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Composite estimators for domain estimation and sensitivity performance interval of their weights

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Abstract

Some composite estimators based on various combinations of two different existing estimators are obtained for domain estimation. The estimation of weights and thus obtaining optimum weights to combine two or more different existing direct and indirect estimators to form composite estimators are not an easy task for practitioners due to many reasons. To account for the absence of optimum weights, we obtained the sensitivity performance intervals for weights with respect to the proposed composite estimator. Subsequently, we determined the sensible values of the involved weights. The aim of this procedure was to confine the superiority for different composite combinations i.e., simple direct vs. direct ratio, simple direct vs. synthetic ratio and direct ratio vs. synthetic ratio composite estimators as compared to the existing estimators.

Key words: domain estimation, synthetic and composite estimation, optimum weight, sensitivity performance interval.

1. Introduction

Generally, sample surveys are used as a cost-effective means for data collection but they are not able to provide estimates with competent precision for domains (subpopulations). Domains may be socio-demographic or geographic subdivision of the population for which separate estimates are required. Direct estimators perform better than synthetic estimators if the sample size is large for the domain while synthetic estimator is better in terms of mean square error (MSE) than direct estimator if the sample size is small for the domain along with the corresponding synthetic assumptions being satisfied, i.e., smaller area resembles larger area in their properties (Gonzalez, 1973). Further, the composite estimator is used, which is a weighted sum of two or more

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estimators having smaller MSE in comparison with MSE of either its component estimators. Evaluation of the optimal weight for the composite estimators is generally difficult and complex in domain estimation. One of the many causes is to involve population parameters in the formula used for its estimation.

Sometimes a difficulty occurs with the weights due to sampling frame problems, which results in some sampled elements being selected with less desired probabilities. The main purpose of weighting adjustments is to reduce the bias in the survey estimates that non-response and non-coverage can cause. Also, a challenging task in the construction of composite estimator is to set the weights of each input variable. Basically, an irritant that needs to be tackled lies in assuming the knowledge of the optimum value of the weighting factor which involves the population quantities. Thus, the main concern of the present article is to develop the performance intervals of weight which ensure the superiority of composite estimators as compared to its individual component estimators.

In the absence of optimum weights, we need an interval of weight with a view to maintaining the efficiency of the composite estimator as compared to its component estimators.In this direction many works are in progress while a very rich literature is available based on estimation of weights. Agrawal and Roy (1999) discussed the performance of efficient estimators of small domains. The generalized class of composite estimator is developed and analyzed by Tikkiwal and Ghiya (2004), including group of estimators which are convexly combined with weights. Further, Pandey and Tikkiwal (2006) also discussed the generalized class of composite estimators under Lahiri-Midzuno sampling scheme. Tikkiwal and Rai (2009) also proposed composite estimators and their sensitivity interval for small domains. King-Jong Lui (2020) discussed notes on the use of the composite estimator for improvement of the ratio estimator.

Here, in the present work we considered the situation of absence of optimum weights and thus obtained the sensitivity performance intervals for weights in respect to the proposed composite estimators and figured out sensible values of the involved weights with a view to confining superiority for different composite combinations.

2. Notations and Formulation of the Problem

Suppose a finite population U={1, 2, ..., i, ..., N} is divided into ‘A’ domains U_a having size N_a (a=1, …, A). We represent the study characteristic by ‘y’ and auxiliary characteristic by ‘x’. A random sample ‘s’ of size ‘n’ is drawn using simple random sampling without replacement (SRSWOR) from population U such that 'n_a' units in the sample 's' comes from domain U_a (a=1, …, A). We denote

$$\sum_{a=1}^{A} N_a = N \quad \text{and} \quad \sum_{a=1}^{A} n_a = n$$

Notations used are given as follows:

$\overline{X}$: Mean of the population based on 'N' observations of x.

$\overline{X}_a$: Mean of the domain 'a' based on 'N_a' observations of x.

$\overline{x}$: Mean of the sample 's' based on 'n' observations of x.

$\overline{x}_a$: Mean of the sample of domain 'a' based on 'n_a' observations of x.

$\overline{Y}$: Mean of the population based on 'N' observations of y.

$\overline{Y}_a$: Mean of the domain 'a' based on 'N_a' observations of y.

$\overline{y}$: Mean of the sample 's' based on 'n' observations of y.

$\overline{y}_a$: Mean of the sample of domain 'a' based on 'n_a' observations of y.

