STATISTICS IN TRANSITION new series, December 2023
Vol. 24, No. 5, pp. 89–108, https://doi.org/10.59170/stattrans-2023-065
Received – 03.08.2021; accepted – 09.03.2023

Analysis for the xgamma distribution based on record values and inter-record times with application to prediction of rainfall and COVID-19 records

Zahra Khoshkhoo Amiri[1], S. M. T. K. MirMostafaee[2]

ABSTRACT

Recently, Sen et al. (2016) introduced a new lifetime distribution, called “xgamma distribution", which can be used as an alternative to other lifetime distributions, like the exponential one. In this paper, we study the problem of classical and Bayesian estimation of the unknown parameter of the xgamma distribution based on record values and inter-record times. The problem of Bayesian prediction of future record values based on record values and inter-record times is also discussed. A small simulation study has been performed to compare the performance of the proposed estimators and the approximate Bayes predictors. Two real data sets related to rainfall and COVID-19 records have been analysed. We considered four one-parameter lifetime distributions as the base models for each data set and compared the goodness-of-fit results. Then, the numerical results of estimation of the parameter and prediction of future records based on the xgamma and exponential records and inter-record times were presented. We observed that the record values and inter-record times from the xgamma distribution could predict future records in a relatively satisfactory way.

Key words: COVID-19 records, lower record values, Bayes predictive distribution, rainfall records, xgamma distribution.

1. Introduction

The xgamma distribution was first introduced by Sen et al. (2016) and its probability density function (PDF) is given by

$$f(x;\theta)=\frac{\theta^2}{1+\theta}\left(1+\frac{\theta}{2}x^2\right)\mathrm{e}^{-\theta x},\quad x>0,\quad \theta>0. \tag{1}$$

The corresponding cumulative distribution function (CDF) is given by

$$F(x;\theta)=1-\frac{1+\theta+\theta x+\frac{\theta^2x^2}{2}}{1+\theta}\mathrm{e}^{-\theta x},\quad x>0,\quad \theta>0.$$

If the PDF of a random variable X is expressed by (1), then we write $X\sim \text{xgamma}(\theta)$. Indeed, the xgamma distribution is a special mixture of the exponential and gamma distributions. The hazard rate function (HRF) of the xgamma distribution can be bathtub-shaped,

[1]Department of Statistics, University of Mazandaran, Babolsar, Iran. E-mail: z.khoshkhoo@stu.umz.ac.ir. ORCID: https://orcid.org/0000-0001-8838-5870.
[2]Department of Statistics, University of Mazandaran, Babolsar, Iran. E-mail: m.mirmostafaee@umz.ac.ir. ORCID: https://orcid.org/0000-0003-2796-4427.

which makes it very suitable for many real lifetime phenomena. In recent years, several studies have been carried out on the inferential problems pertaining to the xgamma distribution; see for example Sen et al. (2018) and Yadav et al. (2019).

Let $\{X_n, n = 1, 2, \cdots\}$ be a sequence of identical and independent random variables. An observation X_j is called a lower record value if $X_j < X_i$ for all $i < j$. A similar definition can be given for upper record values. The sequence of lower record values along with the inter-record times can be given by $(\mathbf{R}, \mathbf{K}) = \{R_1, K_1, R_2, K_2, \cdots, R_{m-1}, K_{m-1}, R_m\}$ where R_i is the i-th record value and K_i is the i-th inter-record time, which is the number of observations after occurrence of R_i that are needed to obtain a new record value R_{i+1}. Record data arise in a wide variety of practical situations; see for example Arnold et al. (1998). Record values and the related subjects have been studied by many authors; see for example Ahmadi and MirMostafaee (2009), MirMostafaee et al. (2016) and Fallah et al. (2018). Record values, along with inter-record times, have become a favourite subject for many researchers in recent decades. Samaniego and Whitaker (1986) discussed the estimation problem of the mean parameter of the exponential distribution based on records and inter-record times. For other examples of recent studies in this regard, see Nadar and Kızılaslan (2015), Kızılaslan and Nadar (2016), Amini and MirMostafaee (2016), Pak and Dey (2019), Kumar et al. (2020) and Bastan and MirMostafaee (2022).

In this paper, first, we obtain the maximum likelihood (ML) estimate of the unknown parameter of the xgamma distribution based on lower record values and inter-record times, and then we construct an asymptotic confidence interval (ACI) for the xgamma parameter in Section 2. Next, we work on the Bayesian estimation of the parameter in Section 3. We approximate the Bayes estimates with the help of the Tierney and Kadane (TK) method, importance sampling (IS) method and Metropolis-Hastings (M-H) algorithm. We also discuss the Bayesian prediction problem of future record values arising from the xgamma distribution based on observed lower record values and inter-record times in Section 4. In Section 5, a small simulation study is conducted to compare the performances of the proposed estimators and approximate Bayes predictors. We analyse two real data sets that are related to rainfall and COVID-2019 phenomena. Section 6 concludes the paper with some remarks.

2. Maximum Likelihood Estimation

In this section, we focus on the ML estimate and an ACI for the parameter. Let $\{R_1, K_1, R_2, K_2, \cdots, R_{m-1}, K_{m-1}, R_m\}$ be a sequence of record data from xgamma(θ). Then, the likelihood function of θ given the observed lower records and inter-record times becomes

$$L(\theta|\mathbf{r},\mathbf{k}) = \prod_{i=1}^{m} f(r_i;\theta)\left[1 - F(r_i;\theta)\right]^{k_i-1} = \frac{\mathrm{e}^{-\theta\sum_{i=1}^{m} k_i r_i}\theta^{2m}}{(1+\theta)^{\sum_{i=1}^{m} k_i}} \prod_{i=1}^{m}\left(1 + \frac{\theta}{2} r_i^2\right)\left[\psi(\theta, r_i)\right]^{k_i-1}, \quad (2)$$

where $r_1 > \cdots > r_m$, $k_m = 1$, $\psi(\theta, r_i) = 1 + \theta + \theta r_i + \dfrac{\theta^2 r_i^2}{2}$ and $\mathbf{r} = \{r_1, \cdots, r_m\}$ and $\mathbf{k} = \{k_1, \cdots, k_{m-1}\}$ are the observed sets of $\mathbf{R} = \{R_1, \cdots, R_{m-1}, R_m\}$ and $\mathbf{K} = \{K_1, \cdots, K_{m-1}\}$, respectively. Note that we always take k_m equal to one for simplicity of the equations.

