifetime data. *Journal of Statistical Theory and Applications*, 16(3), pp. 382–400.

Alzaatreh, A., Lee, C., and Famoye, F., (2013). A new method for generating families of continuous distributions. *Metron*, 71(1), pp. 63–79.

Asgharzadeh, A., Bakouch, H. S., Nadarajah, S., Sharafi, F., et al., (2016). A new weighted lindley distribution with application. *Brazilian Journal of Probability and Statistics*, 30(1), pp. 1–27.

Asgharzadeh, A., Nadarajah, S., and Sharafi, F., (2018). Weibull lindley distribution. *REVSTAT Statistical Journal*, 16(1), pp. 87–113.

Ashour, S. K. and Eltehiwy, M. A., (2015). Exponentiated power lindley distribution. *Journal of advanced research*, 6(6), pp. 895–905.

Calabria, R. and Pulcini, G., (1996). Point estimation under asymmetric loss functions for left-truncated exponential samples. *Communications in Statistics Theory and Methods*, 25(3), pp. 585–600.

Congdon, P., (2001). Bayesian statistical analysis. *Wiley, New York.*

Corless, R. M., Gonnet, G. H., Hare, D. E., Jeffrey, D. J., and Knuth, D. E., (1996). On the lambertw function. *Advances in Computational mathematics*, 5(1), pp. 329–359.

De Haan, L., Ferreira, A., and Ferreira, A., (2006). Extreme value theory: an introduction, *Springer*, volume 21.

Doomik, J., (2007). Object-Oriented Matrix Programming Using OX. *International Thomson Business Press and Oxford, London*.

Geman, S. and Geman, D., (1984). Stochastic relaxation, gibbs distributions, and the bayesian restoration of images. *IEEE Transactions on pattern analysis and machine intelligence*,6, pp. 721–741.

Ghitany, M., Atieh, B., and Nadarajah, S., (2008). Lindley distribution and its application. *Mathematics and computers in simulation*, 78(4), pp. 493–506.

Gleaton, J. U. and Lynch, J. D., (2010). Extended generalized log- logistic families of lifetime distributions with an application. *J. Probab. Stat. Sci*, 8, pp. 1–17.

Gomes-Silva, F. S., Percontini, A., de Brito, E., Ramos, M. W., Ven^ancio, R., and Cordeiro, G. M., (2017). The odd lindley-g family of distributions. *Austrian Journal of Statistics*, 46(1), pp. 65–87.

Gradshteyn, I. and Ryzhik, I., (2007). Table of Integrals, Series, and Products. *Elsevier/Academic Press*.

Hastings,W. K., (1970). Monte Carlo sampling methods using Markov chains and their applications. *Oxford University Press*.

Jodr´ a, P., (2010). Computer generation of random variables with lindley or poisson–lindley distribution via the lambert w function. *Mathematics and Computers in Simulation*, 81(4), pp. 851–859.

Leadbetter, M. R., Lindgren, G., and Rootz´en, H., (2012). Extremes and related properties of random sequences and processes. *Springer Science and Business Media*.

Lindley, D. V., (1958). Fiducial distributions and bayes' theorem. *Journal of the Royal Statistical Society. Series B (Methodological)*, 20(1), pp.102–107.

Lo, G. S., Ngom, M., Kpanzou, T. A., and Diallo, M., (2018). Weak convergence (iia)-functional and random aspects of the univariate extreme value theory. *arXiv preprint arXiv:1810.01625*.

Merovci, F. and Sharma, V. K., (2014). The beta-lindley distribution: properties and applications. *Journal of Applied Mathematics*.

Metropolis, N., Rosenbluth, A. W., Rosenbluth, M. N., Teller, A. H., and Teller, E., (1953). Equation of state calculations by fast computing machines. *The journal of chemical physics*, 21(6), pp. 1087–1092.

Murthy, D. P., Xie, M., and Jiang, R., (2004). *Weibull models, volume 505. John Wiley and Sons*.