Let X_{ai} (a=1, …, A; i=1, …, N_a) denote the i[th] observation of a[th] domain for the characteristic x and Y_{ai} (a=1, …, A; i=1, …, N_a) denote the i[th] observation of a[th] domain for the characteristic y. The corresponding various mean squares and coefficient of variations of domain U_a for direct estimators for study and auxiliary characteristics are given as follows:

$$S'^2_{x_a} = \frac{1}{(N_a - 1)} \sum_{i=1}^{N_a} (X_{ai} - \overline{X}_a)^2 \text{ , } C'_{x_a} = \frac{S'_{x_a}}{\overline{X}_a}$$

$$S'^2_{y_a} = \frac{1}{(N_a - 1)} \sum_{i=1}^{N_a} (Y_{ai} - \overline{Y}_a)^2 \text{ , } C'_{y_a} = \frac{S'_{y_a}}{\overline{Y}_a}$$

$$S'_{x_a y_a} = \frac{1}{(N_a - 1)} \sum_{i=1}^{N_a} (X_{ai} - \overline{X}_a)(Y_{ai} - \overline{Y}_a) \text{ , } C'_{x_a y_a} = \frac{S'_{x_a y_a}}{\overline{X}_a \overline{Y}_a}$$

The corresponding various mean squares and coefficient of variations of domain U_a for synthetic estimators are given as follows:

$$S^2_x = \frac{1}{(N-1)} \sum_{i=1}^{N} (X_i - \overline{X}_a)^2 \text{ , } C_x = \frac{S_x}{\overline{X}_a}$$

$$S^2_y = \frac{1}{(N-1)} \sum_{i=1}^{N} (Y_i - \overline{Y}_a)^2 \text{ , } C_y = \frac{S_y}{\overline{Y}_a}$$

$$S_{xy} = \frac{1}{(N-1)} \sum_{i=1}^{N} (X_i - \overline{X}_a)(Y_i - \overline{Y}_a) \text{ , } C_{xy} = \frac{S_{xy}}{\overline{X}_a \overline{Y}_a}$$

3. Domain Estimator under Study

As we have discussed, separate estimates are required for the domain under study. There are different direct and indirect methods of estimation for the study of domain of interest. For our case, we consider the composite estimators for the estimation of domains.

3.1. Composite Estimators

The following three cases of composite estimators for a[th] domain are considered:

(i) Simple direct estimator with direct ratio estimator

$$\bar{y}_{c,a(1)} = \omega \bar{y}_{d,a} + (1-\omega)\bar{y}_{d,r,a}$$

where $\bar{y}_{d,a}$ = simple direct estimator and $\bar{y}_{d,r,a}$ = direct ratio estimator.

Here, the bias and MSE terms of $\bar{y}_{d,a}$ and $\bar{y}_{d,r,a}$ can be obtained as,

$$Bias(\bar{y}_{d,a}) = 0 \tag{3.1.1}$$

$$MSE(\bar{y}_{d,a}) = \bar{Y}_a^2 \frac{(N_a - n_a)}{(N_a n_a)} \frac{S'^2_{y_a}}{\bar{Y}_a^2} \tag{3.1.2}$$

$$Bias(\bar{y}_{d,r,a}) = \bar{Y}_a \frac{(N_a - n_a)}{(N_a n_a)} \left[\frac{S'^2_{x_a}}{\bar{X}_a^2} - \frac{S'_{x_a y_a}}{\bar{X}_a \bar{Y}_a} \right] \tag{3.1.3}$$

$$MSE(\bar{y}_{d,r,a}) = \bar{Y}_a^2 \frac{(N_a - n_a)}{(N_a n_a)} \left[\frac{S'^2_{x_a}}{\bar{X}_a^2} + \frac{S'^2_{y_a}}{\bar{Y}_a^2} - 2\frac{S'_{x_a y_a}}{\bar{X}_a \bar{Y}_a} \right] \tag{3.1.4}$$

(ii) Simple direct estimator with synthetic ratio estimator

$$\bar{y}_{c,a(2)} = \omega \bar{y}_{d,a} + (1-\omega)\bar{y}_{syn,r,a}$$

where $\bar{y}_{d,a}$ = simple direct estimator and $\bar{y}_{syn,r,a}$ = synthetic ratio estimator.