Consequently, the corresponding log-likelihood function is

$$\ell(\theta|\mathbf{r},\mathbf{k})=2m\ln(\theta)-\sum_{i=1}^{m}k_i\ln(1+\theta)-\theta\sum_{i=1}^{m}k_ir_i+\sum_{i=1}^{m}\ln\left(1+\frac{\theta r_i^2}{2}\right)+\sum_{i=1}^{m}(k_i-1)\ln\left(\psi(\theta,r_i)\right).$$

We take the first partial derivative of log-likelihood function with respect to (w.r.t.) θ and then equate it with zero. Thus, we have

$$\frac{\partial\ell(\theta|\mathbf{r},\mathbf{k})}{\partial\theta}=\frac{2m}{\theta}-\sum_{i=1}^{m}\frac{k_i}{1+\theta}-\sum_{i=1}^{m}k_ir_i+\sum_{i=1}^{m}\frac{r_i^2}{2+\theta r_i^2}+\sum_{i=1}^{m}\frac{(k_i-1)\left(1+r_i+\theta r_i^2\right)}{\psi(\theta,r_i)}=0.$$

It seems that no explicit solution for the above equation exists, and we may use numerical techniques to calculate the ML estimate of θ.
Next, we aim at finding an ACI for the parameter θ. Here, the Fisher information is defined by $I(\theta)=-E\left\{\frac{\partial^2\ln f_\theta(\mathbf{R},\mathbf{K})}{\partial\theta^2}\right\}$, where $f_\theta(\mathbf{r},\mathbf{k})$ is the joint probability function of $R_1,K_1,R_2,K_2,\cdots,R_{m-1},K_{m-1},R_m$, provided that the related integral exists. We have

$$\frac{\partial^2\ell(\theta|\mathbf{r},\mathbf{k})}{\partial\theta^2}=-\frac{2m}{\theta^2}+\sum_{i=1}^{m}\frac{k_i}{(1+\theta)^2}-\sum_{i=1}^{m}\frac{r_i^4}{(2+\theta r_i^2)^2}+\sum_{i=1}^{m}\frac{(k_i-1)\left(\psi(\theta,r_i)r_i^2-(1+r_i+\theta r_i^2)^2\right)}{\left[\psi(\theta,r_i)\right]^2}.$$

Let $\widehat{\theta}_{ML}$ denote the ML estimator (MLE) of θ and z_γ be the γ-th upper quantile of the standard normal distribution. Then, the $100(1-\alpha)\%$ modified asymptotic two-sided equi-tailed confidence interval (MATE CI) for θ is given by (see for example Lehmann and Casella, 1998)

$$\left(\max\left\{0,\widehat{\theta}_{ML}-\frac{z_{\frac{\alpha}{2}}}{\sqrt{\tilde{I}(\widehat{\theta}_{ML})}}\right\},\widehat{\theta}_{ML}+\frac{z_{\frac{\alpha}{2}}}{\sqrt{\tilde{I}(\widehat{\theta}_{ML})}}\right),$$

where $\tilde{I}(\widehat{\theta}_{ML})=-\frac{\partial^2\ell(\theta|\mathbf{R},\mathbf{K})}{\partial\theta^2}\Big|_{\theta=\widehat{\theta}_{ML}}$.

3. Bayesian Estimation

In the context of Bayesian estimation, the information of the experimenter can be revealed in the form of a probability function for the parameter, which is called the prior distribution. Since the parameter of the xgamma distribution is positive, we consider the popular gamma prior for θ with the following PDF

$$\pi(\theta)=\frac{b^a}{\Gamma(a)}\theta^{a-1}\mathrm{e}^{-b\theta},\qquad \theta>0, \tag{3}$$

where a and b are positive hyperparameters that can be determined by the prior knowledge of the experimenter. From (2) and (3), the posterior density can be obtained as

$$\pi(\theta|\mathbf{r},\mathbf{k}) = \frac{1}{D}\frac{\theta^{2m+a-1}}{(1+\theta)^{\sum_{i=1}^{m}k_i}}\mathrm{e}^{-\theta(\sum_{i=1}^{m}k_ir_i+b)}\prod_{i=1}^{m}\left(1+\frac{\theta}{2}r_i^2\right)\left[\psi(\theta,r_i)\right]^{k_i-1},$$

where $D = \int_0^\infty \frac{\theta^{2m+a-1}}{(1+\theta)^{\sum_{i=1}^m k_i}} \mathrm{e}^{-\theta(\sum_{i=1}^m k_i r_i + b)} \prod_{i=1}^m \left(1+\frac{\theta}{2} r_i^2\right) \left[\psi(\theta, r_i)\right]^{k_i - 1} \mathrm{d}\theta$. A very popular quadratic loss function is the squared error loss function (SELF). However, the SELF is not appropriate in many real situations, as it gives identical weights to underestimation and overestimation. One asymmetric loss function is the linear-exponential loss function (LELF), which was introduced by Varian (1975) and is defined by

$$L_{LE}(\theta, \hat{\theta}) = b\left[\exp\{c(\hat{\theta}-\theta)\} - c(\hat{\theta}-\theta) - 1\right], \quad b > 0, \quad c \neq 0,$$

where $\hat{\theta}$ is an estimator of θ. Without loss of generality, we assume $b = 1$. The sign and magnitude of parameter c must be properly determined. If c is bigger than zero, then overestimation is more serious than underestimation and vice versa (Zellner, 1986). The Bayes estimates of θ under the SELF and LELF, become

$$\widehat{\theta}_{SE} = \int_0^\infty \theta \pi(\theta|\mathbf{r},\mathbf{k}) \mathrm{d}\theta = \frac{1}{D} \int_0^\infty \frac{\theta^{2m+a}}{(1+\theta)^{\sum_{i=1}^m k_i}} \mathrm{e}^{-\theta(\sum_{i=1}^m k_i r_i + b)} \prod_{i=1}^m \left(1+\frac{\theta}{2} r_i^2\right) \left[\psi(\theta, r_i)\right]^{k_i - 1} \mathrm{d}\theta,$$

and

$$\begin{aligned} \widehat{\theta}_{LE} &= -\frac{1}{c} \ln M_\theta(-c|\mathbf{r},\mathbf{k}) = -\frac{1}{c} \ln\left(\int_0^\infty \exp(-c\theta)\, \pi(\theta \mid \mathbf{r},\mathbf{k}) \mathrm{d}\theta\right) \\ &= -\frac{1}{c} \ln\left(\frac{1}{D} \int_0^\infty \frac{\theta^{2m+a-1}}{(1+\theta)^{\sum_{i=1}^m k_i}} \mathrm{e}^{-\theta(\sum_{i=1}^m k_i r_i + b + c)} \prod_{i=1}^m \left(1+\frac{\theta}{2} r_i^2\right) \left[\psi(\theta, r_i)\right]^{k_i - 1} \mathrm{d}\theta\right), \end{aligned}$$

respectively, provided that the integrals exist.

It seems that the above Bayes estimates of θ cannot be obtained in closed forms. So, we use three methods to find the approximate Bayes estimates of parameter θ.

3.1. Tierney and Kadane's Approximation

For a one-parameter model, Tierney and Kadane (1986) proposed a technique to approximate Bayes estimates. Let $v_0(\theta) = \frac{1}{n} \ln \pi(\theta \mid \mathbf{r},\mathbf{k})$ and $v^*(\theta) = v_0(\theta) + \frac{1}{n} \ln g(\theta)$. Then, according to the TK approximation method, the approximated Bayes estimate of θ is

$$\widehat{\theta}_{BT} = \sqrt{\frac{\tau^*}{\tau_0}} \exp\left\{ n\left[v^*(\theta^*) - v_0(\theta_0)\right]\right\},$$

where θ^* and θ_0 maximize $v^*(\theta)$ and $v_0(\theta)$, respectively, and τ^* and τ_0 are minus the inverse of the second derivatives of $v^*(\theta)$ and $v_0(\theta)$ at the points θ^* and θ_0, respectively.

