STATISTICS IN TRANSITION new series, September 2025
Vol. 26, No. 3, pp. 59–79, https://doi.org/10.59139/stattrans-2025-027
Received – 30.12.2022; accepted – 30.12.2024

Bayesian estimation of two-parameter power Rayleigh distribution and its application

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Abstract

This paper explores classical and Bayesian approaches to the estimation of unknown parameters and reliability functions for the power Rayleigh distribution. The maximum likelihood estimator (MLE) method is considered in classical estimation. The Bayesian estimation, on the other hand uses several loss functions under informative and non-informative prior distributions, utilizing the Lindley technique and Markov chain Monte Carlo (MCMC) methods for Bayesian computations. Approximate confidence intervals are established based on the MLEs using the delta technique, while Bayes credible intervals are determined using the MCMC method. A simulation study is conducted to compare the performance of these methods in terms of biases and mean square errors, revealing that Bayesian estimators outperform their classical counterparts. Additionally, two real datasets are presented for illustrative purposes.

Key words: Power Rayleigh distribution, delta method, Lindley approximation, Metropolis-Hasting algorithm, highest posterior density credible intervals, Monte Carlo simulation, coverage probability, goodness of fit.

1. Introduction

Parameter estimation is a fundamental aspect of statistics, playing a crucial role in various statistical analyses and decision-making processes. Estimating parameters in statistical distributions involves two primary methodologies: frequentist and Bayesian. In the frequentist paradigm, estimates are derived from observed data, treating parameters as fixed but unknown values. A common technique within this framework is maximum likelihood estimation (MLE), where parameter values are selected to maximize the likelihood function. In contrast, the Bayesian approach treats parameters as random variables with associated probability distributions, acknowledging the inherent uncertainty. Bayesian parameter estimation combines prior beliefs about parameters with observed data using Bayes' theorem, resulting in a posterior distribution that reflects updated knowledge. While the Bayesian approach provides a systematic means to incorporate prior information and adapt beliefs as more data becomes available, it necessitates specifying a prior distribution, introducing subjectivity that may impact results. Several authors have used different lifetime models to

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study parametric inference under frequentist and Bayesian approaches. For example, Soliman (2000), Shuo-Jye-Wu *et al.* (2006), Soliman and Al-Aboud (2008), Dey (2009), Khan *et al.* (2010), Ahmad *et al.* (2013), Asgharzadeh and Azizpour (2016), Ghazal and Hasaballah (2017), Talukdar (2019), Yilmaz and Kara (2022), Irfan and Sharma (2023), and Irfan and Sharma (2024) have considered different distributions for parameter estimation under classical and Bayesian approaches. Some of them also discussed life testing and reliability estimates under various loss functions.

The Rayleigh distribution, a statistical model, stands out for its unique application in representing the magnitudes of vector components featuring random amplitudes. Named after the British scientist Lord Rayleigh, this distribution finds widespread use in diverse domains like wireless communication, radar systems, and image processing. The Rayleigh distribution is a crucial model used in reliability theory, survival analysis, physical sciences, medical imaging, and various branches of engineering. This study focuses on the power Rayleigh distribution (PRD). PRD is the extension of the Rayleigh distribution and was introduced by Bhatt and Ahmad (2020). They also studied some exciting properties like moment-generating function, hazard rate, mean residual life, order statistics, quantiles, etc. PRD offers flexibility for modelling the real-life dataset with a long-tailed, right-skewed curve. Due to the wide practical utility of the PRD, various scholars have studied it for different purposes. For example, Mahmoud *et al.* (2020) discussed the lifetime performance index of PRD under progressive first failure censored data. Kilany *et al.* (2023) obtained the classical estimates for PRD under a complete sample with COVID-19 application. Migdadi *et al.* (2023) derived the Bayes estimates of the parameters of PRD under adaptive type II progressive censored sample. Migdadi *et al.* (2023) discussed the optimal design for the PRD under censored sample for the k-level step-stress accelerated life test. Further, Migdadi *et al.* (2023) obtained the Bayesian and classical estimates of PRD under joint progressive censoring scheme.

This study estimates the parameters and reliability characteristics of PRD using maximum likelihood and derives approximate confidence intervals (ACIs) via the Fisher information matrix. In the Bayesian framework, Bayes estimators under SELF, GELF, and LLF are computed using informative and non-informative priors. Since closed-form solutions are unavailable, we use MCMC and Lindley approximation methods, with Bayesian credible intervals (BCIs) obtained via the Metropolis-Hastings algorithm. A Monte Carlo simulation evaluates the methods, and two real datasets illustrate their application.

The novelty of this paper comes from the fact that no previous study has been found on the reliability and parameter estimation for PRD in the Bayesian context.

The rest of the paper is structured as follows: Section 2 provides an overview of the power Rayleigh distribution. Section 3 discusses the frequentist method for estimating unknown parameters. Section 4 focuses on obtaining Bayes estimators for the unknown model parameters, employing Lindley's method, and reliability estimation through different loss functions. Additionally, Section 4 incorporates MCMC methods. Section 5 presents a simulation study to assess the performance of the estimators in terms of biases and mean square errors (MSE). Section 6 applies two real datasets for practical demonstration and application purposes. Section 7 concludes the work with some suggestions for future research in this field.

2. Power Rayleigh Distribution

Let Y be a random variable that follows Rayleigh distribution with parameter ν whose cumulative distribution function (CDF) and probability distribution function (PDF) are, respectively, given by

$$F(y) = 1 - exp\left(-\frac{y^2}{2\nu^2}\right); y > 0, \nu > 0, \tag{1}$$

and

$$f(y) = \frac{y}{\nu^2} exp\left(-\frac{y^2}{2\nu^2}\right); y > 0, \nu > 0. \tag{2}$$

Now, the transformation $X = Y^{\frac{1}{\tau}}$ will follow power Rayleigh distribution (PRD) with parameter ν and τ be obtained as

$$\begin{aligned} F(x) &= p(X \leq x) \\ &= p(x^\tau). \end{aligned} \tag{3}$$

Using equations (1), (2), and (3), the cumulative distribution function (CDF) and probability distribution function (PDF) of PRD are, respectively, given as

$$F(x) = 1 - exp\left(-\frac{x^{2\tau}}{2\nu^2}\right), x > 0, \nu, \tau > 0, \tag{4}$$

and

$$f(x) = \frac{\tau}{\nu^2} x^{2\tau-1} exp\left(-\frac{x^{2\tau}}{2\nu^2}\right), x > 0, \nu, \tau > 0, \tag{5}$$

where ν and τ are scale and shape parameters of PRD, respectively.

Remark: If $\tau = 1$ then the equation (4) is reduced to the cumulative distribution function of the Rayleigh distribution.

The survival and hazard rate function of PRD are, respectively, given by

$$R(t) = exp\left(-\frac{t^{2\tau}}{2\nu^2}\right); t > 0, \tag{6}$$

and

$$H(t) = \frac{\tau}{\nu^2} t^{2\tau-1}; t > 0. \tag{7}$$

The plots of the hazard rate function for some values of the parameter are depicted in Figure 1.

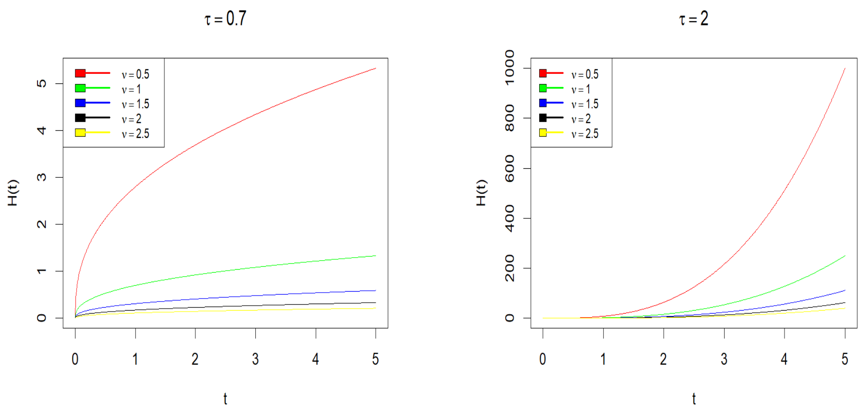

Figure 1: The Hazard rate function fo PRD.

3. Frequentist Approach

In this section, we will obtain PRD's unknown model parameters and survival functions using the maximum likelihood estimation (MLE) method. We will also establish the approximate confidence intervals for the parameters and any functions of them.