The bias and MSE of $\bar{y}_{syn,r,a}$ will be obtained as,

$$Bias(\bar{y}_{syn,r,a}) = \bar{Y}_a \left(\frac{N-n}{Nn} \right) \left[\frac{S_x^2}{\bar{X}_a^2} - \frac{S_{xy}}{\bar{X}_a \bar{Y}_a} \right] \tag{3.1.5}$$

$$MSE(\bar{y}_{syn,r,a}) = \bar{Y}_a^2 \left(\frac{N-n}{Nn} \right) \left[\frac{S_x^2}{\bar{X}_a^2} + \frac{S_y^2}{\bar{Y}_a^2} - 2\frac{S_{xy}}{\bar{X}_a \bar{Y}_a} \right] \tag{3.1.6}$$

(iii) Direct ratio estimator with synthetic ratio estimator

$$\bar{y}_{c,a(3)} = \omega \bar{y}_{d,r,a} + (1-\omega)\bar{y}_{syn,r,a}$$

where $\overline{y}_{d,r,a}$ = direct ratio estimator and $\overline{y}_{syn,r,a}$ = synthetic ratio estimator. The bias and MSE terms of $\overline{y}_{d,r,a}$ and $\overline{y}_{syn,r,a}$ have been already mentioned above.

3.2. Performance Intervals for Weight ω

Let us consider composite estimator t_3 as a linear combination of components t_1 and t_2 i.e.,

$$t_3 = \omega_1 t_1 + \omega_2 t_2 = \omega t_1 + (1-\omega)t_2 \tag{3.2.1}$$

Here $\omega_1 + \omega_2 = 1$, where $\omega_1 = \omega$ and $\omega_2 = 1-\omega$; ω is the assigned weight.

For better performing interval of composite estimator t_3, MSE(t_3) is less than equal to either of MSE(t_1) or MSE(t_2).Now, we have two conditions, the first one is:

$$MSE(t_3) \le MSE(t_1) \tag{3.2.2}$$

$$\Rightarrow MSE\{\omega t_1 + (1-\omega)t_2\} \le MSE(t_1)$$

$$\Rightarrow \omega^2 MSE(t_1) + (1-\omega)^2 MSE(t_2) + 2\omega(1-\omega)c\,\text{ov}(t_1,t_2) \le MSE(t_1)$$

$$\left[\because MSE\{\omega t_1 + (1-\omega)t_2\} = \omega^2 MSE(t_1) + (1-\omega)^2 MSE(t_2) + 2\omega(1-\omega)c\,\text{ov}(t_1,t_2); (\text{Rao, 2003})\right]$$

$$\Rightarrow \omega^2 MSE(t_1) + MSE(t_2) + \omega^2 MSE(t_2) - 2\omega MSE(t_2) + 2\omega cov(t_1,t_2) - 2\omega^2 \text{cov}(t_1,t_2) \le MSE(t_1)$$

$$\Rightarrow \omega^2 \{MSE(t_1) + MSE(t_2) - 2\,\text{cov}(t_1,t_2)\} - 2\omega\{MSE(t_2) - \text{cov}(t_1,t_2)\} + \{MSE(t_2) - MSE(t_1)\} \le 0$$

On solving the above quadratic equation and assuming that the covariance term is small relative to MSE(t_2),we get,

$$\omega = \frac{MSE(t_2) + MSE(t_1)}{MSE(t_2) + MSE(t_1)} = 1 \quad \text{or} \quad \omega = \frac{MSE(t_2) - MSE(t_1)}{MSE(t_2) + MSE(t_1)}$$

As 1 is an integer value of ω, we take the other values of ω as,

$$\omega \ge \frac{MSE(t_2) - MSE(t_1)}{MSE(t_2) + MSE(t_1)} \tag{3.2.3}$$

Again, the second condition is:

$$MSE(t_3) \le MSE(t_2) \tag{3.2.4}$$

Similarly, on solving equation (3.2.4), we get,

$$\omega \le \frac{2MSE(t_2)}{MSE(t_1) + MSE(t_2)} \tag{3.2.5}$$

So, the better performing interval of t_3 estimator is given as,

$$\frac{MSE(t_2) - MSE(t_1)}{MSE(t_2) + MSE(t_1)} \le \omega \le \frac{2MSE(t_2)}{MSE(t_1) + MSE(t_2)} \tag{3.2.6}$$

Now, let us consider the three cases for t_1 and t_2 estimators as discussed before in previous Section 3.1, as follows:

(i) Simple direct estimator with direct ratio estimator

$$\bar{y}_{c,a(1)} = \omega\bar{y}_{d,a} + (1-\omega)\bar{y}_{d,r,a}, \text{ where } t_1 = \bar{y}_{d,a} \text{ and } t_2 = \bar{y}_{d,r,a}.$$