We have

$$\begin{aligned} v_0(\theta) = \frac{1}{n}\Bigg[&- \ln D + (2m+a-1)\ln\theta - \sum_{i=1}^m k_i \ln(1+\theta) - \theta(\sum_{i=1}^m k_i r_i + b) + \sum_{i=1}^m \ln\left(1+\frac{\theta}{2} r_i^2\right) \\ &+ \sum_{i=1}^m (k_i - 1) \ln\left(\psi(\theta, r_i)\right)\Bigg]. \end{aligned}$$

Therefore, θ_0 can be derived from the following equation:

$$\frac{\partial v_0(\theta)}{\partial\theta}=\frac{1}{n}\left[\frac{2m+a-1}{\theta}-\frac{\sum_{i=1}^{m}k_i}{1+\theta}-\sum_{i=1}^{m}k_ir_i-b+\sum_{i=1}^{m}\frac{r_i^2}{2+\theta r_i^2}+\sum_{i=1}^{m}\frac{(k_i-1)(1+r_i+\theta r_i^2)}{\psi(\theta,r_i)}\right]=0.$$

Let ξ_0 be the second order derivative of $v_0(\theta)$ at θ_0, namely

$$\xi_0=\frac{\partial^2}{\partial\theta^2}v_0(\theta)\bigg|_{\theta=\theta_0}=\frac{1}{n}\left[\frac{-(2m+a-1)}{\theta^2}+\frac{\sum_{i=1}^{m}k_i}{(1+\theta)^2}-\sum_{i=1}^{m}\frac{r_i^4}{(2+\theta r_i^2)^2}+\sum_{i=1}^{m}\frac{(k_i-1)\left[\psi(\theta,r_i)r_i^2-(1+r_i+\theta r_i^2)^2\right]}{\left[\psi(\theta,r_i)\right]^2}\right]\Bigg|_{\theta=\theta_0}.$$

Then, set $\tau_0=-\frac{1}{\xi_0}$. First, we derive the approximate Bayes estimate of θ under the SELF. Let $g(\theta)=\theta$ and θ_1^* be the maximum point of the following quantity:

$$v^{*SE}(\theta)=\frac{1}{n}\left[-\ln D+(2m+a)\ln\theta-\sum_{i=1}^{m}k_i\ln(1+\theta)-\theta(\sum_{i=1}^{m}k_ir_i+b)+\sum_{i=1}^{m}\ln\left(1+\frac{\theta}{2}r_i^2\right)+\sum_{i=1}^{m}(k_i-1)\ln\left(\psi(\theta,r_i)\right)\right].$$

Then, θ_1^* can be derived from the following equation:

$$\frac{\partial v^{*SE}(\theta)}{\partial\theta}=\frac{1}{n}\left[\frac{2m+a}{\theta}-\frac{\sum_{i=1}^{m}k_i}{1+\theta}-\sum_{i=1}^{m}k_ir_i-b+\sum_{i=1}^{m}\frac{r_i^2}{2+\theta r_i^2}+\sum_{i=1}^{m}\frac{(k_i-1)(1+r_i+\theta r_i^2)}{\psi(\theta,r_i)}\right]=0.$$

Let ξ_{SE}^* be the second order derivative of $v^{*SE}(\theta)$ at θ_1^*, namely

$$\xi_{SE}^*=\frac{\partial^2}{\partial\theta^2}v^{*SE}(\theta)\bigg|_{\theta=\theta_1^*}=\frac{1}{n}\left[\frac{-(2m+a)}{\theta^2}+\frac{\sum_{i=1}^{m}k_i}{(1+\theta)^2}-\sum_{i=1}^{m}\frac{r_i^4}{(2+\theta r_i^2)^2}+\sum_{i=1}^{m}\frac{(k_i-1)\left[\psi(\theta,r_i)r_i^2-\left(1+r_i+\theta r_i^2\right)^2\right]}{\left[\psi(\theta,r_i)\right]^2}\right]\Bigg|_{\theta=\theta_1^*}.$$

Then, set $\tau_{SE}^*=-\frac{1}{\xi_{SE}^*}$. So, the approximate Bayes estimate of θ under the SELF becomes

$$\widehat{\theta}_{ST}=\sqrt{\frac{\tau_{SE}^*}{\tau_0}}\exp\left\{n\left[v^{*SE}(\theta_1^*)-v_0(\theta_0)\right]\right\}.$$

Next, consider the LELF and let $g(\theta)=\mathrm{e}^{-c\theta}$. Then, we have

$$v^{*LE}(\theta)=\frac{1}{n}\left[-\ln D+(2m+a-1)\ln\theta-\sum_{i=1}^{m}k_i\ln(1+\theta)-\theta(\sum_{i=1}^{m}k_ir_i+b+c)+\sum_{i=1}^{m}\ln\left(1+\frac{\theta}{2}r_i^2\right)+\sum_{i=1}^{m}(k_i-1)\ln\left(\psi(\theta,r_i)\right)\right].$$

The maximum point of $v^{*LE}(\theta)$, denoted by θ_2^*, can be derived from the following equation:

$$\frac{\partial}{\partial\theta}v^{*LE}(\theta)=\frac{1}{n}\left[\frac{2m+a-1}{\theta}-\frac{\sum_{i=1}^m k_i}{1+\theta}-\sum_{i=1}^m k_i r_i-b-c+\sum_{i=1}^m\frac{r_i^2}{2+\theta r_i^2}\right.$$
$$\left.+\sum_{i=1}^m\frac{(k_i-1)(1+r_i+\theta r_i^2)}{\psi(\theta,r_i)}\right]=0.$$

Let ξ_{LE}^* be the second order derivative of $v^{*LE}(\theta)$ at θ_2^*, namely

$$\xi_{LE}^*=\frac{\partial^2}{\partial\theta^2}v^{*LE}(\theta)\bigg|_{\theta=\theta_2^*}=\frac{1}{n}\left[\frac{-(2m+a-1)}{\theta^2}+\frac{\sum_{i=1}^m k_i}{(1+\theta)^2}-\sum_{i=1}^m\frac{r_i^4}{(2+\theta r_i^2)^2}\right.$$
$$\left.+\sum_{i=1}^m\frac{(k_i-1)\left[\psi(\theta,r_i)r_i^2-(1+r_i+\theta r_i^2)^2\right]}{\left[\psi(\theta,r_i)\right]^2}\right]\Bigg|_{\theta=\theta_2^*}.$$

Then, set $\tau_{LE}^*=-\frac{1}{\xi_{LE}^*}$ and the approximate Bayes estimate of θ under the LELF becomes

$$\widehat{\theta}_{LT}=-\frac{1}{c}\ln\left(\sqrt{\frac{\tau_{LE}^*}{\tau_0}}\exp\left\{n\left[v^{*LE}(\theta_2^*)-v_0(\theta_0)\right]\right\}\right).$$

3.2. Importance Sampling Method

Another well-known method of approximating Bayes point estimates is the importance sampling method; see for example Albert (2009, Section 5.9). The posterior density function of θ given $\mathbf{r}$ and $\mathbf{k}$, can be rewritten as

$$\pi(\theta|\mathbf{r},\mathbf{k})=g(\theta|\mathbf{r},\mathbf{k})h(\theta,\mathbf{r},\mathbf{k})=\frac{\theta^{2m+a-1}}{D(1+\theta)^{\sum_{i=1}^m k_i}}\mathrm{e}^{-\theta(\sum_{i=1}^m k_i r_i+b)}\prod_{i=1}^m\left(1+\frac{\theta}{2}r_i^2\right)\left[\psi(\theta,r_i)\right]^{k_i-1},$$

where $g(\theta|\mathbf{r},\mathbf{k})$ is the gamma density with parameters $a+2m$ and $\sum_{i=1}^m k_i r_i+b$ and

$$h(\theta,\mathbf{r},\mathbf{k})=\frac{\Gamma(a+2m)}{D\left(\sum_{i=1}^m k_i r_i+b\right)^{a+2m}(1+\theta)^{\sum_{i=1}^m k_i}}\prod_{i=1}^m\left(1+\frac{\theta}{2}r_i^2\right)\left[\psi(\theta,r_i)\right]^{k_i-1}.$$