3.1. Maximum likelihood estimation

Let $x_1, x_2, \ldots, x_n$ be a random sample drawn from the PRD with PDF (5). Therefore, the likelihood function, $L(\nu, \tau|x)$, is the joint density of the random sample, $x_1, x_2, \ldots, x_n$, and is given by

$$L(\tau, \nu|x) = \prod_{i=1}^{n} \left[\frac{\tau}{\nu^2} x_i^{2\tau-1} exp\left(-\frac{x^{2\tau}}{2\nu^2} \right) \right]. \tag{8}$$

On simplifying, we get

$$L(\tau, \nu|x) = \left(\frac{\tau}{\nu^2} \right)^n \prod_{i=1}^{n} x_i^{2\tau-1} exp\left(-\frac{\sum_{i=1}^{n} x_i^{2\tau}}{2\nu^2} \right). \tag{9}$$

On taking the logarithm of equation (9), the log-likelihood function can be expressed as follows:

$$l(\tau, \nu|x) = n\log(\tau) - 2n\log(\nu) + (2\tau - 1)\sum_{i=1}^{n}\log(x_i) - \frac{\sum_{i=1}^{n} x_i^{2\tau}}{2\nu^2}. \tag{10}$$

Now, differentiating equation (10) with respect to τ and ν and equating the resulting terms to zero, the normalizing equations are given as

$$\frac{\partial l}{\partial \tau} = \frac{n}{\tau} + 2\sum_{i=1}^{n} \log(x_i) - \frac{\sum_{i=1}^{n} x_i^{2\tau} \log(x_i)}{\nu^2} = 0, \tag{11}$$

and

$$\frac{\partial l}{\partial \nu} = -\frac{2n}{\nu} + \frac{\sum_{i=1}^{n} x_i^{2\tau}}{\nu^3} = 0. \tag{12}$$

Let $\hat{\tau}$ and $\hat{\nu}$ represent MLEs of τ and ν, respectively. Then $\hat{\tau}$ is obtained by solving the nonlinear equation (11) using Newton-Rapson iterative procedures. Thereafter, $\hat{\nu}$ can be obtained by substituting $\hat{\tau}$ into equation (12) and solving the resulting expression as follows:

$$\hat{\nu} = \left(\frac{\sum_{i=1}^{n} x_i^{2\hat{\tau}}}{2n} \right)^{\frac{1}{2}}. \tag{13}$$

Using invariance property of the MLE the $R(t)$, say $\hat{R}(t)$, can be written as

$$\hat{R}(t) = exp\left(-\frac{t^{2\hat{\tau}}}{2\hat{\nu}^2} \right). \tag{14}$$

3.2. Approximate confidence intervals

Since all the second-order partial derivatives exist in their domain. Therefore, the interval estimates for the parameters τ and ν are derived using the Fisher information matrix as follows:

$$I(\tau,\nu) = -E\begin{bmatrix} I_{\tau\tau} & I_{\tau\nu} \\ I_{\nu\tau} & I_{\nu\nu} \end{bmatrix}, \tag{15}$$

where $I_{\tau\tau} = \frac{\partial^2 l}{\partial \tau^2} = -\frac{n}{\tau^2} - \frac{2\sum_{i=1}^{n} x_i^{2\tau}(\log(x_i))^2}{\nu^2}$,
$I_{\tau\nu} = I_{\nu\tau} = \frac{\partial^2 l}{\partial\tau\partial\nu} = \frac{\partial^2 l}{\partial\nu\partial\tau} = \frac{2}{\nu^3}\sum_{i=1}^{n} x_i^{2\tau}\log(x_i)$, and $I_{\nu\nu} = \frac{2n}{\nu^2} - \frac{3}{\nu^4}\sum_{i=1}^{n} x_i^{2\tau}$.

From (15), it is observed that the exact solution of expectation is tedious to evaluate. Therefore, approximate variance and covariance matrix $I^{-1}(\hat{\tau},\hat{\nu})$ is given by

$$I^{-1}(\hat{\tau},\hat{\nu}) = \begin{bmatrix} -I_{\tau\tau} & -I_{\tau\nu} \\ -I_{\nu\tau} & -I_{\nu\nu} \end{bmatrix}^{-1}_{(\hat{\tau},\hat{\nu})}. \tag{16}$$

According to Gomez-Deniz (2010), the MLE vector $(\hat{\tau},\hat{\nu})^T$ is consistent and asymptotically normally distributed as follows:

$$\sqrt{n}\left[(\hat{\tau},\hat{\nu})^T - (\tau,\nu)^T\right] \to N(0, I^{-1}(\tau,\nu)).$$

The $100(1-\xi)\%$ confidence intervals τ and ν can be approximated by $\hat{\tau} \pm Z_{\frac{\xi}{2}} \sqrt{var(\hat{\tau})}$ and $\hat{\nu} \pm Z_{\frac{\xi}{2}} \sqrt{var(\hat{\nu})}$, respectively, where the diagonal elements of $I^{-1}(\tau,\nu)$ represent the variance of parameters of τ and ν while the off-diagonal elements represent the covariance between τ and ν, and $Z_{\frac{\xi}{2}}$ is the $(1-\frac{\xi}{2})$ quantile of the standard normal distribution.

To derive the confidence intervals for $R(t)$, we must know their variance, and for this the delta method is adapted to obtain the variances of $R(t)$. For more details about the delta method, one may refer to W. H. Greene (2003). Utilising the delta technique, the approximate variance of $\hat{R}(t)$ (at their MLEs $\hat{\tau}$ and $\hat{\nu}$) can be expressed as follows:

$$var(\hat{R}(t)) = \left[\nabla \hat{R}(t)\right]^T I^{-1}(\hat{\tau},\hat{\nu})[\nabla \hat{R}(t)], \tag{17}$$

where T is stands for transpose operator and $\nabla \hat{R}(t)$ is the gradient of reliability function, respectively, with respect to τ and ν, i.e.,

$$[\nabla R(t)]^T = \left[\frac{\partial R(t)}{\partial \tau}, \frac{\partial R(t)}{\partial \nu}\right]_{(\hat{\tau},\hat{\nu})}. \tag{18}$$

From equation (6) the derivative of $R(t)$ with respect to τ and ν is obtained as follows:

$$\frac{\partial R(t)}{\partial \tau} = -\frac{t^{2\tau}}{\nu^2} \log(t)\, exp\left(-\frac{t^{2\tau}}{2\nu^2}\right), \quad \frac{\partial R(t)}{\partial \nu} = \frac{t^{2\tau}}{\nu^3}\, exp(-\frac{t^{2\tau}}{2\nu^2}).$$

Therefore, the $100(1-\xi)\%$ confidence intervals for $R(t)$ can be derived as follows:

$$\hat{R}(t) \pm Z_{\frac{\xi}{2}} \sqrt{var(\hat{R}(t))}.$$

4. Bayesian inference

In this section, we will consider the Bayesian estimation method to estimate the parameters and any parametric function of PRD based on the complete sample.

4.1. Loss functions

In Bayesian estimation, the loss function quantifies the cost of estimation errors, guiding decision-making to minimize expected loss by balancing precision and bias. Common loss functions include SELF (symmetric), GELF, and LINEX (asymmetric). Let $\tilde{\phi}$ be an estimator of ϕ. The SELF, GELF, and LLF are defined as follows:

$$L_{SE}(\phi,\tilde{\phi}) = (\phi-\tilde{\phi})^2, \tag{19}$$

$$L_{GE}(\phi,\tilde{\phi}) = \left(\frac{\tilde{\phi}}{\phi}\right)^q - q\,\log\left(\frac{\tilde{\phi}}{\phi}\right) - 1, q \neq 0, \tag{20}$$

and

$$L_{LI}(\phi,\tilde{\phi}) = e^{-q(\phi-\tilde{\phi})} - p(\phi-\tilde{\phi}) - 1, q \neq 0. \tag{21}$$

4.2. Prior distribution

In Bayesian analysis, the prior represents initial beliefs about parameters before data integration. Selecting priors is critical but lacks a fixed rule. Here, a non-informative prior is assumed for τ, and a conjugate prior for ν, defined as:

$$\pi_1(\tau) = \frac{1}{\tau}; \;\; \tau > 0,$$

and

$$\pi_2(\nu) = \frac{1}{\nu^{a+1}} exp\left(-\frac{b}{2\nu^2}\right); \;\; a, b > 0,$$

where $a > 0$ and $b > 0$ are hyperparameters. The joint prior distribution for τ and ν is given as:

$$\pi(\tau, \nu) = \frac{1}{\tau \nu^{a+1}} exp\left(-\frac{b}{2\nu^2}\right); \; \tau > 0, \nu, a, b > 0. \tag{22}$$

4.3. Posterior analysis

The posterior density function of τ and ν is obtained by combining equation (9) and (22), which is written as follows:

$$\pi(\tau, \nu|x) = K^{-1} \frac{\tau^{n-1}}{\nu^{2n+a+1}} \prod_{i=1}^{n} x_i^{2\tau-1} exp\left(-\frac{b + \sum_{i=1}^{n} x_i^{2\tau}}{2\nu^2}\right), \tag{23}$$

where $K = \int_0^\infty \int_0^\infty \frac{\tau^{n-1}}{\nu^{2n+a+1}} \prod_{i=1}^{n} x^{2\tau-1} exp\left(-\frac{b+\sum_{i=1}^{n} x_i^{2\tau}}{2\nu^2}\right) d\tau d\nu$, is called the normalising constant. Equation (23) cannot be converted analytically due to the complex form of the likelihood function. In order to derive the Bayes estimates, the Lindley approximation technique is used for further analysis.