Putting the formulae of MSE from the expressions (3.1.2) and (3.1.4)in the expression of the left-hand part and right-hand part of (3.2.6),we get,

$$\frac{\left[\frac{S'^2_{x_a}}{\bar{X}^2_a} - 2\frac{S'_{x_a y_a}}{\bar{X}_a\bar{Y}_a}\right]}{\left[\frac{S'^2_{x_a}}{\bar{X}^2_a} + 2\frac{S'^2_{y_a}}{\bar{Y}^2_a} - 2\frac{S'_{x_a y_a}}{\bar{X}_a\bar{Y}_a}\right]} \le \omega \le \frac{2\left[\frac{S'^2_{x_a}}{\bar{X}^2_a} + \frac{S'^2_{y_a}}{\bar{Y}^2_a} - 2\frac{S'_{x_a y_a}}{\bar{X}_a\bar{Y}_a}\right]}{\left[\frac{S'^2_{x_a}}{\bar{X}^2_a} + 2\frac{S'^2_{y_a}}{\bar{Y}^2_a} - 2\frac{S'_{x_a y_a}}{\bar{X}_a\bar{Y}_a}\right]} \quad (3.2.7)$$

(ii) Simple direct estimator with synthetic ratio estimator

$$\bar{y}_{c,a(2)} = \omega\bar{y}_{d,a} + (1-\omega)\bar{y}_{syn,r,a}, \text{ where } t_1 = \bar{y}_{d,a} \text{ and } t_2 = \bar{y}_{syn,r,a}.$$

After putting the MSE expressions, the expression (3.2.6) provides,

$$\frac{\left(\frac{N-n}{Nn}\right)\left[\frac{S^2_x}{\bar{X}^2_a} + \frac{S^2_y}{\bar{Y}^2_a} - 2\frac{S_{xy}}{\bar{X}_a\bar{Y}_a}\right] - \left(\frac{N_a-n_a}{N_a n_a}\right)\frac{S'^2_{y_a}}{\bar{Y}^2_a}}{\left(\frac{N-n}{Nn}\right)\left[\frac{S^2_x}{\bar{X}^2_a} + \frac{S^2_y}{\bar{Y}^2_a} - 2\frac{S_{xy}}{\bar{X}_a\bar{Y}_a}\right] + \left(\frac{N_a-n_a}{N_a n_a}\right)\frac{S'^2_{y_a}}{\bar{Y}^2_a}} \le \omega \le \frac{2\left(\frac{N-n}{Nn}\right)\left[\frac{S^2_x}{\bar{X}^2_a} + \frac{S^2_y}{\bar{Y}^2_a} - 2\frac{S_{xy}}{\bar{X}_a\bar{Y}_a}\right]}{\left(\frac{N-n}{Nn}\right)\left[\frac{S^2_x}{\bar{X}^2_a} + \frac{S^2_y}{\bar{Y}^2_a} - 2\frac{S_{xy}}{\bar{X}_a\bar{Y}_a}\right] + \left(\frac{N_a-n_a}{N_a n_a}\right)\frac{S'^2_{y_a}}{\bar{Y}^2_a}} \quad (3.2.8)$$

(iii) Direct ratio estimator with synthetic ratio estimator

$$\bar{y}_{c,a(3)} = \omega\bar{y}_{d,r,a} + (1-\omega)\bar{y}_{syn,r,a}, \text{ where } t_1 = \bar{y}_{d,r,a} \text{ and } t_2 = \bar{y}_{syn,r,a}.$$

The MSE of $\bar{y}_{d,r,a}$ and $\bar{y}_{syn,r,a}$ are given by expressions (3.1.4) and (3.1.6) respectively. Thus, expression (3.2.6) provides,

$$\frac{\left(\frac{N-n}{Nn}\right)\left[\frac{S^2_x}{\bar{X}^2_a} + \frac{S^2_y}{\bar{Y}^2_a} - 2\frac{S_{xy}}{\bar{X}_a\bar{Y}_a}\right] - \left(\frac{N_a-n_a}{N_a n_a}\right)\left[\frac{S'^2_{x_a}}{\bar{X}^2_a} + \frac{S'^2_{y_a}}{\bar{Y}^2_a} - 2\frac{S'_{x_a y_a}}{\bar{X}_a\bar{Y}_a}\right]}{\left(\frac{N-n}{Nn}\right)\left[\frac{S^2_x}{\bar{X}^2_a} + \frac{S^2_y}{\bar{Y}^2_a} - 2\frac{S_{xy}}{\bar{X}_a\bar{Y}_a}\right] + \left(\frac{N_a-n_a}{N_a n_a}\right)\left[\frac{S'^2_{x_a}}{\bar{X}^2_a} + \frac{S'^2_{y_a}}{\bar{Y}^2_a} - 2\frac{S'_{x_a y_a}}{\bar{X}_a\bar{Y}_a}\right]} \le \omega \le$$