Nadarajah, S., Bakouch, H. S., and Tahmasbi, R., (2011). A generalized lindley distribution. *Sankhya B*, 73(2), pp.331–359.

Oluyede, B. O., Yang, T., and Makubate, B., (2016). A new class of generalized power lindley distribution with applications to lifetime data. *Asian Journal of Mathematics and Applications*, 2016, pp. 1–34.

Ozel, G., Alizadeh, M., Cakmakyapan, S., Hamedani, G., Ortega, E. M., and Cancho, V. G., (2017). The odd log-logistic lindley poisson model for lifetime data. *Communications in Statistics-Simulation and Computation*, 46(8), pp. 6513–6537.

Ranjbar, V., Alizadeh, M., and Altun, E., (2018). Extended generalized lindley distribution: Properties and applications. *Journal of Mathematical Extension*, 13(1), pp. 117–142.

Zakerzadeh, H. and Dolati, A., (2009). Generalized lindley distribution. *Journal of Mathematical Extension*, 3(2), pp.1–17.

Appendix

Tables and figures of the real data analyses section:

Tables:

Table 3: Parameter ML estimates and theirs standard errors (in parentheses).

Model	α	β	γ	λ
$Lindley(\lambda)$	–	–	–	0.5656(0.0585)
$GL(\alpha,\lambda)$	0.6223(0.1142)	–	–	0.4351(0.0709)
$PL(\beta,\lambda)$	0.7593(0.0792)	–	–	0.7701(0.1088)
$BL(\alpha,\beta,\lambda)$	0.6605(0.1407)	0.4098(0.5014)	–	0.9475(1.0566)
$EPL(\alpha,\beta,\lambda)$	0.6825(0.2692)	1.2376(0.9557)	–	0.9372(0.6396)
$OLLL(\alpha,\lambda)$	0.7099(0.0894)	–	–	0.6317(0.0853)
$KPL(\alpha,\beta,\gamma,\lambda)$	0.7799(0.1003)	1.5335(0.5297)	0.1262(0.0368)	4.3580(1.0558)
$EGL(\alpha,\gamma,\lambda)$	0.6192(0.1068)	0.4135(0.4174)	–	0.3805(0.1489)
$NOLLL(\alpha,\beta,\lambda)$	0.2513(0.1063)	1.4241(0.5008)	–	1.2655(0.3774)
$OLLGL(\alpha,\beta,\lambda)$	0.2575(0.0972)	1.5854(0.5016)	–	5.4620(3.3951)

Table 4: Goodness-of-fit test statistics.

Model	*AIC*	*BIC*	$p-value$	W^*	A^*	$-l$
$Lindley(\lambda)$	215.8801	217.7921	0.0128	0.1358	0.7415	106.9412
$GL(\alpha,\lambda)$	210.5744	214.3985	0.3338	0.1393	0.7576	103.2872
$PL(\beta,\lambda)$	209.6294	213.4534	0.5108	0.1085	0.6061	102.8147
$BL(\alpha,\beta,\lambda)$	212.1457	217.8818	0.3376	0.1318	0.7167	103.0729
$EPL(\alpha,\beta,\lambda)$	211.5485	217.2846	0.5544	0.0992	0.5667	102.7742
$OLLL(\alpha,\lambda)$	209.0254	212.8494	0.4245	0.1212	0.6521	102.5127
$KPL(\alpha,\beta,\gamma,\lambda)$	212.9133	220.5614	0.5858	0.0809	0.4850	215.8257
$EGL(\alpha,\gamma,\lambda)$	212.4044	218.1405	0.4438	0.1281	0.7047	103.2022
$NOLLL(\alpha,\beta,\lambda)$	206.9584	212.6945	0.8271	0.0369	0.2691	100.4792
$OLLGL(\alpha,\beta,\lambda)$	**206.5137**	**212.2498**	**0.8984**	**0.0362**	**0.2569**	**100.2569**

Table 5: The LR test results.