4.4. Lindley's method

Let $u(\tau, \nu)$ be the function of τ and ν, then by using the equation (23), the expected value of $u(\tau, \nu)$ is given by:

$$E(u(\tau, \nu|x)) = \frac{\int_\tau \int_\nu u(\tau, \nu) e^{l(\tau, \nu|\mathbf{\underline{x}}) + \eta(\tau, \nu)} d\tau d\nu}{\int_\tau \int_\nu e^{l(\tau, \nu|\mathbf{\underline{x}}) + \eta(\tau, \nu)} d\tau d\nu}.$$

The Bayes estimator $u(\tau, \nu)$ is the solution of the above equation. Unfortunately, it is very hard to obtain the Bayes estimator analytically due to its dependence on the ratio of two integrals. To overcome this difficulty, the Lindley's technique introduced by Lindley's 1980, is used. Let the I be the approximate value of $E(u(\tau, \nu|x))$ then

$$
\begin{aligned}
I(x) \quad = \quad & \hat{u}(\tau,\nu)+0.5(\hat{u}_{\tau\tau}\hat{\sigma}_{\tau\tau}+\hat{u}_{\nu\nu}\hat{\sigma}_{\nu\nu})+\hat{u}_{\tau\nu}\hat{\sigma}_{\tau\nu}+\hat{u}_{\tau}(\hat{\sigma}_{\tau\tau}\hat{\eta}_{\tau}+\hat{\sigma}_{\nu\tau}\hat{\eta}_{\nu})+ \\
& \hat{u}_{\nu}(\hat{\sigma}_{\tau\nu}\hat{\eta}_{\tau}+\hat{\sigma}_{\nu\nu}\hat{\eta}_{\nu})+0.5\hat{l}_{\tau\tau\tau}(\hat{u}_{\tau}\hat{\sigma}_{\tau\tau}^{2}+\hat{u}_{\nu}\hat{\sigma}_{\tau\tau}\hat{\sigma}_{\tau\nu})+0.5\hat{l}_{\tau\tau\nu} \\
& \left(3\hat{u}_{\tau}\hat{\sigma}_{\tau\tau}\hat{\sigma}_{\tau\nu}+\hat{u}_{\nu}(\hat{\sigma}_{\tau\tau}\hat{\sigma}_{\nu\nu}+2\hat{\sigma}_{\tau\nu}^{2})\right)+0.5\hat{l}_{\tau\nu\nu}\left(\hat{u}_{\tau}(\hat{\sigma}_{\tau\tau}\hat{\sigma}_{\nu\nu}+2\hat{\sigma}_{\tau\nu}^{2})\right. \\
& \left.+3\hat{u}_{\nu}\hat{\sigma}_{\tau\nu}\hat{\sigma}_{\nu\nu}\right)+0.5\hat{l}_{\nu\nu\nu}(\hat{u}_{\tau}\hat{\sigma}_{\tau\nu}\hat{\sigma}_{\tau\tau}+\hat{u}_{\nu}\hat{\sigma}_{\nu\nu}^{2}), \qquad (24)
\end{aligned}
$$

where $\hat{u}(\tau,\nu)$ is the function of τ and ν evaluated at $\hat{\tau}$ and $\hat{\nu}$ and σ_{ij} is the $(i,j)^{th}$ element of matrix $[-\hat{l}_{ij}]^{-1}; i,j=1,2$.

The other notation is interpreted with the following definition, such that:
$\hat{u}_{\tau}=\frac{\partial u}{\partial \tau}$, $\hat{u}_{\nu}=\frac{\partial u}{\partial \nu}$, $\hat{u}_{\nu\tau}=\hat{u}_{\tau\nu}=\frac{\partial^2 u}{\partial \nu \partial \tau}$, $\hat{\eta}_{\nu}=\frac{\partial \log \pi(\tau,\nu)}{\partial \nu}$, $\hat{\eta}_{\tau}=\frac{\partial \log \pi(\tau,\nu)}{\partial \tau}$, $\hat{l}_{\nu}=\frac{\partial l}{\partial \nu}$, $\hat{l}_{\tau}=\frac{\partial l}{\partial \tau}$

$\hat{l}_{\nu\tau}=\hat{l}_{\tau\nu}=\frac{\partial^2 l}{\partial \nu \partial \tau}$, $\hat{l}_{\nu\nu}=\frac{\partial^2 l}{\partial \nu^2}$, $\hat{l}_{\tau\tau}=\frac{\partial^2 l}{\partial \tau^2}$, $\hat{l}_{\nu\nu\nu}=\frac{\partial^3 l}{\partial \nu^3}$, $\hat{l}_{\tau\tau\tau}=\frac{\partial^3 l}{\partial \tau^3}$, $\hat{l}_{\nu\nu\tau}=\frac{\partial^3 l}{\partial \nu^2 \partial \tau}$, and $\hat{l}_{\nu\tau\tau}=\frac{\partial^3 l}{\partial \nu \tau^2}$.

The Bayes estimates of τ, ν, and $S(t)$ under the SELF, GELF, and LLF are obtained in the following subsections.

4.4.1 Bayesian estimation under SELF

One of the very popular symmetric loss function is SELF, which was first addressed by Legendre (1805), which endows equal weight for overestimation and underestimation.

The Bayes estimate (BE) of parameter τ under SELF is found as follows:

$$
\hat{\tau}_{BS}=E(\tau|x)=\frac{\int_0^\infty\int_0^\infty \tau^n \frac{1}{\nu^{2n+a+1}}\prod_i^n x_i^{2\tau-1} exp\left(\frac{b+\sum_i^n x_i^{2\tau}}{2\nu^2}\right)\, d\tau d\nu}{\int_0^\infty\int_0^\infty \tau^{n-1} \frac{1}{\nu^{2n+a+1}}\prod_i^n x_i^{2\tau-1} exp\left(\frac{b+\sum_i^n x_i^{2\tau}}{2\nu^2}\right)\, d\tau d\nu}. \qquad (25)
$$

If $u(\tau,\nu)=\tau$, then $u_{\tau}=1$, $u_{\nu\nu}=u_{\nu}=u_{\tau\tau}=u_{\nu\tau}=u_{\tau\nu}=0$. Then the BE of τ is written as follows:

$$
\begin{aligned}
\hat{\tau}_{BS}=&\hat{\tau}+(\hat{\eta}_{\tau}\hat{\sigma}_{\tau\tau}+\hat{\eta}_{\nu}\hat{\sigma}_{\nu\tau})+0.5\left[\hat{L}_{\tau\tau\tau}\hat{\sigma}_{\tau\tau}^{2}+3\hat{L}_{\tau\tau\nu}\hat{\sigma}_{\tau\nu}\hat{\sigma}_{\tau\tau}+\hat{L}_{\tau\nu\nu}(\hat{\sigma}_{\nu\nu}\hat{\sigma}_{\tau\tau}+2\hat{\sigma}_{\tau\nu}^{2})\right. \\
&\left.+\hat{L}_{\nu\nu\nu}\hat{\sigma}_{\tau\nu}\hat{\sigma}_{\nu\nu}\right]. \qquad (26)
\end{aligned}
$$

The Bayes estimate (BE) of parameter ν under SELF is found as follows:

$$
\hat{\nu}_{BS}=E(\nu|x)=\frac{\int_0^\infty\int_0^\infty \tau^{n-1} \frac{1}{\nu^{2n+a}}\prod_i^n x_i^{2\tau-1} exp\left(\frac{b+\sum_i^n x_i^{2\tau}}{2\nu^2}\right)\, d\tau d\nu}{\int_0^\infty\int_0^\infty \tau^{n-1} \frac{1}{\nu^{2n+a+1}}\prod_i^n x_i^{2\tau-1} exp\left(\frac{b+\sum_i^n x_i^{2\tau}}{2\nu^2}\right)\, d\tau d\nu}.
$$

If $u(\tau,\nu)=\nu$ then $u_{\nu}=1$, $u_{\nu\nu}=u_{\tau}=u_{\tau\tau}=u_{\nu\tau}=u_{\tau\nu}=0$.

Thus, the BE of ν under SELF is obtained as follows:

$$\begin{aligned}\hat{\nu}_{BS} =&\hat{\nu} + (\hat{\eta}_{\nu}\hat{\sigma}_{\nu\nu} + \hat{\eta}_{\tau}\hat{\sigma}_{\tau\nu}) + 0.5\left[\hat{L}_{\tau\tau\tau}\hat{\sigma}_{\tau\tau}\hat{\sigma}_{\tau\nu} + 3\hat{L}_{\tau\nu\nu}\hat{\sigma}_{\nu\nu}\hat{\sigma}_{\tau\nu}\right.\\ &\left.+\hat{L}_{\tau\tau\nu}(\hat{\sigma}_{\nu\nu}\hat{\sigma}_{\tau\tau} + 2\hat{\sigma}^2_{\tau\nu}) + \hat{L}_{\nu\nu\nu}\hat{\sigma}^2_{\nu\nu}\right].\end{aligned} \tag{27}$$

If $u(\tau,\nu) = exp\left(-\frac{t^{2\tau}}{2\nu^2}\right)$ then $u_{\nu} = \frac{t^{2\tau}}{\nu^3}exp\left(-\frac{t^{2\tau}}{2\nu^2}\right)$,
$u_{\nu\nu} = -\frac{3t^{2\tau}}{\nu^4}exp\left(-\frac{t^{2\tau}}{2\nu^2}\right) + \left(\frac{t^{2\tau}}{\nu^3}\right)^2 exp\left(-\frac{t^{2\tau}}{2\nu^2}\right)$,
$u_{\tau} = -\frac{t^{2\tau}\log(t)}{2\nu^2}exp\left(-\frac{t^{2\tau}}{2\nu^2}\right)$, $u_{\tau\tau} = -\frac{2t^{2\tau}(\log(t))^2}{\nu^2}exp\left(-\frac{t^{2\tau}}{2\nu^2}\right) + \left(\frac{2t^{2\tau}\log(t)}{\nu^2}\right)^2 exp\left(-\frac{t^{2\tau}}{2\nu^2}\right)$,
and $u_{\nu\tau} = u_{\tau\nu} = \frac{t^{2\tau}\log(t)}{\nu^3}exp\left(-\frac{t^{2\tau}}{2\nu^2}\right) - \frac{t^{4\tau}\log(t)}{\nu^5}exp\left(-\frac{t^{2\tau}}{2\nu^2}\right)$.