$$\frac{2\left(\frac{N-n}{Nn}\right)\left[\frac{S^2_x}{\bar{X}^2_a} + \frac{S^2_y}{\bar{Y}^2_a} - 2\frac{S_{xy}}{\bar{X}_a\bar{Y}_a}\right]}{\left(\frac{N-n}{Nn}\right)\left[\frac{S^2_x}{\bar{X}^2_a} + \frac{S^2_y}{\bar{Y}^2_a} - 2\frac{S_{xy}}{\bar{X}_a\bar{Y}_a}\right] + \left(\frac{N_a-n_a}{N_a n_a}\right)\left[\frac{S'^2_{x_a}}{\bar{X}^2_a} + \frac{S'^2_{y_a}}{\bar{Y}^2_a} - 2\frac{S'_{x_a y_a}}{\bar{X}_a\bar{Y}_a}\right]} \quad (3.2.9)$$

3.3. Sensitivity Performance Intervals for Weight ω

Let us consider 'P' as the proportional inflation in the *MSE* of t_3 due to use of some ω other than $\omega_{opt.}$,i.e.

$$P = \frac{MSE(t_3) - MSE_{opt.}(t_3)}{MSE_{opt.}(t_3)} \tag{3.3.1}$$

For the sake of convenience neglecting the covariance term which does not hampered the equation and on substituting the formula of MSE of t_3 under ω and $\omega_{opt.}$, we get:

$$P = \frac{\left\{\omega^2 MSE(t_1) + (1-\omega)^2 MSE(t_2)\right\} - \left\{\omega_{opt.}{}^2 MSE(t_1) + (1-\omega_{opt.})^2 MSE(t_2)\right\}}{\left\{\omega_{opt.}{}^2 MSE(t_1) + (1-\omega_{opt.})^2 MSE(t_2)\right\}} \tag{3.3.2}$$

Divide numerator and denominator by $(1-\omega_{opt.})^2$ and taking $(1-\omega)^2$ common from first term of numerator, we have;

$$P = \left(\frac{1-\omega}{1-\omega_{opt.}}\right)^2 P_1 - 1 \tag{3.3.3}$$

where,

$$P_1 = \frac{\left(\frac{\omega}{1-\omega}\right)^2 MSE(t_1) + MSE(t_2)}{\left(\frac{\omega_{opt.}}{1-\omega_{opt.}}\right)^2 MSE(t_1) + MSE(t_2)}$$

As *P* is a ratio of two positive quantity (as numerator and denominator of *P* are positive quantity) so, $P \geq 0$, which implies

$$P_1 \geq \left(\frac{1-\omega_{opt.}}{1-\omega}\right)^2 \tag{3.3.4}$$

Therefore,

$$\frac{\left(\frac{\omega}{1-\omega}\right)^2 MSE(t_1) + MSE(t_2)}{\left(\frac{\omega_{opt.}}{1-\omega_{opt.}}\right)^2 MSE(t_1) + MSE(t_2)} \geq \left(\frac{1-\omega_{opt.}}{1-\omega}\right)^2 \tag{3.3.5}$$

On simplifying equation (3.3.5), we have;

$$(\omega + \omega_{opt.}) \geq \frac{2MSE(t_2)}{MSE(t_1) + MSE(t_2)} \tag{3.3.6}$$

Now, we have to find the optimum weight for composite estimator t_3.

$$t_3 = \omega t_1 + (1-\omega) t_2$$

$$MSE(t_3) = \omega^2 MSE(t_1) + (1-\omega)^2 MSE(t_2) + 2\omega(1-\omega)\operatorname{cov}(t_1, t_2) \tag{3.3.7}$$

On differentiating eq. (3.3.7) with respect to ω and equating it to zero after neglecting the covariance term, assuming that the covariance term is relatively small, we get,

$$\omega_{opt.} = \frac{MSE(t_2)}{MSE(t_1)+MSE(t_2)} \tag{3.3.8}$$

Using (3.3.6) and (3.3.8), we have,

$$\omega \geq \frac{MSE(t_2)}{MSE(t_1)+MSE(t_2)} \tag{3.3.9}$$

Thus, the sensitivity performance interval for ω is given as:

$$\frac{MSE(t_2)}{MSE(t_1)+MSE(t_2)} \leq \omega \leq \frac{2MSE(t_2)}{MSE(t_1)+MSE(t_2)} \tag{3.3.10}$$

Now, the sensitivity performance interval of the involved weight for the above three composite estimators as discussed before in previous Section 3 are given as follows:

(i) Simple direct estimator with direct ratio estimator

$\bar{y}_{c,a(1)} = \omega\bar{y}_{d,a} + (1-\omega)\bar{y}_{d,r,a}$, where $t_1 = \bar{y}_{d,a}$ and $t_2 = \bar{y}_{d,r,a}$.