Algorithm 1:
Step 1. Generate θ_1 from $g(\theta|\mathbf{r},\mathbf{k})$.
Step 2. Repeat Step 1, N times to obtain $\theta_1,\ldots,\theta_N$, where N is a large number.
Step 3. The approximate Bayes estimates of θ under the SELF and LELF are given by

$$\widehat{\theta}_{SI}=\frac{\sum_{i=1}^N h(\theta_i,\mathbf{r},\mathbf{k})\theta_i}{\sum_{j=1}^N h(\theta_j,\mathbf{r},\mathbf{k})}=\sum_{i=1}^N\theta_i w_i,$$

and

$$\widehat{\theta}_{LI} = -\frac{1}{c}\ln\left(\frac{\sum_{i=1}^{N} e^{-c\theta_i} h(\theta_i,\mathbf{r},\mathbf{k})}{\sum_{j=1}^{N} h(\theta_j,\mathbf{r},\mathbf{k})}\right) = -\frac{1}{c}\ln\left(\sum_{i=1}^{N}\exp(-c\theta_i)w_i\right),$$

respectively, where

$$w_i = \frac{h(\theta_i,\mathbf{r},\mathbf{k})}{\sum_{j=1}^{N} h(\theta_j,\mathbf{r},\mathbf{k})}, \quad i = 1,\cdots,N. \tag{4}$$

Let $\{\theta_1,\cdots,\theta_N\}$ be the generated sample using the IS method and $\theta_{(1)} \le \cdots \le \theta_{(N)}$ be the corresponding ordered values of $\{\theta_1,\cdots,\theta_N\}$. Let

$$w_i^* = \frac{h(\theta_{(i)},\mathbf{r},\mathbf{k})}{\sum_{j=1}^{N} h(\theta_j,\mathbf{r},\mathbf{k})}, \quad i = 1,\cdots,N.$$

Consider the following intervals

$$L_j(N) = \left(\widehat{\theta}^{(\frac{j}{N})}, \widehat{\theta}^{(\frac{j+[(1-\alpha)N]}{N})}\right), \qquad j = 1,2,\ldots,N-[(1-\alpha)N],$$

where $[x]$ denotes the integer part of x and $\widehat{\theta}^{(\alpha)} = \theta_{(i)}$ if $\sum_{j=1}^{i-1} w_j^* < \alpha \le \sum_{j=1}^{i} w_j^*$. Then, the $100(1-\alpha)\%$ Chen and Shao shortest width credible interval (CSSW CrI) for θ is given by $L_q(N)$, where q is selected so that (Chen and Shao, 1999)

$$\widehat{\theta}^{(\frac{q+[(1-\alpha)N]}{N})} - \widehat{\theta}^{(\frac{q}{N})} = \min_{1\le j\le N-[(1-\alpha)N]} \widehat{\theta}^{(\frac{j+[(1-\alpha)N]}{N})} - \widehat{\theta}^{(\frac{j}{N})}.$$

3.3. Metropolis-Hastings Method

The Metropolis-Hastings (M-H) method was originally proposed by Metropolis et al. (1953) and then was generalized by Hastings (1970). One M-H algorithm for our case can be summarized as follows.

Algorithm 2:
Step 1. Start with an initial guess $\theta_0 = \widehat{\theta}_{ML}$ and set $t = 1$.
Step 2. Given θ_{t-1}, generate θ^* from the truncated-normal distribution, $N(\theta_{t-1},\sigma^2)I_{\{\theta>0\}}$. Then, set $\theta_t = \theta^*$ with probability

$$P = \min\left\{\frac{\pi(\theta^* \mid \mathbf{r},\mathbf{k})q(\theta^{t-1} \mid \theta^*)}{\pi(\theta^{t-1} \mid \mathbf{r},\mathbf{k})q(\theta^* \mid \theta^{t-1})}, 1\right\},$$

where $q(x \mid b)$ is the density of $N(b,\sigma^2)I_{\{x>0\}}$, otherwise set $\theta_t = \theta_{t-1}$.
Step 3. Set $t = t+1$ and repeat Step 2, T times, where T is a large number. Then, $\{\theta_{M+1},\theta_{M+2},\cdots,\theta_T\}$ is the generated sample, where M is a burn-in period. Now, the ap-

proximated Bayes point estimates of θ under the SELF and LELF are given by

$$\widehat{\theta}_{SM} = \frac{1}{M^*}\sum_{t=M+1}^{T}\theta_t, \qquad \text{and} \qquad \widehat{\theta}_{LM} = -\frac{1}{c}\ln\left(\frac{1}{M^*}\sum_{t=M+1}^{T} \mathrm{e}^{-c\theta_t}\right),$$

respectively, where $M^* = T - M$. We have taken $\sigma^2 = 1$ in Section 5.

Let $\theta_{(1)} \leq \cdots \leq \theta_{(M^*)}$ be the corresponding ordered values of $\{\theta_{M+1}, \cdots, \theta_T\}$. Consider the intervals $L_j(M^*) = \left(\theta_{(j)}, \theta_{(j+[(1-\alpha)M^*])}\right)$ for $j = 1,2,...,M^* - [(1-\alpha)M^*]$, then the $100(1-\alpha)\%$ CSSW CrI for θ can be reported as $L_q(M^*)$, where q is selected so that (Chen and Shao, 1999)

$$\theta_{(q+[(1-\alpha)M^*])} - \theta_{(q)} = \min_{1\leq j\leq M^*-[(1-\alpha)M^*]} \theta_{(j+[(1-\alpha)M^*])} - \theta_{(j)}.$$

4. Bayesian Prediction

Suppose that $R_1, K_1, R_2, K_2, \cdots, R_{m-1}, K_{m-1}, R_m$ are a sequence of available record data from xgamma(θ) and we wish to predict the s-th unobserved record value, denoted by $R_s (s > m)$. The conditional density function of R_s given $\mathbf{r}$ and $\mathbf{k}$ is given by

$$\begin{aligned} f(r_s|\theta,\mathbf{r},\mathbf{k}) &= \frac{f(r_s;\theta)[Q(r_s,\theta)-Q(r_m,\theta)]^{s-m-1}}{\Gamma(s-m)F(r_m;\theta)} \\ &= \frac{[Q(r_s,\theta)-Q(r_m,\theta)]^{s-m-1}}{\Gamma(s-m)}\left(1-\frac{\psi(\theta,r_m)}{(1+\theta)}\mathrm{e}^{-\theta r_m}\right)^{-1}\frac{\theta^2}{1+\theta}\left(1+\frac{\theta}{2}r_s^2\right)\mathrm{e}^{-\theta r_s}, \end{aligned} \tag{5}$$

where $0 < r_s < r_m$, $Q(r_s,\theta) = -\ln F(r_s;\theta)$ and $\psi(\theta, r_m) = 1+\theta+\theta r_m + \frac{\theta^2 r_m^2}{2}$.