Then BE of the survival function under SELF is given by:

$$\begin{aligned}\hat{R}_{BS}(t) \;=\;& \hat{R} + 0.5(\hat{u}_{\tau\tau}\hat{\sigma}_{\tau\tau} + \hat{u}_{\nu\nu}\hat{\sigma}_{\nu\nu}) + \hat{u}_{\tau\nu}\hat{\sigma}_{\tau\nu} + \hat{u}_{\tau}(\hat{\sigma}_{\tau\tau}\hat{\eta}_{\tau} + \hat{\sigma}_{\nu\tau}\hat{\eta}_{\nu}) +\\ &\hat{u}_{\nu}(\hat{\sigma}_{\tau\nu}\hat{\eta}_{\tau} + \hat{\sigma}_{\nu\nu}\hat{\eta}_{\nu}) + 0.5\hat{L}_{\tau\tau\tau}(\hat{u}_{\tau}\hat{\sigma}^2_{\tau\tau} + \hat{u}_{\nu}\hat{\sigma}_{\tau\tau}\hat{\sigma}_{\tau\nu}) + 0.5\hat{L}_{\tau\tau\nu}\\ &\left(3\hat{u}_{\tau}\hat{\sigma}_{\tau\tau}\hat{\sigma}_{\tau\nu} + \hat{u}_{\nu}(\hat{\sigma}_{\tau\tau}\hat{\sigma}_{\nu\nu} + 2\hat{\sigma}^2_{\tau\nu})\right) + 0.5\hat{L}_{\tau\nu\nu}\left(\hat{u}_{\tau}(\hat{\sigma}_{\tau\tau}\hat{\sigma}_{\nu\nu} + 2\hat{\sigma}^2_{\tau\nu})\right.\\ &\left.+3\hat{u}_{\nu}\hat{\sigma}_{\tau\nu}\hat{\sigma}_{\nu\nu}\right) + 0.5\hat{L}_{\tau\tau\tau}(\hat{u}_{\tau}\hat{\sigma}_{\tau\nu}\hat{\sigma}_{\tau\tau} + \hat{u}_{\nu}\hat{\sigma}^2_{\nu\nu}).\end{aligned} \tag{28}$$

4.4.2 Bayesian estimation under GELF

GELF is an asymmetric loss function and it was proposed by Calabria and Pulcini (1994). BE of τ under this loss function is derived as follows:

$$\hat{\tau}_{BG} = [E(\tau^{-q}|x)]^{-\frac{1}{q}};\;\; q \neq 0. \tag{29}$$

If $u(\tau,\nu) = \tau^{-q}$ then $u_{\tau} = -q\tau^{-(q+1)}$, $u_{\tau\tau} = q(q+1)\tau^{-(q+2)}$, $u_{\tau} = u_{\nu\nu} = u_{\tau\nu} = u_{\nu\tau} = 0$.

Then using equation (24) we have

$$\begin{aligned}E(\tau^{-q}|x) \;=\;& \hat{\tau}^{-q} + 0.5\hat{u}_{\tau\tau}\hat{\sigma}_{\tau\tau} + \hat{u}_{\tau}(\hat{\sigma}_{\tau\tau}\hat{\eta}_{\tau} + \hat{\sigma}_{\nu\tau}\hat{\eta}_{\nu}) + 0.5\hat{L}_{\tau\tau\tau}\hat{u}_{\tau}\hat{\sigma}^2_{\tau\tau}\\ &+1.5\hat{L}_{\tau\tau\nu}\hat{u}_{\tau}\hat{\sigma}_{\tau\tau}\hat{\sigma}_{\tau\nu} + 0.5\hat{L}_{\tau\nu\nu}\left(\hat{u}_{\tau}(\hat{\sigma}_{\tau\tau}\hat{\sigma}_{\nu\nu} + 2\hat{\sigma}^2_{\tau\nu})\right) + 0.5\hat{L}_{\nu\nu\nu}\hat{u}_{\tau}\hat{\sigma}_{\tau\nu}\hat{\sigma}_{\nu\nu}.\end{aligned}$$

If $u(\tau,\nu) = \nu^{-q}$, $u_{\nu} = -q\nu^{-(q+1)}$, $u_{\nu\nu} = q(q+1)\nu^{-(q+2)}$, $u_{\tau} = u_{\tau\tau} = u_{\tau\nu} = u_{\nu\tau} = 0$. Then the BE of ν under GELF is obtained as:

$$\hat{\nu}_{BG} = [E(\nu^{-q}|x)]^{-\frac{1}{q}};\;\; q \neq 0, \tag{30}$$

where

$$\begin{aligned}E(\nu^{-q}|x) \;=\;& \hat{\nu}^{-q} + 0.5\hat{u}_{\nu\nu}\hat{\sigma}_{\nu\nu} + \hat{u}_{\nu}(\hat{\sigma}_{\nu\nu}\hat{\eta}_{\nu} + \hat{\sigma}_{\tau\nu}\hat{\eta}_{\tau}) + 0.5\hat{L}_{\tau\tau\tau}\hat{u}_{\nu}\hat{\sigma}_{\tau\nu}\hat{\sigma}_{\tau\tau}\\ &+1.5\hat{L}_{\tau\nu\nu}\hat{u}_{\nu}\hat{\sigma}_{\nu\nu}\hat{\sigma}_{\tau\nu} + 0.5\hat{L}_{\tau\tau\nu}\left(\hat{u}_{\nu}(\hat{\sigma}_{\tau\tau}\hat{\sigma}_{\nu\nu} + 2\hat{\sigma}^2_{\tau\nu})\right) + 0.5\hat{L}_{\nu\nu\nu}\hat{u}_{\nu}\hat{\sigma}^2_{\nu\nu}.\end{aligned}$$

If $u(\tau,\nu)=exp\left(\frac{qt^{2\tau}}{2\nu^2}\right)$, $u_\nu=-\frac{qt^{2\tau}}{\nu^3}exp\left(\frac{qt^{2\tau}}{2\nu^2}\right)$,

$u_{\nu\nu}=\frac{3qt^{2\tau}}{\nu^4}exp\left(\frac{qt^{2\tau}}{2\nu^2}\right)+\left(\frac{qt^{2\tau}}{\nu^3}\right)^2 exp\left(\frac{qt^{2\tau}}{2\nu^2}\right), u_\tau=\frac{qt^{2\tau}log(t)}{2\nu^2}exp\left(\frac{qt^{2\tau}}{\nu^2}\right)$,

$u_{\tau\tau}=\frac{2qt^{2\tau}(log(t))^2}{\nu^2}exp\left(\frac{qt^{2\tau}}{2\nu^2}\right)+\left(\frac{qt^{2\tau}log(t)}{\nu^2}\right)^2 exp\left(\frac{qt^{2\tau}}{2\nu^2}\right)$,

$u_{\nu\tau}=u_{\tau\nu}=-\frac{2qt^{2\tau}log(t)}{\nu^3}exp\left(\frac{qt^{2\tau}}{2\nu^2}\right)-\frac{q^2t^{4\tau}log(t)}{\nu^5}exp\left(\frac{qt^{2\tau}}{2\nu^2}\right)$.

Then, BE of $R(t)$ under GELF is obtained as

$$\hat{R}(t)_{BG}=[E(R(t)^{-q}|x)]^{-\frac{1}{q}};\;\; q\neq 0, \tag{31}$$

where

$$\begin{aligned} E(R(t)^{-q}|x) \quad = \quad & \hat{R}(t)^{-q}+0.5\hat{u}_{\tau\tau}\hat{\sigma}_{\tau\tau}+\hat{u}_\tau(\hat{\sigma}_{\tau\tau}\hat{\eta}_\tau+\hat{\sigma}_{\nu\tau}\hat{\eta}_\nu)+\\ & 0.5\hat{L}_{\tau\tau\tau}\hat{u}_\tau\hat{\sigma}^2_{\tau\tau}+1.5\hat{L}_{\tau\tau\nu}\hat{u}_\tau\hat{\sigma}_{\tau\tau}\hat{\sigma}_{\tau\nu}+0.5\hat{L}_{\tau\nu\nu}\left(\hat{u}_\tau(\hat{\sigma}_{\tau\tau}\hat{\sigma}_{\nu\nu}+2\hat{\sigma}^2_{\tau\nu})\right)\\ & +0.5\hat{L}_{\nu\nu\nu}\hat{u}_\tau\hat{\sigma}_{\tau\nu}\hat{\sigma}_{\nu\nu}. \end{aligned}$$

4.4.3 Bayesian estimation under LLF

LLF is an asymmetric loss function that proposed by Klebanov (1972) and later used by Varian (1975).