	Hypotheses	LR	p-value
OLLGL versus Lindley	$H_0 : \alpha = \beta = 1$	13.3663	0.00125
OLLGL versus OLL-L	$H_0 : \beta = 1$	4.5116	0.03366
OLLGL versus GL	$H_0 : \alpha = 1$	6.0607	0.01382

Table 6: Bayesian estimates $\widehat{\theta}$ and their posterior risks $r_{\widehat{\theta}}$ of the parameters under different loss functions.

Data	First data set		
Bayesian estimation			
Loss function	$\widehat{\alpha}$ $(r_{\widehat{\alpha}})$	$\widehat{\beta}$ $(r_{\widehat{\beta}})$	$\widehat{\lambda}$ $(r_{\widehat{\lambda}})$
SELF	274.818 (10.648)	0.3393 (0.0021)	6.4192 (0.1030)
WSELF	274.336 (0.4825)	0.3332 (0.0061)	6.4032 (0.0160)
MSELF	273.853 (0.0018)	0.3270 (0.0185)	6.3872 (0.0025)
PLF	275.059 (0.4824)	0.3423 (0.0060)	6.4272 (0.0160)
KLF	274.577 (0.0018)	0.3362 (0.0183)	6.4112 (0.0025)

Table 7: Credible and *HPD* intervals of the parameters α, β and λ.

	Credible interval	HPD interval
α	(266.8, 282.9)	(252.8, 296.9)
β	(0.3081, 0.3687)	(0.2466, 0.4236)
λ	(6.196, 6.638)	(5.819, 7.066)

Figures:

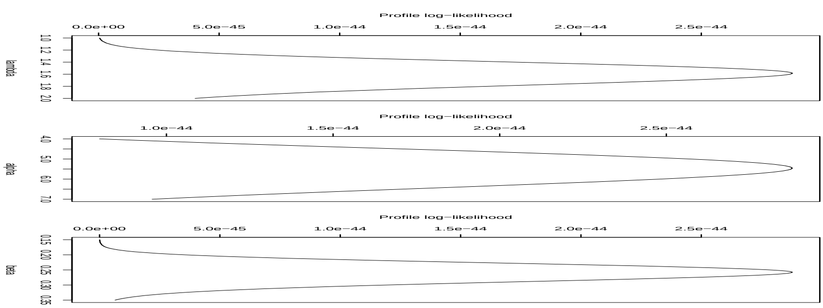

Figure 4: The profile log-likelihood functions of the OLLGL distribution.

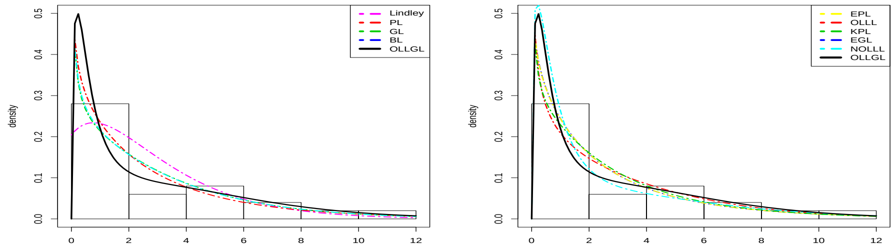

Figure 5: Fitted densities of distributions.

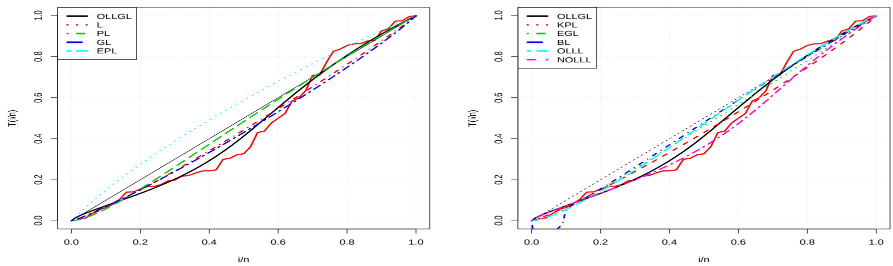

Figure 6: TTT plots of distributions.