Then, from (5), the Bayes predictive density function of R_s is derived as

$$h(r_s \mid \mathbf{r},\mathbf{k}) = \int_0^\infty f(r_s \mid \theta,\mathbf{r},\mathbf{k})\pi(\theta \mid \mathbf{r},\mathbf{k})\mathrm{d}\theta.$$

The predictions of R_s under the SELF and LELF can be given by $\widehat{R}_s^S = \int_0^{r_m} r_s h(r_s \mid \mathbf{r},\mathbf{k})\mathrm{d}r_s$, and $\widehat{R}_s^L = -\frac{1}{c}\ln\int_0^{r_m}\mathrm{e}^{-cr_s}h(r_s \mid \mathbf{r},\mathbf{k})\mathrm{d}r_s$, respectively. It seems that the Bayes predictive density function of R_s cannot be obtained analytically. Therefore, we approximate $h(r_s \mid \mathbf{r},\mathbf{k})$ using the IS and M-H methods. Assume that $\{\theta_i, i = 1,...,N\}$ and $\{\theta_t, t = M+1,...,T\}$ are the generated samples using the IS and M-H procedures, respectively. Then, the estimates of $h(r_s \mid \mathbf{r},\mathbf{k})$ using these generated samples are given by

$$\widehat{h}_{IM}(r_s \mid \mathbf{r},\mathbf{k}) = \sum_{i=1}^{N} w_i f(r_s \mid \theta_i,\mathbf{r},\mathbf{k}), \qquad \text{and} \qquad \widehat{h}_{MH}(r_s \mid \mathbf{r},\mathbf{k}) = \frac{1}{M^*}\sum_{t=M+1}^{T} f(r_s \mid \theta_t,\mathbf{r},\mathbf{k}),$$

respectively, where w_i is defined in (4).

Now, using the generated sample obtained by the IS method, the approximate predictions of R_s under the SELF and LELF are, respectively, given by (provided that they exist)

$$\widehat{R}_s^{SI} = \sum_{i=1}^{N} w_i \int_0^{r_m} r_s f(r_s \mid \theta_i, \mathbf{r}, \mathbf{k}) \mathrm{d}r_s, \quad \text{and} \quad \widehat{R}_s^{LI} = -\frac{1}{c} \ln\left\{ \sum_{i=1}^{N} w_i \int_0^{r_m} \mathrm{e}^{-cr_s} f(r_s|\theta_i, \mathbf{r}, \mathbf{k}) \mathrm{d}r_s \right\}.$$

Using the generated sample obtained by the M-H method, the approximate predictions of R_s under the SELF and LELF are, respectively, given by (provided that they exist)

$$\widehat{R}_s^{SM} = \frac{1}{M^*} \sum_{t=M+1}^{T} \int_0^{r_m} r_s f(r_s \mid \theta_t, \mathbf{r}, \mathbf{k}) \mathrm{d}r_s,$$

and

$$\widehat{R}_s^{LM} = -\frac{1}{c} \ln\left\{ \frac{1}{M^*} \sum_{t=M+1}^{T} \int_0^{r_m} \mathrm{e}^{-cr_s} f(r_s \mid \theta_t, \mathbf{r}, \mathbf{k}) \mathrm{d}r_s \right\}.$$

The $100(1-\alpha)\%$ Bayesian prediction interval (BPI) for R_s is given by $(L(\mathbf{R},\mathbf{K}), U(\mathbf{R},\mathbf{K}))$, where the prediction limits $L(\mathbf{r},\mathbf{k})$ and $U(\mathbf{r},\mathbf{k})$ can be obtained by solving the following nonlinear equations simultaneously (see for example Pak and Dey (2019))

$$\int_0^{L(\mathbf{r},\mathbf{k})} h(r_s \mid \mathbf{r}, \mathbf{k}) \mathrm{d}r_s = \frac{\alpha}{2}, \quad \text{and} \quad \int_0^{U(\mathbf{r},\mathbf{k})} h(r_s \mid \mathbf{r}, \mathbf{k}) \mathrm{d}r_s = 1 - \frac{\alpha}{2}.$$

Therefore, the $100(1-\alpha)\%$ approximate BPI (ABPI) for R_s, denoted by (L^*, U^*), based on the IS method, can be obtained by solving the following equations simultaneously:

$$\sum_{i=1}^{N} w_i \int_0^{L^*} f(r_s \mid \theta_i, \mathbf{r}, \mathbf{k}) \mathrm{d}r_s = \frac{\alpha}{2}, \quad \text{and} \quad \sum_{i=1}^{N} w_i \int_0^{U^*} f(r_s \mid \theta_i, \mathbf{r}, \mathbf{k}) \mathrm{d}r_s = 1 - \frac{\alpha}{2}.$$

Besides, the $100(1-\alpha)\%$ ABPI for R_s, denoted as (L^{**}, U^{**}), based on the M-H method, can be obtained by solving the following equations simultaneously (see for example AL-Hussaini and Al-Awadhi (2010)):

$$\frac{1}{M^*} \sum_{t=M+1}^{T} \int_0^{L^{**}} f(r_s \mid \theta_t, \mathbf{r}, \mathbf{k}) \mathrm{d}r_s = \frac{\alpha}{2}, \quad \text{and} \quad \frac{1}{M^*} \sum_{t=M+1}^{T} \int_0^{U^{**}} f(r_s \mid \theta_t, \mathbf{r}, \mathbf{k}) \mathrm{d}r_s = 1 - \frac{\alpha}{2}.$$

5. Numerical Illustration

In this section, we provide a simulation study and two real data examples.

5.1. Simulation Study

Here, we perform a Monte Carlo simulation to assess the point and interval estimators and approximate predictors that are developed in this paper. In this simulation study, the number of replications is taken to be $N^* = 5000$. We generate $(m+1)$ records and their corresponding inter-record times from xgamma(θ) in each replication, where two values are considered for m, namely $m = 5$ and 7 and the parameter values are selected to be $\theta = 0.5$ and 1.5. In the context of the Bayesian inference, we use two priors: **Prior I**: In this prior, the hyperparameters are determined so that the prior mean equals the true value of the parameter and the prior variance equals 1. Thus, for $\theta = 0.5$, we have $(a,b) = (0.25, 0.5)$, and for $\theta = 1.5$, we have $(a,b) = (2.25, 1.5)$. **Prior II**: In this prior, the hyperparameters are determined so that the prior mean equals the true value of the parameter and the prior variance equals 100. Thus, for $\theta = 0.5$ we have $(a,b) = (0.0025, 0.005)$, and for $\theta = 1.5$ we have $(a,b) = (0.0225, 0.015)$. We compute the ML estimates and the approximate Bayes estimates under the TK, IS and M-H methods based on the generated first m records and $(m-1)$ record times. Besides, the Geweke test (see Geweke, 1992), Raftery and Lewis's diagnostic (see Raftery and Lewis, 1992, 1996) and Heidelberger and Welch's convergence diagnostic (see Heidelberger and Welch, 1983) are used to check the convergence of the generated M-H Markov chains. Note that Heidelberger and Welch (1983) used or hinted the results of Schruben et al. (1980), Heidelberger and Welch (1981a, 1981b), Schruben (1982) and Schruben et al. (1983). In some cases, we have taken every second or third sampled value (and increase the number of sampled values accordingly) to achieve a convergent M-H Markov chain. All the final chains have sizes equal to 10000. The performance of the competitive estimators has been compared in terms of their estimated risks (ERs). In addition, the average width (AW) and coverage probability (CP) criteria have been employed to evaluate the interval estimators and predictors. Let $\widehat{\theta}$ be an estimator of θ and $\widehat{\theta}_i$ be the corresponding estimate derived in the i-th replication. Then, the ERs of $\widehat{\theta}$ w.r.t. the SELF and LELF functions are, respectively, given by

$$ER_S(\widehat{\theta}) = \frac{1}{N^*}\sum_{i=1}^{N^*}(\widehat{\theta}_i - \theta)^2, \text{ and } ER_L(\widehat{\theta}) = \frac{1}{N^*}\sum_{i=1}^{N^*}\Big(\exp[c(\widehat{\theta}_i - \theta)] - c(\widehat{\theta}_i - \theta) - 1\Big). \quad (6)$$

The approximate point and interval predictions for the $(m+1)$-th record value, namely R_{m+1} are calculated as well. In the context of prediction, the evaluation is based on the estimated prediction risks (EPRs) w.r.t. to the SELF and LELF for the point predictors, which are formulated similarly to (6). The simulation results have been presented in Tables 1-3 for the point estimators and in Table 4 for the interval estimators. The results for the prediction have been presented in Tables 5-8. From Tables 1-8, we extract the following conclusions:

- The ERs of Prior I are less than those of Prior II in the most cases, as expected, since Prior I is more informative than Prior II, however, the EPRs of Prior I are very close to those of Prior II. Besides, the Bayesian credible intervals of Prior I have smaller AWs than those of Prior II, however, the AWs of the ABPIs of Prior I are very close to those of Prior II.