If $u(\tau,\nu)=exp(-q\tau),\; u_\tau=-q\; exp(-q\tau),\; u_{\tau\tau}=q^2\; exp(-q\tau),\; u_\nu=u_{\nu\nu}=u_{\tau\nu}=u_{\nu\tau}=0$. Then, BE of τ under LLF is obtained as:

$$\hat{\tau}_{BL}=-\frac{1}{q}\; ln\, E(exp(-q\tau)|x));\; q\neq 0, \tag{32}$$

where

$$\begin{aligned} E(exp(-q\tau)|x) \quad = \quad & exp(-q\hat{\tau})+0.5\hat{u}_{\tau\tau}\hat{\sigma}_{\tau\tau}+\hat{u}_\tau(\hat{\sigma}_{\tau\tau}\hat{\eta}_\tau+\hat{\sigma}_{\nu\tau}\hat{\eta}_\nu)+0.5\hat{L}_{\tau\tau\tau}\hat{u}_\tau\hat{\sigma}^2_{\tau\tau}\\ & +1.5\hat{L}_{\tau\tau\nu}\hat{u}_\tau\hat{\sigma}_{\tau\tau}\hat{\sigma}_{\tau\nu}+0.5\hat{L}_{\tau\nu\nu}\left(\hat{u}_\tau(\hat{\sigma}_{\tau\tau}\hat{\sigma}_{\nu\nu}+2\hat{\sigma}^2_{\tau\nu})\right)\\ & +0.5\hat{L}_{\nu\nu\nu}\hat{u}_\tau\hat{\sigma}_{\tau\nu}\hat{\sigma}_{\tau\tau}. \end{aligned}$$

If $u(\tau,\nu)=exp(-q\nu),\; u_\nu=-q\; exp(-q\nu),\; u_{\nu\nu}=q^2\; exp(-q\nu),\; u_\tau=u_{\tau\tau}=u_{\tau\nu}=u_{\nu\tau}=0$. Then

$$\hat{\nu}_{BL}=-\frac{1}{q}\; ln\, E(exp(-q\nu)|x);\; q\neq 0, \tag{33}$$

where

$$\begin{aligned} E(exp(-q\nu)|x) \quad = \quad & exp(-q\hat{\nu})+0.5\hat{u}_{\nu\nu}\hat{\sigma}_{\nu\nu}+\hat{u}_\nu(\hat{\sigma}_{\nu\nu}\hat{\eta}_\nu+\hat{\sigma}_{\tau\nu}\hat{\eta}_\tau)+0.5\hat{L}_{\tau\tau\tau}\hat{u}_\nu\hat{\sigma}_{\tau\nu}\hat{\sigma}_{\tau\tau}\\ & +1.5\hat{L}_{\tau\nu\nu}\hat{u}_\nu\hat{\sigma}_{\tau\tau}\hat{\sigma}_{\tau\nu}+0.5\hat{L}_{\tau\tau\nu}\left(\hat{u}_\nu(\hat{\sigma}_{\tau\tau}\hat{\sigma}_{\nu\nu}+2\hat{\sigma}^2_{\tau\nu})\right)+0.5\hat{L}_{\nu\nu\nu}\hat{u}_\nu\hat{\sigma}^2_{\nu\nu}. \end{aligned}$$

If $u(\tau,\nu) = exp\left(-q\, exp\left(-\frac{t^{2\tau}}{2\nu^2}\right)\right)$, $u_\nu = -\frac{qt^{2\tau}}{\nu^3} exp\left[-\left\{\left(q\, exp\left(-\frac{t^{2\tau}}{2\nu^2}\right)\right) + \frac{t^{2\tau}}{2\nu^2}\right\}\right]$,

$$u_{\nu\nu} = \frac{qt^{2\tau}}{\nu^3} exp\left[-\left\{\left(q\, exp\left(-\frac{t^{2\tau}}{2\nu^2}\right)\right) + \frac{t^{2\tau}}{2\nu^2}\right\}\right] \left\{\frac{3}{\nu} + q\left(exp\left(-\frac{t^{2\tau}}{2\nu^2}\right) + \frac{t^{2\tau}}{2\nu^2}\right)\left(qexp(-\frac{t^{2\tau}}{2\nu^2})\frac{t^{2\tau}}{\nu^3} - \frac{t^{2\tau}}{\nu^3}\right)\right\}, \tag{34}$$

$$u_\tau = \frac{qt^{2\tau}log(t)}{2\nu^2} exp\left[-\left\{\left(q\, exp\left(-\frac{t^{2\tau}}{2\nu^2}\right)\right) + \frac{t^{2\tau}}{2\nu^2}\right\}\right],$$

$$u_{\tau\tau} = \frac{qt^{2\tau}(log(t))^2}{\nu^2} exp\left[-\left\{\left(q\, exp\left(-\frac{t^{2\tau}}{2\nu^2}\right)\right) + \frac{t^{2\tau}}{2\nu^2}\right\}\right]\left\{2 + \frac{qt^{2\tau}}{\nu^2}\left(exp(\left(-\frac{t^{2\tau}}{\nu^2}\right) - 1\right)\right\},$$

and

$$u_{\nu\tau} = u_{\tau\nu} = -\, exp\left[-\left\{\left(q\, exp\left(-\frac{t^{2\tau}}{2\nu^2}\right)\right) + \frac{t^{2\tau}}{2\nu^2}\right\}\right] \left\{\frac{2qt^{2\tau}log(t)}{\nu^3} - \frac{qt^{2\tau}log(t)}{\nu^2}\left(\frac{qt^{2\tau}}{\nu^3} exp\left(-\frac{t^{2\tau}}{2\nu^2}\right) - \frac{t^{2\tau}}{\nu^3}\right)\right\}. \tag{35}$$

BE of $R(t)$ under LLF is given by

$$\hat{R}(t)_{BL} = -\frac{1}{q}\, ln\, \left(E(exp(-q\, R(t))|x)\right);\;\; q \neq 0, \tag{36}$$

where

$$\begin{aligned} E(exp(-qR(t))|x) \;=\; & \hat{R}(t)^{-q} + 0.5\hat{u}_{\tau\tau}\hat{\sigma}_{\tau\tau} + \hat{u}_\tau(\hat{\sigma}_{\tau\tau}\hat{\eta}_\tau + \hat{\sigma}_{\nu\tau}\hat{\eta}_\nu) + 0.5\hat{L}_{\tau\tau\tau}\hat{u}_\tau\hat{\sigma}^2_{\tau\tau} \\ & + 1.5\hat{L}_{\tau\tau\nu}\hat{u}_\tau\hat{\sigma}_{\tau\tau}\hat{\sigma}_{\tau\nu} + 0.5\hat{L}_{\tau\nu\nu}\left(\hat{u}_\tau(\hat{\sigma}_{\tau\tau}\hat{\sigma}_{\nu\nu} + 2\hat{\sigma}^2_{\tau\nu})\right) \\ & + 0.5\hat{L}_{\nu\nu\nu}\hat{u}_\tau\hat{\sigma}_{\tau\nu}\hat{\sigma}_{\nu\nu}. \end{aligned}$$

Using Lindley's approximation, the Bayes estimates of the parameters τ, ν, and $R(t)$ are derived, but the highest posterior density (HPD) credible interval is not possible to be constructed. Therefore, the Metropolis-Hasting (M-H) algorithm is introduced to compute the Bayes estimates as well as HPD credible intervals.

4.4.4 Metropolis-Hastings (M-H) Algorithm

The Metropolis-Hastings algorithm is a Markov Chain Monte Carlo (MCMC) technique used for sampling from complex probability distributions. It operates by iteratively proposing candidate states, evaluating their acceptance probability based on a defined proposal distribution and target distribution, and accepting or rejecting them accordingly. The algorithm addresses challenges in sampling from distributions that are difficult to directly sample from by introducing a transition probability function. For more about the M-H algorithm one may refers to Metropolis *et al.* (1953) and Hastings (1970). Using the M-H algorithm, the conditional posterior density function of parameters τ and ν are given by

$$\pi_1(\tau|\nu,x) \propto \tau^{n-1}\prod_{i=1}^{n} x_i^{2\tau-1}\ exp\left(-\frac{\sum_{i=1}^{n} x_i^{2\tau}}{2\nu^2}\right), \tag{37}$$

and

$$\pi_2(\nu|\tau,x) \propto \frac{1}{\nu^{2n+a+1}}\ exp\left(-\frac{b+\sum_{i=1}^{n} x_i^{2\tau}}{2\nu^2}\right). \tag{38}$$

respectively. Since the conditional posterior distributions of the parameters τ and ν in the equations (35) and (36) are unknown. Therefore, we can use M-H algorithm with normal proposal distribution to generate the posterior sample from (35) and (36) respectively. The M-H algorithm consists of the following steps:

Step 1: Set $j = 1$ and start with initial values $\tau^{(0)} = \hat{\tau}$ and $\nu^{(0)} = \hat{\nu}$.
Step 2: Use the M-H algorithm steps to generate posterior samples for $\tau^{(j)}$ and $\nu^{(j)}$ from the conditional distributions $\pi_1(\tau^{(j-1)}|\nu^{(j-1)},x)$ and $\pi_2(\nu^{(j-1)}|\tau^{(j-1)},x)$ using normal proposal distributions $N(\tau^{(j-1)},var(\tau))$ and $N(\nu^{(j-1)},var(\nu))$ respectively.
Step 3: Set j = j+1.
Step 4: Replicate steps 2-3 N times to extract samples $\phi^{(j)} = (\tau^{(j)},\nu^{(j)},R^{(j)}(t))$ for $j = 1,2,...,N$.
Step 5: The Bayes estimates of the parameters τ, ν, and $R(t)$ under SELF, GELF, and LLF can be obtained from the following expressions:

$$\hat{\phi}_{BS} = \frac{1}{N-M}\sum_{j=M+1}^{N}\phi^{(j)}, \tag{39}$$

$$\hat{\phi}_{BG} = \left(\frac{1}{N-M}\sum_{j=M+1}^{N}(\phi^{(j)})^{-q}\right)^{-\frac{1}{q}}, \tag{40}$$

$$\hat{\phi}_{BL} = -\frac{1}{q}\ \ln\left(\frac{1}{N-M}\sum_{j=M+1}^{N} exp(-q\phi^{(j)})\right), \tag{41}$$

where M is the burn in period of the Markov chain.

Step 6: To construct the HPD credible interval of $\phi = (\tau,\nu,R(t))$ order the MCMC sample of ϕ, where $\phi^{(j)} = (\tau^{(j)},\nu^{(j)},R^{(j)}(t))$, $j = 1,2,...,N$, for sufficiently large N, then for arbitrary, $0 < \xi < 1$ the $100(1-\xi)\%$ credible interval ϕ can be obtained as $(\phi^{[k]},\phi^{[k+N-(\xi N+1)})$, where $k = 1,2,...,[N\xi]$. Therefore, the $100(1-\xi)\%$ credible interval can be constructed based on the condition given below:

$$(\phi^{[k^*+N-(\xi N+1)} - \hat{\tau}^{[k^*]]}) = min_{k=1}^{N\xi}\ (\phi^{[k+N-(\xi N+1)} - \phi^{[k]]}). \tag{42}$$

where [z] denotes the greatest integer less or equal to z.