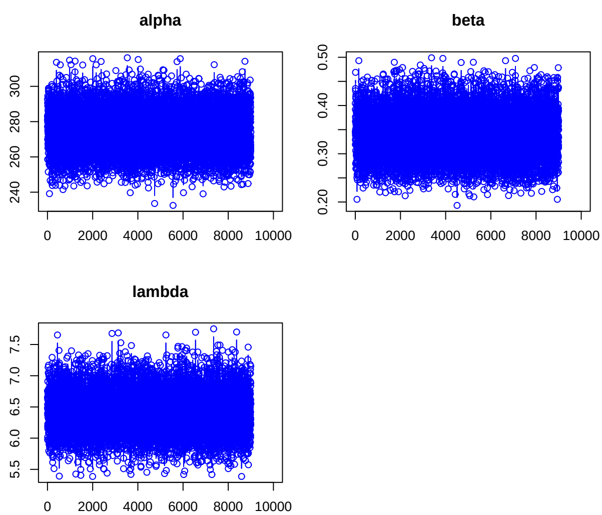

Figure 7: Plots of Bayesian analysis and performance of Gibbs sampling. Trace plots of each parameter of *OLLGL* distribution.

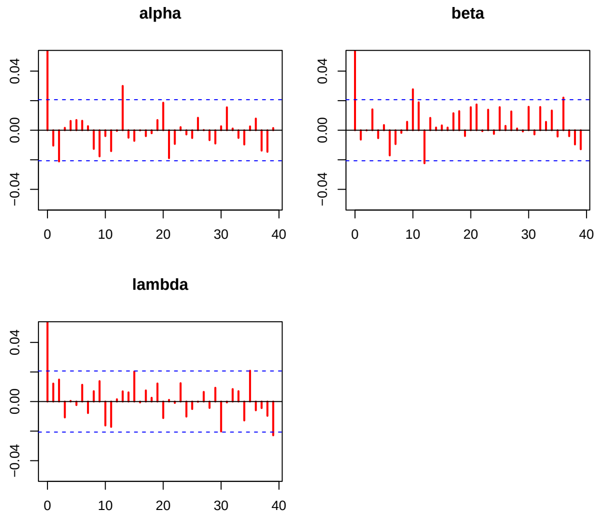

Figure 8: Plots of Bayesian analysis and performance of Gibbs sampling. Autocorrelation plots of each parameter of *OLLGL* distribution.

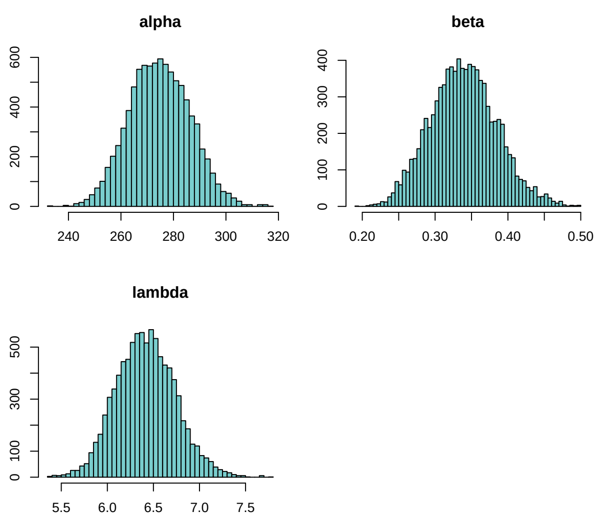

Figure 9: Plots of Bayesian analysis and performance of Gibbs sampling. Histogram plots of each parameter of *OLLGL* distribution.