- The ERs of the point estimators and EPRs of the predictors are decreasing w.r.t. the number of records in the most cases, as expected. Besides, the AWs of the interval estimators and the ABPIs are decreasing w.r.t. the number of records.

Table 1: The ERs of the point estimators of θ when $\theta = 0.5$.

	Prior I			Prior II		
$m=5$	ER_S	ER_L $c=0.5$	ER_L $c=-0.5$	ER_S	ER_L $c=0.5$	ER_L $c=-0.5$
$\widehat{\theta}_{ML}$	0.0357	0.0048	0.0042	0.0357	0.0048	0.0042
$\widehat{\theta}_{ST}$	0.0329	0.0044	0.0039	0.0358	0.0048	0.0042
$\widehat{\theta}_{LT}(c=0.5)$	0.0303	0.0040	0.0036	0.0328	0.0044	0.0038
$\widehat{\theta}_{LT}(c=-0.5)$	0.0357	0.0048	0.0042	0.0390	0.0053	0.0045
$\widehat{\theta}_{SI}$	0.0330	0.0044	0.0039	0.0359	0.0049	0.0042
$\widehat{\theta}_{LI}(c=0.5)$	0.0305	0.0041	0.0036	0.0330	0.0044	0.0039
$\widehat{\theta}_{LI}(c=-0.5)$	0.0359	0.0048	0.0042	0.0392	0.0053	0.0045
$\widehat{\theta}_{SM}$	0.0186	0.0024	0.0023	0.0356	0.0048	0.0041
$\widehat{\theta}_{LM}(c=0.5)$	0.0176	0.0023	0.0021	0.0327	0.0044	0.0038
$\widehat{\theta}_{LM}(c=-0.5)$	0.0198	0.0026	0.0024	0.0388	0.0053	0.0045
$m=7$						
$\widehat{\theta}_{ML}$	0.0220	0.0029	0.0026	0.0220	0.0029	0.0026
$\widehat{\theta}_{ST}$	0.0209	0.0027	0.0025	0.0220	0.0029	0.0026
$\widehat{\theta}_{LT}(c=0.5)$	0.0198	0.0026	0.0024	0.0208	0.0027	0.0025
$\widehat{\theta}_{LT}(c=-0.5)$	0.0220	0.0029	0.0026	0.0232	0.0031	0.0028
$\widehat{\theta}_{SI}$	0.0212	0.0028	0.0025	0.0223	0.0029	0.0027
$\widehat{\theta}_{LI}(c=0.5)$	0.0201	0.0026	0.0025	0.0211	0.0028	0.0025
$\widehat{\theta}_{LI}(c=-0.5)$	0.0224	0.0030	0.0027	0.0236	0.0031	0.0028
$\widehat{\theta}_{SM}$	0.0124	0.0016	0.0015	0.0218	0.0029	0.0026
$\widehat{\theta}_{LM}(c=0.5)$	0.0121	0.0015	0.0015	0.0207	0.0027	0.0025
$\widehat{\theta}_{LM}(c=-0.5)$	0.0128	0.0016	0.0016	0.0231	0.0030	0.0027

- We see that the M-H method leads to the smaller ERs in comparison with the TK and IS methods in the most cases. Besides, the M-H method produces estimators that have ERs which are smaller than or close to those of the ML method in more than 50% of the cases.

5.2. Example 1 (Real Data Set 1)

Here, we consider the following data on the amount of rainfall (in inches) recorded at the Los Angeles Civic Center in February from 1998 to 2018; see the website of Los Angeles Almanac: www.laalmanac.com/weather/we08aa.php.

```
0.56, 5.54,  8.87, 0.29,  4.64, 4.89, 11.02, 2.37, 0.92,  1.64,
3.57, 4.27,  3.29, 0.16,  0.20, 3.58, 0.83,  0.79, 4.17,  0.03.
```

First, we compare the fit of the xgamma distribution with three other one-parameter lifetime distributions, listed as follows:
(i) The exponential distribution with a scale parameter θ.
(ii) The Lindley distribution with parameter θ whose PDF is given by (Lindley, 1958 and Ghitany et al., 2008)

$$f_{Lindley}(x) = \frac{\theta^2}{1+\theta}(1+x)e^{-\theta x}, \qquad \theta > 0, \quad x > 0.$$

Table 2: The ERs of the point estimators of θ when $\theta = 1.5$.

	Prior I			Prior II		
		ER_L	ER_L		ER_L	ER_L
$m=5$	ER_S	$c=0.5$	$c=-0.5$	ER_S	$c=0.5$	$c=-0.5$
$\widehat{\theta}_{ML}$	0.5901	0.1251	0.0549	0.5901	0.1251	0.0549
$\widehat{\theta}_{ST}$	0.2312	0.0337	0.0255	0.6104	0.1293	0.0566
$\widehat{\theta}_{LT}(c=0.5)$	0.1808	0.0253	0.0207	0.4152	0.0721	0.0418
$\widehat{\theta}_{LT}(c=-0.5)$	0.3026	0.0461	0.0323	1.0359	2.1959	0.0814
$\widehat{\theta}_{SI}$	0.2299	0.0335	0.0254	0.6078	0.1316	0.0563
$\widehat{\theta}_{LI}(c=0.5)$	0.1810	0.0254	0.0207	0.4205	0.0751	0.0421
$\widehat{\theta}_{LI}(c=-0.5)$	0.3028	0.0462	0.0323	1.0289	1.4503	0.0814
$\widehat{\theta}_{SM}$	0.3965	0.0446	0.0555	0.5465	0.1078	0.0520
$\widehat{\theta}_{LM}(c=0.5)$	0.4156	0.0466	0.0583	0.3810	0.0640	0.0390
$\widehat{\theta}_{LM}(c=-0.5)$	0.3771	0.0425	0.0526	0.8752	0.3042	0.0738
$m=7$						
$\widehat{\theta}_{ML}$	0.3073	0.0510	0.0319	0.3073	0.0510	0.0319
$\widehat{\theta}_{ST}$	0.1741	0.0248	0.0196	0.3160	0.0526	0.0327
$\widehat{\theta}_{LT}(c=0.5)$	0.1453	0.0201	0.0168	0.2481	0.0384	0.0268
$\widehat{\theta}_{LT}(c=-0.5)$	0.2119	0.0311	0.0233	0.4148	0.0794	0.0407
$\widehat{\theta}_{SI}$	0.1738	0.0247	0.0196	0.3146	0.0524	0.0326
$\widehat{\theta}_{LI}(c=0.5)$	0.1456	0.0201	0.0168	0.2487	0.0385	0.0268
$\widehat{\theta}_{LI}(c=-0.5)$	0.2123	0.0311	0.0233	0.4152	0.0795	0.0408
$\widehat{\theta}_{SM}$	0.4584	0.0512	0.0645	0.2868	0.0463	0.0302
$\widehat{\theta}_{LM}(c=0.5)$	0.4723	0.0527	0.0666	0.2289	0.0346	0.0250
$\widehat{\theta}_{LM}(c=-0.5)$	0.4443	0.0497	0.0625	0.3743	0.0675	0.0375

Table 3: The AWs and CPs of the 95% interval estimators of θ.