5. Simulation Study

We conduct an extensive Monte Carlo simulation study to assess the relative precision of the proposed estimates. We generate 10^4 samples from PRD with parameters τ and ν for each combination of different parameter values and sample size. Here, we choose the parameters as $\tau = 0.5, 1, 2$, $\beta = 1, 1.5, 1.5$ and the corresponding values of $R(t)$ at $t = 0.5, 1.50, 1.25$ are 0.7788, 0.6065, and 0.5813, and sample sizes as $n = 20, 40, 60, 80, 100$. The MLEs and Bayes estimates using Lindley and MCMC techniques are applied to simulate the data. In the Bayes estimates, we consider the non-informative prior for scale parameter τ and informative gamma prior for shape parameter τ. In addition, the values of hyper-parameters $a = 2, 3$ and $b = 2, 2$ are chosen such that the prior mean is equal to the true value of the parameter. Moreover, the Bayes estimates are obtained using SELF, GELF, and LINEX loss functions. For GELF and LLF, the constant q is taken to be -0.5 and 0.5, respectively. The evaluation of the estimates has been conducted with consideration given to the following standpoint:

- **Average absolute bias (AAB):** Let ψ and $\hat{\psi}$ denote the actual and predicted value of the parameters, and N represent the total number of replications. Then the average absolute value is defined as follows:

$$AAB = \frac{1}{N}\sum_{i=1}^{N} |\psi_i - \hat{\psi}_i|.$$

  A smaller AAB value suggests the experimental data exhibits higher accuracy with the predictive model.

- **Mean squared error (MSE):** The MSE is defined as follows:

$$MSE = \frac{1}{N}\sum_{i=1}^{N} (\psi_i - \hat{\psi})^2.$$

  The smaller value signifies superior performance of the estimates.

- **Average length (AL):** AL of the interval estimates at a significance level of ξ has been assessed. A shorter length indicates superior performance in the estimation of intervals.

- **Coverage probability (CP):** The probability of containing the actual parameter values within the estimated interval ranges.

The average absolute bias (AAB) and mean square error (MSE) of τ, ν and $R(t)$ are presented in Tables 1, 3, and 5, respectively. Moreover, the associated 95% approximate confidence and HPD credible intervals are also obtained and listed in Tables 2, 4, and 6, respectively. The following interpretations can be obtained from these Tables:
(i) Tables 1, 3, and 5 show that as the sample size n increases, the average absolute bias (AAB) and mean square error (MSE) of all estimates decrease as expected. This suggests

that all the estimators are asymptotically efficient.
(ii) The performance of the Bayes estimates is better than the classical approach in terms of AAB and MSE.
(iii) In Bayesian estimation, the MCMC approach provides superior outcomes to the Lindley approximation techniques regarding AAB and MSE. Comparing all the estimators, the following mathematical relation can be obtained:

$$AAB_{MLE} > AAB_{Lindley} > AAB_{MCMC},$$

and

$$MSE_{MLE} > MSE_{Lindley} > MSE_{MCMC}.$$

(iv) Concerning the loss functions, GELF ($q = 0.5$) under the MCMC approach is a better choice for the parameters τ and ν and LLF ($q = -0.5$) is better for the reliability function in terms of AAB and MSE.
(v) In Tables 2, 4, and 6, the highest posterior density (HPD) credible interval is shorter than the approximate confidence intervals (ACIs) in terms of the average length of confidence/credible intervals.
(vi) The coverage probability (CP) of HPD credible interval is higher than the approximate confidence intervals.

Table 1: Simulation results for classical and Bayes estimator of parameters $\tau = 0.5$, $\nu = 1$ and $R(0.5) = 0.7788$ with biases (first row) and MSEs (second row)

			Lindley					MCMC				
	n	MLE	SELF	GELF		LLF		SELF	GELF		LLF	
				q = -0.5	q = 0.5	q =-0.5	q = 0.5		q = -0.5	q =0.5	q = -0.5	q = 0.5
τ	20	0.0769	0.0740	0.0730	0.0722	0.0750	0.0792	0.0737	0.0728	0.0713	0.0745	0.0729
		0.0114	0.0110	0.0102	0.0106	0.0108	0.0109	0.0103	0.0100	0.0095	0.0105	0.0100
ν		0.1371	0.1281	0.1288	0.1258	0.1312	0.1295	0.1266	0.1244	0.1207	0.1302	0.1234
		0.0355	0.0326	0.0312	0.0288	0.0330	0.0300	0.0317	0.0303	0.0277	0.0344	0.0295
$R(t)$		0.0610	0.0542	0.0550	0.0560	0.0541	0.0552	0.0538	0.0542	0.0553	0.0536	0.0541
		0.0058	0.0046	0.0048	0.0049	0.0046	0.0048	0.0046	0.0046	0.0048	0.0045	0.0046
τ	40	0.0551	0.0544	0.0538	0.0534	0.0540	0.0539	0.0534	0.0529	0.0522	0.0537	0.0530
		0.0052	0.0047	0.0048	0.0047	0.0050	0.0049	0.0049	0.0048	0.0046	0.0049	0.0048
ν		0.0870	0.0851	0.0846	0.0822	0.0859	0.0849	0.0844	0.0835	0.0819	0.0856	0.0832
		0.0127	0.0120	0.0118	0.0118	0.0127	0.0120	0.0120	0.0117	0.0112	0.0123	0.0116
$R(t)$		0.0434	0.0417	0.0416	0.0410	0.0414	0.0412	0.0403	0.0404	0.0407	0.0402	0.0403
		0.0029	0.0027	0.0027	0.0028	0.0027	0.0027	0.0026	0.0026	0.0026	0.0025	0.0026
τ	60	0.0422	0.0419	0.0418	0.0410	0.0417	0.0416	0.0413	0.0411	0.0408	0.0415	0.0411
		0.0030	0.0029	0.0029	0.0028	0.0029	0.0029	0.0029	0.0028	0.0028	0.0029	0.0029
ν		0.0721	0.0711	0.0710	0.0712	0.0718	0.0704	0.0710	0.0705	0.0696	0.0717	0.0704
		0.0088	0.0086	0.0085	0.0084	0.0088	0.0085	0.0086	0.0084	0.0082	0.0088	0.0084
$R(t)$		0.0350	0.0339	0.0338	0.0336	0.0339	0.0337	0.0335	0.0335	0.0338	0.0334	0.0335
		0.0019	0.0019	0.0019	0.0019	0.0018	0.0018	0.0018	0.0018	0.0018	0.0017	0.0018
τ	80	0.0360	0.0358	0.0356	0.0359	0.0360	0.0357	0.0355	0.0354	0.0352	0.0356	0.0354
		0.0021	0.0021	0.0021	0.0021	0.0021	0.0021	0.0021	0.0020	0.0020	0.0021	0.0021
ν		0.0057	0.0057	0.0055	0.0055	0.0057	0.0055	0.0056	0.0055	0.0054	0.0056	0.0055
		0.0298	0.0288	0.0285	0.0290	0.0289	0.0290	0.0287	0.0287	0.0289	0.0286	0.0287
$R(t)$		0.0298	0.0290	0.0290	0.0289	0.0289	0.0288	0.0287	0.0287	0.0289	0.0286	0.0287
		0.0014	0.0013	0.0013	0.0013	0.0013	0.0013	0.0013	0.0013	0.0013	0.0013	0.0013
τ	100	0.0331	0.0329	0.0326	0.0326	0.0329	0.0328	0.0326	0.0326	0.0324	0.0327	0.0326
		0.0018	0.0018	0.0018	0.0018	0.0018	0.0018	0.0018	0.0017	0.0017	0.0018	0.0018
ν		0.0554	0.0551	0.0550	0.0544	0.0549	0.0549	0.0543	0.0541	0.0536	0.0546	0.0540
		0.0050	0.0050	0.0048	0.0048	0.0049	0.0048	0.0048	0.0048	0.0047	0.0049	0.0048
$R(t)$		0.0277	0.0271	0.0266	0.0269	0.0268	0.0268	0.0266	0.0266	0.0267	0.0266	0.0266
		0.0012	0.0012	0.0012	0.0011	0.0012	0.0011	0.0011	0.0011	0.0011	0.0011	0.0011

Table 2: The 95% confidence interval and HPD credible interval for parameters $\tau = 0.5$, $\nu = 1$ and $R(0.5) = 0.7788$

Parameters	n	ACI	CPs	HPD	CPs
τ	20	0.3679	0.9410	0.3350	0.9450
ν		0.6281	0.9390	0.5652	0.9460
$R(0.5)$		0.2849	0.9010	0.2460	0.9220
τ	40	0.2527	0.9430	0.2368	0.9550
ν		0.4243	0.9370	0.3957	0.9450
$R(0.5)$		0.2051	0.9280	0.1864	0.9340
τ	60	0.2030	0.9510	0.1916	0.9410
ν		0.3466	0.9510	0.3277	0.9350
$R(0.5)$		0.1677	0.9220	0.1557	0.9250
τ	60	0.1742	0.9470	0.1652	0.9560
ν		0.2950	0.9540	0.2790	0.9670
$R(0.5)$		0.1464	0.9500	0.1365	0.9520
τ	100	0.1560	0.9370	0.1479	0.9380
ν		0.2638	0.9440	0.2497	0.9580
$R(0.5)$		0.1309	0.9430	0.1226	0.9530