Here, the expression of sensitivity interval is obtained as

$$\frac{\left[\frac{S'^2_{x_a}}{\bar{X}^2_a} + \frac{S'^2_{y_a}}{\bar{Y}^2_a} - 2\frac{S'_{x_a y_a}}{\bar{X}_a\bar{Y}_a}\right]}{\left[\frac{S'^2_{x_a}}{\bar{X}^2_a} + 2\frac{S'^2_{y_a}}{\bar{Y}^2_a} - 2\frac{S'_{x_a y_a}}{\bar{X}_a\bar{Y}_a}\right]} \leq \omega \leq \frac{2\left[\frac{S'^2_{x_a}}{\bar{X}^2_a} + \frac{S'^2_{y_a}}{\bar{Y}^2_a} - 2\frac{S'_{x_a y_a}}{\bar{X}_a\bar{Y}_a}\right]}{\left[\frac{S'^2_{x_a}}{\bar{X}^2_a} + 2\frac{S'^2_{y_a}}{\bar{Y}^2_a} - 2\frac{S'_{x_a y_a}}{\bar{X}_a\bar{Y}_a}\right]} \tag{3.3.11}$$

(ii) Simple direct estimator with synthetic ratio estimator

$\bar{y}_{c,a(2)} = \omega\bar{y}_{d,a} + (1-\omega)\bar{y}_{syn,r,a}$, where $t_1 = \bar{y}_{d,a}$ and $t_2 = \bar{y}_{syn,r,a}$

The sensitivity performance interval for ω in this case is obtained as

$$\frac{\left(\frac{N-n}{Nn}\right)\left[\frac{S^2_x}{\bar{X}^2_a} + \frac{S^2_y}{\bar{Y}^2_a} - 2\frac{S_{xy}}{\bar{X}_a\bar{Y}_a}\right]}{\left(\frac{N-n}{Nn}\right)\left[\frac{S^2_x}{\bar{X}^2_a} + \frac{S^2_y}{\bar{Y}^2_a} - 2\frac{S_{xy}}{\bar{X}_a\bar{Y}_a}\right] + \left(\frac{N_a-n_a}{N_a n_a}\right)\frac{S'^2_{y_a}}{\bar{Y}^2_a}} \leq \omega \leq \frac{2\left(\frac{N-n}{Nn}\right)\left[\frac{S^2_x}{\bar{X}^2_a} + \frac{S^2_y}{\bar{Y}^2_a} - 2\frac{S_{xy}}{\bar{X}_a\bar{Y}_a}\right]}{\left(\frac{N-n}{Nn}\right)\left[\frac{S^2_x}{\bar{X}^2_a} + \frac{S^2_y}{\bar{Y}^2_a} - 2\frac{S_{xy}}{\bar{X}_a\bar{Y}_a}\right] + \left(\frac{N_a-n_a}{N_a n_a}\right)\frac{S'^2_{y_a}}{\bar{Y}^2_a}} \tag{3.3.12}$$

(iii) Direct ratio estimator with synthetic ratio estimator

$$\bar{y}_{c,a(3)} = \omega \bar{y}_{d,r,a} + (1-\omega)\bar{y}_{syn,r,a} \text{ , where } t_1 = \bar{y}_{d,r,a} \text{ and } t_2 = \bar{y}_{syn,r,a} .$$