			Prior I		Prior II	
θ	m	MATE CI	IS method	M-H method	IS method	M-H method
0.5	5	0.6162 (0.9662)	0.5783 (0.9336)	0.5377 (0.9498)	0.5877 (0.9326)	0.5965 (0.9548)
	7	0.4987 (0.9566)	0.4652 (0.9140)	0.4421 (0.9408)	0.4691 (0.9120)	0.4860 (0.9486)
1.5	5	2.2977 (0.9650)	1.8428 (0.9746)	0.9582 (0.2540)	2.1983 (0.9542)	2.1713 (0.9542)
	7	1.8139 (0.9598)	1.5702 (0.9670)	0.7843 (0.0246)	1.7560(0.9516)	1.7359 (0.9518)

Table 4: The EPRs of approximate predictors of R_{m+1} when $\theta = 0.5$.

	Prior I			Prior II		
$m=5$	EPR_S	EPR_L $c=0.5$	EPR_L $c=-0.5$	EPR_S	EPR_L $c=0.5$	EPR_L $c=-0.5$
$\widehat{R}^{SI}_{m+1}$	0.0180	0.0023	0.0023	0.0180	0.0023	0.0023
$\widehat{R}^{LI}_{m+1}(c=0.5)$	0.0184	0.0022	0.0024	0.0184	0.0022	0.0024
$\widehat{R}^{LI}_{m+1}(c=-0.5)$	0.0184	0.0024	0.0022	0.0184	0.0024	0.0022
$\widehat{R}^{SM}_{m+1}$	0.0180	0.0023	0.0023	0.0180	0.0023	0.0023
$\widehat{R}^{LM}_{m+1}(c=0.5)$	0.0183	0.0022	0.0024	0.0184	0.0022	0.0024
$\widehat{R}^{LM}_{m+1}(c=-0.5)$	0.0184	0.0024	0.0022	0.0184	0.0024	0.0022
$m=7$						
$\widehat{R}^{SI}_{m+1}$	0.0018	0.0002	0.0002	0.0018	0.0002	0.0002
$\widehat{R}^{LI}_{m+1}(c=0.5)$	0.0017	0.0002	0.0002	0.0017	0.0002	0.0002
$\widehat{R}^{LI}_{m+1}(c=-0.5)$	0.0019	0.0003	0.0002	0.0019	0.0003	0.0002
$\widehat{R}^{SM}_{m+1}$	0.0018	0.0002	0.0002	0.0018	0.0002	0.0002
$\widehat{R}^{LM}_{m+1}(c=0.5)$	0.0017	0.0002	0.0002	0.0017	0.0002	0.0002
$\widehat{R}^{LM}_{m+1}(c=-0.5)$	0.0019	0.0003	0.0002	0.0019	0.0003	0.0002

Table 5: The EPRs of approximate predictors of R_{m+1} when $\theta = 1.5$.

	Prior I			Prior II		
$m=5$	EPR_S	EPR_L $c=0.5$	EPR_L $c=-0.5$	EPR_S	EPR_L $c=0.5$	EPR_L $c=-0.5$
$\widehat{R}^{SI}_{m+1}$	0.0005	0.0001	0.0001	0.0005	0.0001	0.0001
$\widehat{R}^{LI}_{m+1}(c=0.5)$	0.0005	0.0001	0.0001	0.0005	0.0001	0.0001
$\widehat{R}^{LI}_{m+1}(c=-0.5)$	0.0005	0.0001	0.0001	0.0005	0.0001	0.0001
$\widehat{R}^{SM}_{m+1}$	0.0005	0.0001	0.0001	0.0005	0.0001	0.0001
$\widehat{R}^{LM}_{m+1}(c=0.5)$	0.0005	0.0001	0.0001	0.0005	0.0001	0.0001
$\widehat{R}^{LM}_{m+1}(c=-0.5)$	0.0005	0.0001	0.0001	0.0005	0.0001	0.0001
$m=7$						
$\widehat{R}^{SI}_{m+1}$	0.00007	0.00001	0.00001	0.00007	0.00001	0.00001
$\widehat{R}^{LI}_{m+1}(c=0.5)$	0.00007	0.00001	0.00001	0.00007	0.00001	0.00001
$\widehat{R}^{LI}_{m+1}(c=-0.5)$	0.00007	0.00001	0.00001	0.00007	0.00001	0.00001
$\widehat{R}^{SM}_{m+1}$	0.00007	0.00001	0.00001	0.00007	0.00001	0.00001
$\widehat{R}^{LM}_{m+1}(c=0.5)$	0.00007	0.00001	0.00001	0.00007	0.00001	0.00001
$\widehat{R}^{LM}_{m+1}(c=-0.5)$	0.00007	0.00001	0.00001	0.00007	0.00001	0.00001

Table 6: The AWs and CPs of the 95% ABPIs of R_{m+1}.

		Prior I		Prior II	
θ	m	IS method	M-H method	IS method	M-H method
0.5	5	0.2001 (0.9514)	0.2001 (0.9514)	0.2001 (0.9514)	0.2001 (0.9514)
	7	0.0477 (0.9566)	0.0477 (0.9566)	0.0477 (0.9566)	0.0477 (0.9566)
1.5	5	0.0355 (0.9500)	0.0356 (0.9498)	0.0355 (0.9500)	0.0355 (0.9500)
	7	0.0087 (0.9500)	0.0087 (0.9500)	0.0087 (0.9500)	0.0087 (0.9500)

(iii) The Shanker distribution with the following PDF (Shanker, 2015):

$$f_{Shanker}(x) = \frac{\theta^2}{1+\theta^2}(\theta + x)\mathrm{e}^{-\theta x}, \qquad \theta > 0, \quad x > 0.$$

We use the ML method to obtain the parameter estimate. We use the Kolmogorov-Smirnov (K-S) test and the corresponding p-value, Akaike information criterion (AIC), Bayesian information criterion (BIC) and Hannan-Quinn information criterion (HQIC) to compare the fits of the considered distributions and the results have been given in Table 9. From Table 9, we observe that the xgamma and exponential models fit the data better than the Lindley and Shanker models, as they have smaller AICs, BICs, HQICs and K-S test statistics.

Table 7: MLEs and goodness-of-fit statistics for Example 1.