Table 3: Simulation results for classical and Bayes estimator of parameters $\tau = 1$, $\nu = 1.5$ and $R(1.5) = 0.6065$ with biases (first row) and MSEs (second row)

			Lindley						MCMC				
	n	MLE	SELF	GELF		LLF		SELF	GELF		LLF	
				q = -0.5	q = 0.5	q = -0.5	q = 0.5		q = -0.5	q = 0.5	q = -0.5	q = 0.5
τ	20	0.1616	0.1430	0.1428	0.1442	0.1456	0.1425	0.1362	0.1362	0.1370	0.1376	0.1352
		0.0477	0.0322	0.0312	0.0311	0.0319	0.0305	0.0303	0.0300	0.0297	0.0312	0.0295
ν		0.2942	0.2433	0.2372	0.2352	0.2525	0.2388	0.2329	0.2291	0.2240	0.2462	0.2232
		0.1850	0.1083	0.1032	0.0874	0.1163	0.0879	0.0989	0.0936	0.0852	0.1177	0.0864
$R(1.5)$		0.0748	0.0686	0.0690	0.0722	0.0674	0.0681	0.0671	0.0684	0.0713	0.0666	0.0676
		0.0087	0.0073	0.0074	0.0082	0.0073	0.0074	0.0073	0.0075	0.0081	0.0072	0.0074
τ	40	0.1054	0.0982	0.0982	0.0986	0.0990	0.0971	0.0968	0.0968	0.0970	0.0973	0.0964
		0.0185	0.0156	0.0154	0.0157	0.0152	0.0151	0.0145	0.0145	0.0144	0.0148	0.0144
ν		0.1879	0.1754	0.1747	0.1745	0.1785	0.1737	0.1673	0.1659	0.1638	0.1715	0.1639
		0.0618	0.0485	0.0472	0.0451	0.0502	0.0448	0.0463	0.0452	0.0434	0.0494	0.0439
$R(1.5)$		0.0528	0.0504	0.0512	0.0519	0.0507	0.0506	0.0498	0.0503	0.0513	0.0496	0.0500
		0.0044	0.0041	0.0041	0.0043	0.0040	0.0042	0.0040	0.0041	0.0043	0.0040	0.0040
τ	60	0.0850	0.0811	0.0811	0.0813	0.0818	0.0812	0.0808	0.0808	0.0808	0.0811	0.0805
		0.0123	0.0111	0.0114	0.0116	0.0112	0.0113	0.0106	0.0105	0.0104	0.0107	0.0104
ν		0.1478	0.1409	0.1407	0.1405	0.1484	0.1481	0.1388	0.1380	0.1366	0.1413	0.1368
		0.0384	0.0348	0.0332	0.0322	0.0350	0.0332	0.0326	0.0319	0.0308	0.0342	0.0312
$R(1.5)$		0.0416	0.0412	0.0414	0.0420	0.0419	0.0420	0.0407	0.0410	0.0416	0.0406	0.0408
		0.0027	0.0026	0.0027	0.0027	0.0026	0.0026	0.0026	0.0026	0.0027	0.0026	0.0026
τ	80	0.0722	0.0698	0.0694	0.0694	0.0698	0.0698	0.0688	0.0687	0.0688	0.0690	0.0686
		0.0083	0.0076	0.0076	0.0076	0.0076	0.0076	0.0075	0.0074	0.0074	0.0075	0.0074
ν		0.1244	0.1252	0.1236	0.1197	0.1188	0.1199	0.1187	0.1182	0.1174	0.1202	0.1175
		0.0259	0.0238	0.0235	0.0229	0.0240	0.0232	0.0228	0.0225	0.0220	0.0236	0.0222
$R(1.5)$		0.0357	0.0354	0.0354	0.0356	0.0351	0.0352	0.0350	0.0352	0.0356	0.0349	0.0351
		0.0020	0.0020	0.0020	0.0020	0.0020	0.0020	0.0019	0.0020	0.0020	0.0019	0.0019
τ	100	0.0622	0.0615	0.0614	0.0616	0.0619	0.0620	0.0608	0.0608	0.0609	0.0609	0.0607
		0.0063	0.0059	0.0059	0.0059	0.0060	0.0059	0.0059	0.0059	0.0059	0.0059	0.0058
ν		0.1111	0.1096	0.1089	0.1090	0.1098	0.1097	0.1081	0.1079	0.1077	0.1089	0.1075
		0.0208	0.0191	0.0194	0.0193	0.0194	0.0193	0.0190	0.0188	0.0185	0.0195	0.0186
$R(1.5)$		0.0325	0.0325	0.0324	0.0325	0.0323	0.0324	0.0324	0.0325	0.0329	0.0323	0.0325
		0.0017	0.0016	0.0016	0.0017	0.0017	0.0016	0.0016	0.0016	0.0017	0.0016	0.0016

Table 4: The average lengths of 95% confidence intervals and coverage probability for parameters $\tau = 1$, $\nu = 1.5$ and $R(1.5) = 0.6065$

Parameters	n	**ACI**	**CPs**	**HPD**	**CPs**
τ		0.7401	0.9400	0.6385	0.9460
ν	20	1.403	0.949	1.109	0.9150
$R(0.5)$		0.3441	0.9150	0.3094	0.9270
τ		0.5027	0.9560	0.4550	0.9580
ν	40	0.9016	0.9390	0.7859	0.9470
$R(0.5)$		0.2464	0.9260	0.2277	0.9330
τ		0.4067	0.9380	0.3759	0.9130
ν	60	0.7220	0.9560	0.6534	0.9220
$R(0.5)$		0.2020	0.9480	0.1897	0.9420
τ		0.3483	0.9590	0.3238	0.9460
ν	60	0.6164	0.9530	0.5636	0.9540
$R(0.5)$		0.1752	0.9420	0.1647	0.9310
τ		0.3107	0.9520	0.2909	0.9570
ν	100	0.5452	0.9560	0.5026	0.9580
$R(0.5)$		0.1570	0.9460	0.1482	0.9530

Table 5: Simulation results for classical and Bayes estimator of parameters $\tau = 2$, $\nu = 1.5$ and $R(1.25) = 0.5813$ with biases (first row) and MSEs (second row)

			Lindley					**MCMC**				
	n	**MLE**	**SELF**	**GELF**		**LLF**		**SELF**	**GELF**		**LLF**	
				q = -0.5	q = 0.5	q = -0.5	q = 0.5		q = -0.5	q = 0.5	q = -0.5	q = 0.5
τ		0.3327	0.2815	0.2822	0.2826	0.2889	0.2807	0.2679	0.2672	0.2674	0.2756	0.2628
		0.2065	0.1430	0.1419	0.1329	0.1494	0.1436	0.1239	0.1217	0.1187	0.1348	0.1158
ν		0.2989	0.2417	0.2403	0.2377	0.2499	0.2381	0.2275	0.2224	0.2143	0.2442	0.2153
	20	0.1896	0.1093	0.1023	0.0911	0.1384	0.0923	0.0992	0.0922	0.0813	0.1288	0.0824
$R(1.25)$		0.0748	0.0667	0.0671	0.0670	0.0665	0.0672	0.0648	0.0658	0.0688	0.0645	0.0651
		0.0087	0.0070	0.0073	0.0080	0.0070	0.0071	0.0068	0.0071	0.0080	0.0068	0.0069
τ		0.2126	0.2078	0.2060	0.2066	0.2080	0.2023	0.1935	0.1933	0.1934	0.1959	0.1918
		0.0752	0.0611	0.0606	0.0606	0.0622	0.0599	0.0590	0.0586	0.0581	0.0612	0.0574
ν		0.1917	0.1888	0.1756	0.1758	0.1871	0.1781	0.1709	0.1695	0.1674	0.1754	0.1674
	40	0.0665	0.0514	0.0495	0.0480	0.0542	0.0480	0.0494	0.0479	0.0456	0.0533	0.0462
$R(1.25)$		0.0537	0.0521	0.0525	0.0536	0.0517	0.0521	0.0510	0.0515	0.0526	0.0509	0.0512
		0.0045	0.0041	0.0043	0.0043	0.0042	0.0042	0.0040	0.0041	0.0043	0.0040	0.0041
τ		0.1648	0.1671	0.1671	0.1689	0.1683	0.1695	0.1690	0.1575	0.1581	0.1582	0.1570
		0.0443	0.0411	0.0409	0.0411	0.0406	0.0396	0.0390	0.0390	0.0390	0.0396	0.0386
ν		0.1444	0.1390	0.1389	0.1393	0.1399	0.1381	0.1366	0.1361	0.1358	0.1382	0.1353
	60	0.0358	0.0324	0.0321	0.0302	0.0324	0.0304	0.0314	0.0301	0.0294	0.0316	0.0296
$R(1.25)$		0.0427	0.0424	0.0426	0.0422	0.0420	0.0423	0.0421	0.0405	0.0412	0.0409	0.0412
		0.0029	0.0028	0.0028	0.0029	0.0028	0.0028	0.0027	0.0028	0.0028	0.0027	0.0027
τ		0.1509	0.1430	0.1427	0.1425	0.1442	0.1420	0.1334	0.1326	0.1324	0.1348	0.1321
		0.0366	0.0330	0.0328	0.0326	0.0336	0.0325	0.0319	0.0317	0.0316	0.0316	0.0314
ν		0.1309	0.1236	0.1228	0.1213	0.1257	0.1219	0.1040	0.1130	0.1152	0.1159	0.1128
	80	0.0281	0.0249	0.0245	0.0238	0.0258	0.0241	0.0229	0.0238	0.0228	0.0238	0.0220
$R(1.25)$		0.0372	0.0360	0.0361	0.0363	0.0359	0.0360	0.0350	0.0350	0.0351	0.0340	0.0349
		0.0021	0.0020	0.0020	0.0021	0.0020	0.0020	0.0019	0.0019	0.0019	0.0019	0.0019
τ		0.1302	0.1259	0.1259	0.1260	0.1265	0.1255	0.1157	0.1144	0.1170	0.1163	0.1098
		0.0269	0.0248	0.0247	0.0247	0.0251	0.0245	0.0235	0.0237	0.0238	0.0243	0.0235
ν		0.1117	0.1074	0.1069	0.1060	0.1087	0.1063	0.0974	0.0968	0.0961	0.0984	0.0961
	100	0.0204	0.0185	0.0183	0.0179	0.0190	0.0180	0.0174	0.0173	0.0180	0.0179	0.0171
$R(1.25)$		0.0319	0.0312	0.0313	0.0315	0.0312	0.0312	0.0305	0.0304	0.0304	0.0299	0.0305
		0.0016	0.0016	0.0016	0.0016	0.0016	0.0016	0.0015	0.0015	0.0015	0.0015	0.0016