Here, the sensitivity performance interval will be obtained as

$$\frac{\left(\frac{N-n}{Nn}\right)\left[\frac{S_x^2}{\bar{X}_a^2}+\frac{S_y^2}{\bar{Y}_a^2}-2\frac{S_{xy}}{\bar{X}_a\bar{Y}_a}\right]}{\left(\frac{N-n}{Nn}\right)\left[\frac{S_x^2}{\bar{X}_a^2}+\frac{S_y^2}{\bar{Y}_a^2}-2\frac{S_{xy}}{\bar{X}_a\bar{Y}_a}\right]+\left(\frac{N_a-n_a}{N_a n_a}\right)\left[\frac{S_{x_a}'^2}{\bar{X}_a^2}+\frac{S_{y_a}'^2}{\bar{Y}_a^2}-2\frac{S_{x_a y_a}'}{\bar{X}_a\bar{Y}_a}\right]} \le \omega \le$$

$$\frac{2\left(\frac{N-n}{Nn}\right)\left[\frac{S_x^2}{\bar{X}_a^2}+\frac{S_y^2}{\bar{Y}_a^2}-2\frac{S_{xy}}{\bar{X}_a\bar{Y}_a}\right]}{\left(\frac{N-n}{Nn}\right)\left[\frac{S_x^2}{\bar{X}_a^2}+\frac{S_y^2}{\bar{Y}_a^2}-2\frac{S_{xy}}{\bar{X}_a\bar{Y}_a}\right]+\left(\frac{N_a-n_a}{N_a n_a}\right)\left[\frac{S_{x_a}'^2}{\bar{X}_a^2}+\frac{S_{y_a}'^2}{\bar{Y}_a^2}-2\frac{S_{x_a y_a}'}{\bar{X}_a\bar{Y}_a}\right]} \tag{3.3.13}$$

4. Numerical Illustration

We consider the data from Sarndal et al. (1992), Appendix B of Sweden Municipalities which are classified into eight geographical regions. We take all eight geographical regions for the study purpose with sizes 25, 48, 32, 38, 56, 41, 15 and 29 respectively and select a sample of 5, 10, 6, 8, 11, 8, 3, 6sampling units from each domain respectively. We take the study variable 'y' as RMT85 (Revenues from the 1985 municipal taxation (in millions of kronor)) and the auxiliary variable 'x' as P85 (1985 population (in thousands)). The performance intervals for weight derived in equations (3.2.7), (3.2.8), (3.2.9)and sensitivity performance intervals for weight derived in equations (3.3.11), (3.3.12), (3.3.13)are presented in Table 4.1 and 4.2 respectively.

Table 4.1: Performance intervals for weight of three different composite estimators

Domain	Simple Direct with Direct Ratio $\bar{y}_{c,a(1)}$	Simple Direct with Synthetic Ratio $\bar{y}_{c,a(2)}$	Direct Ratio with Synthetic Ratio $\bar{y}_{c,a(3)}$
	$\omega_1 \le \omega \le \omega_2$	$\omega_1 \le \omega \le \omega_2$	$\omega_1 \le \omega \le \omega_2$
1	[-0.9843,0.0157]	[-0.9950,0.0050]	[-0.5191,0.4808]
2	[-0.9867,0.0132]	[-0.8200,0.1799]	[0.8737,1.8737]
3	[-0.8767,0.1232]	[-0.6234,0.3765]	[0.5587,1.5587]
4	[-0.6600,0.3399]	[-0.9626,0.0374]	[-0.8297,0.1703]
5	[-0.6903,0.3097]	[-0.9818,0.0182]	[-0.9043,0.0957]
6	[-0.9712,0.0288]	[-0.4356,0.5644]	[0.9284,1.9284]
7	[-0.9748,0.0252]	[-0.8565,0.1435]	[0.7166,1.7166]
8	[-0.9703,0.0297]	[-0.7278,0.2722]	[0.8253,1.8253]

Table 4.2: Sensitivity performance intervals for weight of three composite estimators

Domain	Simple Direct with Direct Ratio $\overline{y}_{c,a(1)}$	Simple Direct with Synthetic Ratio $\overline{y}_{c,a(2)}$	Direct Ratio with Synthetic Ratio $\overline{y}_{c,a(3)}$
	$\omega_1 \leq \omega \leq \omega_2$	$\omega_1 \leq \omega \leq \omega_2$	$\omega_1 \leq \omega \leq \omega_2$
1	[0.0078,0.0157]	[0.0025,0.0050]	[0.2404,0.4808]
2	[0.0066,0.0132]	[0.0899,0.1799]	[0.9368,1.8737]
3	[0.0616,0.1232]	[0.1882,0.3765]	[0.7793,1.5587]
4	[0.1699,0.3399]	[0.0187,0.0374]	[0.0852,0.1703]
5	[0.1548,0.3097]	[0.0091,0.0182]	[0.0478,0.0957]
6	[0.0144,0.0288]	[0.2822,0.5644]	[0.9642,1.9284]
7	[0.0126,0.0252]	[0.0718,0.1435]	[0.8583, 1.7166]
8	[0.0148,0.0297]	[0.1361,0.2722]	[0.9126,1.8253]