Distribution	MLE	AIC	BIC	HQIC	K-S	p-value
xgamma	0.6895	87.0868	88.0826	87.2812	0.1925	0.3978
exponential	0.3245	87.0166	88.0123	87.2110	0.1561	0.6575
Lindley	0.5358	89.0368	90.0325	89.2312	0.2068	0.3141
Shanker	0.5874	90.4794	91.4752	90.6738	0.2010	0.3465

We have extracted the lower records and the corresponding inter-record times as follows:

i	1	2	3	4
r_i	0.56	0.29	0.16	0.03
k_i	3	10	6	1

We have considered both the exponential and xgamma models, as we see that these models fit the data well. Here, we have used the approximate non-informative prior with the prior mean equal to 1.5 and the prior variance equal to 100, so we have $(a,b) = (0.0225, 0.0150)$. We have calculated the ML and approximate Bayes point estimates, as well as the 95% interval estimates of the parameter for both exponential and xgamma distributions. The point predictions and 95% ABPIs for the next future record, namely R_5, have been obtained as well. We have used the M-H method for the Bayesian estimation and prediction for the xgamma distribution. However, the Bayesian estimates of the unknown parameter for the exponential distribution have explicit forms, and we have also used the function `integrate` in R to evaluate predictions and ABPIs for the exponential distribution. The numerical results of this example have been given in Table 10. From Table 10, we see that all the predictions and ABPIs are too close to each other in both exponential and xgamma distributions. Here, we predict that the next lowest amount of rainfall (after the year 2018) would be approximately 0.015 inches, which is the predicted 5-th lower record value since 1998.

Table 8: The numerical results of Example 1.

Estimation	$\widehat{\theta}_{ML}$	$\widehat{\theta}_{SE}$	$\widehat{\theta}_{LE}$ $(c = 0.5)$	$\widehat{\theta}_{LE}$ $(c = -0.5)$	MATE CI
xgamma	1.3467	1.3293	1.2749	1.3904	(0.3999, 2.2935)
exponential	0.7181	0.7202	0.6898	0.7545	(0.0144, 1.4219)
Prediction		$\widehat{R}_5^S$	$\widehat{R}_5^L$ $(c = 0.5)$	$\widehat{R}_5^L$ $(c = -0.5)$	ABPI
xgamma		0.0148	0.0149	0.0149	(0.0007, 0.0292)
exponential		0.0149	0.0149	0.0150	(0.0007, 0.0292)

5.3. Example 2 (Real Data Set 2)

The second data set includes daily numbers of deaths due to the COVID-19 virus in Poland from 1 September 2020 to 1 October 2020; see the website of COVID-19 data: https://ourwordindata. org/coronavirus-source-data. The data are:

```
19, 20, 14, 8, 13, 7, 4, 12, 11, 12, 10, 13, 6, 15, 24, 10,
16, 17, 12, 11, 5, 18, 28, 25, 23, 32, 8, 15, 36, 30, 30.
```

Despite the fact that the daily numbers of deaths possess a discrete nature, several authors have fitted continuous distributions to this type of data sets; see for example El-Monsef et al. (2021). Now, if we observe the number of deaths on a specified day, then we may ask what the next lower number would be and become interested in predicting the next future lower record. Once again, we fitted the xgamma, exponential, Lindley and Shanker models to the above data and the related results have been given in Table 11. We see that the xgamma distribution fits the data best among the four considered models. Note that the exponential model does not fit the data significantly based on the K-S test at the level $\alpha = 0.05$. However, proceeding the same line of Example 1, we consider both xgamma and exponential models to analyse the record data. We observe that the lowest record occurred on 7 September 2020, so we can consider only the following data:

```
19, 20, 14, 8, 13, 7, 4.
```

The extracted lower records and the corresponding inter-record times from the above data are as follows:

i	1	2	3	4	5
r_i	19	14	8	7	4
k_i	2	1	2	1	1

We intend to predict the 5-th lower record value based on the first 4 observed records. Here, we have used informative priors whose prior means are taken to be the corresponding ML estimates approximately and the prior variance is equal to a value close to 0.4 (see Yadav et al., 2019). So, we obtain a prior with $(a,b) = (0.0702, 0.4255)$ for the xgamma distribution and another prior with $(a,b) = (0.0094, 0.1537)$ for the exponential model. We have calculated the ML and approximate Bayes point estimates, as well as the 95% interval estimates of the parameter for both the exponential and xgamma distributions. The point predictions and 95% ABPIs for the next future record, namely R_5, have been obtained as well. The numerical results of Example 2 have been given in Table 12. From Table 12, we see that all the approximate predictions are somehow close to 4, which is the true value of R_5. Besides, the ABPIs contain the true value of R_5.

Table 9: MLEs and goodness-of-fit statistics for Example 2.

Distribution	MLE	AIC	BIC	HQIC	K-S	p-value
xgamma	0.1702	221.2816	222.7156	221.749	0.1334	0.6393
exponential	0.0615	236.8925	238.3265	237.36	0.2658	0.0249
Lindley	0.1165	223.8098	225.2438	224.2773	0.1694	0.3358
Shanker	0.1218	221.9618	223.3958	222.4293	0.2177	0.1057

Table 10: The numerical results of Example 2.

Estimation	$\widehat{\theta}_{ML}$	$\widehat{\theta}_{SE}$	$\widehat{\theta}_{LE}$ $(c = 0.5)$	$\widehat{\theta}_{LE}$ $(c = -0.5)$	MATE CI
xgamma	0.1750	0.1705	0.1699	0.1711	(0.0780, 0.2719)
exponential	0.0533	0.0554	0.0552	0.0556	(0.0011, 0.1056)
Prediction		$\widehat{R}_5^S$	$\widehat{R}_5^L$ $(c = 0.5)$	$\widehat{R}_5^L$ $(c = -0.5)$	ABPI
xgamma		3.8454	2.8450	4.7229	(0.2050, 6.8675)
exponential		3.2757	2.3813	4.2395	(0.1453, 6.7882)

6. Concluding Remarks

Recently, the xgamma distribution, which is a flexible distribution for lifetime phenomena, has been introduced by Sen et al. (2016). In this paper, first, we derived the ML estimate of the xgamma parameter based on record values and the corresponding inter-record times. Then, we focused on the Bayesian estimation of the parameter, and we used a symmetric loss function as well as an asymmetric one. The Bayesian point estimates involve complicated integrals that do not seem to have closed forms, so we have used the TK, IS and M-H methods, to evaluate them. We have also become involved in predicting future records, as the prediction of future records has attracted the researchers' attention in applied situations.

A simulation study has been conducted in order to assess the point and interval estimators of the unknown parameter of the xgamma distribution and the point and interval predictors of a future lower record value. From the simulation study, we conclude that the number of observed records and the values of hyperparameters affect the performance of the estimators and predictors. Two real data sets have been analysed, where the first one includes the rainfall data. Here, a lower record value can be a warning about a future drought. The second data set involves the daily numbers of deaths in Poland due to the COVID-19 virus, where the lower records can show whether the virus can become under control or not. We compared the fits of the xgamma distribution with three other one-parameter lifetime distributions, and we observed the xgamma fitted both data sets well enough. Taking this information into account, we analysed the record data with the help of both the xgamma and exponential distributions through classical and Bayesian methods. We observe that the prediction results based on both the xgamma and exponential distributions are too close to each other for the rainfall data, and they are not much different from each other for the COVID-19 data. Besides, the obtained predicted values of the 5-th lower record are close to the true value of R_5 for the COVID-19 data, which confirms that the theoretical results of the paper may perform well in prediction. Summing up, we may conclude that the results of this paper may be useful in the estimation and prediction in real phenomena. All the computations of the paper were done using Maple 2016 and the statistical software R (R Core Team, 2020), and the packages coda (see Plummer et al., 2006, 2018), nleqslv (see Hasselman, 2018), truncnorm (see Mersmann et al., 2018) and AdequacyModel (see Marinho et al., 2013) therein.

Acknowledgements

We would like to express our sincere thanks to the Editor, Professor Patryk Barszcz, and the two anonymous referees for their valuable comments that highly improved the paper.

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