Table 6: The average lengths of 95% confidence intervals and coverage probability for the parameters $\tau = 2$, $\nu = 1.5$ and $R(1.25) = 0.5813$

Parameters	n	ACI	CPs	HPD	CPs
τ	20	1.498	0.939	1.337	0.948
ν		1.431	0.956	1.165	0.9560
$R(1.25)$		0.3471	0.9210	0.3195	0.9410
τ	40	1.0089	0.9540	0.9824	0.9590
ν		0.9134	0.9470	0.8443	0.9450
$R(1.25)$		0.2483	0.9290	0.2414	0.9440
τ	60	0.8065	0.9540	0.7753	0.9580
ν		0.7117	0.9460	0.6901	0.9400
$R(1.25)$		0.2038	0.9430	0.2035	0.9450
τ	80	0.6999	0.9480	0.6439	0.9480
ν		0.6212	0.9510	0.5988	0.9550
$R(1.25)$		0.1766	0.9420	0.1835	0.9460
τ	100	0.6219	0.9470	0.5914	0.9540
ν		0.5483	0.9540	0.5846	0.9530
$R(1.25)$		0.1582	0.9490	0.1494	0.9550

6. Application

In this section, we employ two real datasets to demonstrate the process of calculating estimators for the unknown model parameters.

Dataset I

The dataset considered by Bing Long (2023) represents the failure time of the mechanical components. To check whether dataset I better fits the considered model or not, we utilized different goodness of fit criteria, namely the Akaike information criterion (AIC), Bayesian information criterion (BIC), Hannan-Quinn information criterion (HQIC), corrected Akaike information criterion (AICC). In addition, K-S distance and p-value are obtained and listed in Table 7. Afterwards, MLEs of the parameters τ and ν are completed and presented in Table 7. For comparison purposes, we have considered different lifetime models such as the Rayleigh distribution (RD), Exponentiated Rayleigh distribution (ERD), Weibull Rayleigh distribution (WRD), and Transmuted Rayleigh distribution (TRD), and fitted them to the empirical dataset depicted in Figure 2. It has been observed that the PRD is a good fit for dataset I in comparison with other lifetime models. The estimated values of τ, ν and $R(t)$ under frequentist and Bayesian approaches for the real dataset I are listed in Table 8. The interval estimates are also derived and presented in Table 9.

Table 7: Goodness of fit measures and MLEs for the dataset I

Model	$\hat{\tau}$	$\hat{\nu}$	**-2 log(l)**	**AIC**	**BIC**	**HQIC**	**AICC**	**K-S**	**p-value**
PRD	1.2089	1.1055	48.55	52.55	55.35	53.45	52.99	0.07901	0.985
RD	-	0.9897	50.23	52.23	53.63	52.67	52.37	0.09540	0.924
ERD	1.4863	0.6511	48.02	52.02	54.82	52.91	52.46	0.07908	0.984
WRD	0.8641	0.6840	48.09	52.09	54.89	52.99	52.54	0.38940	0.00013
TRD	0.85215	-0.68747	48.85	52.85	55.65	53.74	53.29	0.08019	0.982

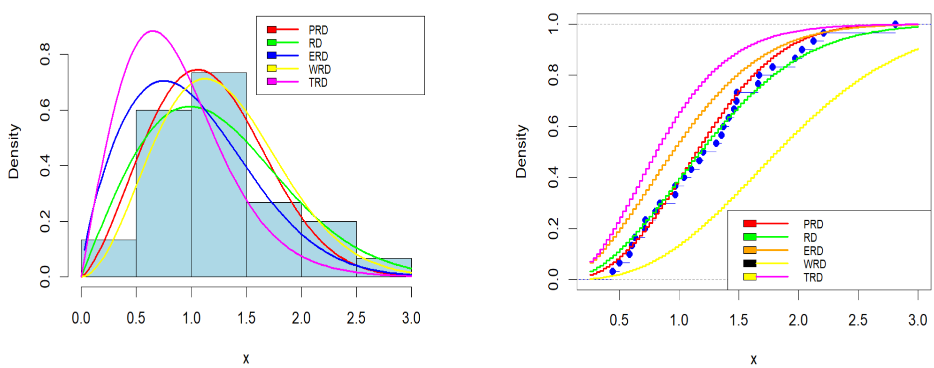

Figure 2: The estimated density and fitted plot of PRD and different lifetime models for dataset I.

Table 8: Estimated values of τ, ν and $R(0.5)$ based on the real dataset I

		Lindley					**MCMC**				
	MLE	**SELF**	**GELF**		**LLF**		**SELF**	**GELF**		**LLF**	
			q = -0.5	q = 0.5	q =-0.5	q = 0.5		q = -0.5	q =0.5	q = -0.5	q = 0.5
τ	1.2089	1.0817	1.0437	1.1832	1.0807	1.0709	1.2052	1.1988	1.1861	1.2131	1.1975
ν	1.1055	1.2123	1.2083	1.2176	1.1125	1.1074	1.1321	1.1269	1.1169	1.1384	1.12616
$R(0.5)$	0.9263	0.9110	0.9106	0.9102	0.9126	0.9106	0.9211	0.9207	0.9201	0.9214	0.9208

Table 9: The 95% ACI and HPD credible intervals of the parameters τ, ν and $R(t)$ based on the real dataset I

Parameters	**ACI**	**HPD**
τ	(0.8785,1.5394)	(0.8824,1.5346)
ν	(0.8220,1.3889)	(0.8718,1.3112)
$R(0.5)$	(0.8624,0.9901)	(0.8603,0.9787)

Dataset II

Dataset II represents the breaking stress of carbon fibers of 50 mm length (GPa) that has been analyzed by Al-Aqtash et al. (2014). Later on, Bhat and Ahmad (2020) considered the same dataset and demonstrated that PRD best fit to dataset II based on the different goodness of fit tests and various graphs. The MLEs and BEs of the parameters τ, ν, and $R(t)$ at $t = 2$ are presented in Table 10. For BEs, 5000 MCMC samples are generated, and the first 500 samples are discarded to avoid the initial guess. Note that non-informative prior information is considered because no prior information is available in this experiment. It has been observed that the MLEs and BEs estimates are pretty close to each other. The 95% ACI/HPD credible intervals are constructed and listed in Table 11. It can be seen that HPD credible intervals are more faithful than ACI.

Table 10: Estimated values of τ, ν and $R(2)$ based on the real dataset II

		Lindley					MCMC					
	MLE	**SELF**	**GELF**		**LLF**		**SELF**	**GELF**		**LLF**		
			q = -0.5	q = 0.5	q =-0.5	q = 0.5		q = -0.5	q =0.5	q = -0.5	q = 0.5	
τ	1.721	1.694	1.684	1.751	1.726	1.725	1.749	1.743	1.733	1.757	1.740	
ν	4.850	5.043	4.956	5.369	4.864	4.802	5.258	5.179	5.027	5.736	4.887	
$R(2)$	0.7939	0.7976	0.7939	0.7967	7698	0.7667	0.8010	0.8004	0.7993	0.8015	0.8006	

Table 11: The 95% ACI and HPD credible intervals for the parameters τ, ν and $R(2)$ based on the dataset II

Parameters	**ACI**	**HPD**	
τ	(1.3965, 2.0445)	(1.5514, 1.8801)	
ν	(2.8181, 6.8833)	(3.6821, 5.9394)	
$R(2)$	(0.6813, 0.8872)	(0.7115, 0.8611)	

7. Conclusions

This paper explores frequentist and Bayesian inference for the parameters and reliability estimation of the power Rayleigh distribution using a complete sample. The maximum likelihood estimates and approximate confidence intervals of the parameters have been computed using iterative procedures via the “nleqslv" package. Further, two approximation techniques have been considered: the Lindley and M-H algorithm for Bayesian computation under various loss functions. The performance of all estimators has been assessed through a Monte Carlo simulation study. The simulation experiment demonstrates that the Bayes approach dominates the frequentist approach based on absolute bias and mean square error. When comparing the Lindley and MCMC methods, the MCMC approach is more efficient than the Lindley technique. Moreover, the performance of the GELF and LINEX loss functions is better for the parameters and survival function in all cases, respectively. Finally, the Bayesian MCMC approach is recommended. In future studies, the scope of this study could be extended to hierarchical and empirical Bayesian estimation methods, especially for censored data and record values.

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