From the above two tables we see that the performance intervals for weight of $\overline{y}_{c,a(1)}, \overline{y}_{c,a(2)}$ and $\overline{y}_{c,a(3)}$ are ranging from -0.9867 to 0.3399, -0.9950 to 0.5644 and -0.9043 to 1.9284 respectively. It means all three composite estimators retain its superiority for values of ω ranging from -0.9950 to 1.9284. Also, we observe that the length of the performance intervals for weight of composite estimators is one which follows from the expression (3.2.6). Table 4.2 clearly shows that the sensitivity performance interval for weight of composite estimators lies between 0.0025 to 1.9284.

5. Conclusions

Composite estimators provide efficient estimates for the unknown population parameters as compared to their constituent estimators. The estimation of weights in the composite estimators are not easy task and due to this reason, this is not a popular estimator among users and practitioners. Here, in the present study an effort is made to get sensitivity performance intervals of the weight that guarantee the superiority of the proposed composite estimator with respect to its component estimators in the field of domain estimation also.

From the above analysis of three different composite estimators, we obtain the performance intervals of weight which ensure supremacy of composite estimators as compared to their component estimators. As an example, we show that the combination of direct ratio estimator with synthetic ratio estimator performs better within performance intervals obtained in Table 4.1in terms of MSE. It is also concluded that the composite estimators for the weights lie in the sensitivity performance intervals are less varying in terms of MSE. The outcomes of the study will be useful to develop efficient composite estimators for the domain estimation in general and for small area estimation in particular.

References

Agrawal, M. C., Roy, D. C., (1999). Efficient Estimators for Small Domains. *Journal of the Indian Society of Agricultural Statistics*, 52(3), pp. 327–337.

Ghosh, M., Rao, J. N. K., (1994). Small Area Estimation: An Appraisal. *Statistical Science*, 91, pp. 55–93.

Gonzalez, M. E., (1973). Use and Evaluation of Synthetic Estimates. *Proceedings of the Social Statistical Section of American Statistical Association*, pp. 33–36.

Hedayat, A. S., Sinha, B. K., (1991). Design and Inference in Finite Population Sampling. *John Wiley & Sons, New York.*

Holt, D., Smith T. M. F., and Tomberlin, T. J., (1979). A Model-Based Approach to Estimation for Small Subgroups of a Population. *Journal of the American Statistical Association*, 74, pp. 405–410.

Kalton, G., Flores-Cervantes, I., (2003). Weighting Methods. *Journal of Official Statistics*, 19(2), pp. 81–97.

L. Kung-Jong, (2020). Notes on Use of the Composite Estimator: an Improvement of the Ratio Estimator. *Journal of Official Statistics*, 36(1), pp. 137–149.

Pandey, K. K., (2010). Aspects of Small Area Estimation Using Auxiliary Information. *VDM Verlag Dr.Muller e.K.*

Pandey, P. S., Kathuria, O. P., (1995). Some Composite Estimators for Small Area Estimation. *Journal of the Indian Society of Agricultural Statistics*, 47(3), pp. 262–272.

Platek, R., Rao, J. N. K., Sarndal, C. E., and Singh, M. P., (1981). Small Area Statistics: An International Symposium. *John Wiley & Sons, New York.*

Purcell, N. J., Kish, L., (1979). Estimation for Small Domains. *Biometrics*, 35, pp. 365–384.

Rai, P. K., Pandey K. K., (2013). Synthetic Estimators Using Auxiliary Information in Small Domains, *Statistics in Transition-new series*, 14(1), pp. 31–44.

Rao, J. N. K., (2003). Small Area Estimation. *Wiley Inter-Science, New Jersey, John Wiley and Sons.*

Sarndal, C. E., Swensson, B., and Wretman, J., (1992). Model assisted survey sampling. *Springer-Verlag.*

Schaible, W. L., (1979). A Composite Estimator for Small Area Statistics. *Synthetic Estimates for Small Areas:Statistical Workshop Papers and Discussion*, National Institute on Drug Abuse (NIDA) Research Monograph 24, pp. 36–53.

Tikkiwal, G .C., Ghiya, A., (2004). A generalized Class of Composite Estimators with Application to Crop Acreage Estimation for Small Domains. *Statistics in Transition*, 6, pp. 697–711.

Tikkiwal, G. C., Rai, P. K., (2009). A Composite Estimator for Small Domains and its Sensitivity Interval for Weights α. *Statistics in Transition*, 10(2), pp. 269–275.