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Kumaraswamy family of generalized Type II Exponentiated Half Logistic-G distribution: properties and statistical inference

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Abstract

We present a new family of distributions (FDs) named the Kumaraswamy type II exponentiated half logistic-G (K-TIIEHL-G). Notably, this FDs can be represented as an infinite linear combination of exponentiated-G densities enabling the derivation of important statistical properties. The density and hazard rate geometries were investigated for some special cases. The model parameters were determined using the maximum likelihood technique and Monte Carlo simulation experiments were carried out to validate the consistency of the model parameters. The Kumaraswamy type II exponentiated half logistic-Weibull (K-TIIEHL-W), a member of the K-TIIEHL-G family, was applied to two sets of real data. The results indicated that the K-TIIEHL-W model significantly outperformed several established contending equi-parameter models in terms of data fitting.

Key words: Kumaraswamy-G, Type II Exponentiated Half logistic-G, Exponentiated-G, Maximum Likelihood Estimation, Monte Carlo Simulations.

1. Introduction

Not all real-world lifespan data can be adequately modeled using existing distributions. To enhance flexibility, classical distributions often require modifications through the inclusion of one or more parameters. These parameters help address kurtosis and skewness, leading to what are termed generalized or modified distributions, which offer improved flexibility in data modeling. Cordeiro and de Castro (2011) developed the Kumaraswamy-G FDs with the cumulative distribution function (cdf) given by

$$F(x) = 1 - [1 - G^c(x;\eta)]^\lambda \tag{1}$$

and probability distribution function (pdf)

$$f(x) = c\lambda g(x;\eta)G^{c-1}(x;\eta)\,[1 - G^c(x;\eta)]^{\lambda-1}, \tag{2}$$

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respectively, where $G(x;\eta)$ is the parent distribution, $g(x;\eta)=\frac{dG(x;\eta)}{dx}$, η is the parent distribution vector of parameters, and c and λ are two positive shape parameters. It is worthy noting that, when $c=\lambda=1$, the parent distribution is obtained. The supplementary parameters c and λ are intended to control the skewness and tail weight of the resulting distribution.

The Kumaraswamy FDs is a flexible and valuable tool for modeling data bounded on the interval [0,1]. It features closed-form expressions for its cumulative and probability density functions, facilitating straightforward computations and statistical inference. Additionally, the Kumaraswamy distribution is robust, effectively handling various types of skewness and kurtosis, and is widely used in fields like finance, hydrology, and environmental studies, particularly in Bayesian statistics. These advantages have inspired numerous researchers to extend the Kumaraswamy generator, leading to significant contributions such as the Kumaraswamy half logistic introduced by Usman, Ahsan-ul-Haq and Talib (2017), the Kumaraswamy Weibull by Cordeiro, Ortega and Nadarajah (2010), and the Topp-Leone Kumaraswamy-G distribution proposed by Ibrahim et al. (2020). Additionally, notable advancements include the Topp-Leone Kumaraswamy Fréchet by Bashiru (2024), Kumaraswamy odd Lindley-G by Chipepa, Oluyede and Makubate (2019) and the Kumaraswamy generalized power Lomax distribution by Nagarjuna, Vardhan and Vardhan (2021), among others. These advancements significantly enhance distribution theory by providing more flexible models that can accommodate diverse data behaviors, thereby improving the accuracy of statistical inference in various applications.

In this article, we propose a novel FDs called the K-TIIEHL-G. Our motivations for developing this distribution include:

- developing a new generalized distribution that, effectively accommodates variations in skewness, and kurtosis across a wide range of data types.
- generating various hazard rate geometries.
- fostering new insights in statistical modeling, paving way for future research and applications that require robust and adaptable methodologies.

We will structure our paper as follows: Section 2 introduces the new FDs, including a discussion on notable special cases. In Section 3, we will examine the statistical properties of the proposed FDs. Section 4 focuses on estimation of the unknown parameters using the maximum likelihood technique. In Section 5, we present the results obtained from Monte Carlo simulation experiments. Section 6 will showcase the practical applications of the proposed model using empirical data. Finally, Section 7 will summarize our findings and discuss their broader implications.

2. The new family of distributions

We develop the Kumaraswamy type II exponentiated half logistic-G (K-TIIEHL-G) FDs in this section, including an analysis of some specific special cases.

2.1. The K-TIIEHL-G distribution

Al-Mofleh et al. (2020) proposed the type II exponentiated half logistic-G (TIIEHL-G) FDs with cdf

$$F(x) = 1 - \left[\frac{1 - G^{\delta}(x, \varpi)}{1 + G^{\delta}(x, \varpi)}\right]^{\alpha} \tag{3}$$

and pdf

$$f(x) = \frac{2\alpha\delta g(x;\varpi)G^{\delta-1}(x;\varpi)}{(1+G^{\delta}(x;\varpi))^{\alpha+1}}\left(1 - G^{\delta}(x;\varpi)\right)^{\alpha-1}, \tag{4}$$

where $\alpha, \delta > 0$ are both shape parameters, and ϖ defines the vector of parameters from the parent distribution $G(.)$.

Replacing the parent cdf in Equation (1) with the TIIEHL-G distribution and setting $\delta = 1$ yields a new FDs called K-TIIEHL-G FDs with cdf given by

$$F(x) = 1 - \left[1 - \left(1 - [K_G(x, \varpi)]^{\alpha}\right)^{c}\right]^{\lambda}, \tag{5}$$

for $\alpha, c, \lambda > 0$ where $K_G(x, \varpi) = \frac{\bar{G}(x,\varpi)}{1+G(x,\varpi)}$ and $\bar{G}(x, \varpi) = 1 - G(x, \varpi)$. Sub-models of the K-TIIEHL-G can be obtained by setting individual parameters to unit. The pdf and hazard rate function (hrf) of the K-TIIEHL-G FDs are respectively given by

$$\begin{aligned} f(x) &= \frac{2\alpha c\lambda g(x, \varpi)\bar{G}^{\alpha-1}(x, \varpi)}{[1 + G(x, \varpi)]^{\alpha+1}}\left(1 - [K_G(x, \varpi)]^{\alpha}\right)^{c-1} \\ &\times \left[1 - \left(1 - [K_G(x, \varpi)]^{\alpha}\right)^{c}\right]^{\lambda-1} \end{aligned} \tag{6}$$

and

$$\begin{aligned} h(x) &= \frac{2\alpha c\lambda g(x, \varpi)\bar{G}^{\alpha-1}(x, \varpi)}{[1 + G(x, \varpi)]^{\alpha+1}}\left(1 - [K_G(x, \varpi)]^{\alpha}\right)^{c-1} \\ &\times \left[1 - \left(1 - [K_G(x, \varpi)]^{\alpha}\right)^{c}\right]^{-1}, \end{aligned} \tag{7}$$

where $\alpha, c, \lambda > 0$, and parent parameter vector ϖ.

2.2. Model identifiability

Identifiability refers to the ability to uniquely determine model parameters from data, which is crucial for accurate parameter estimates and reliable inferences in statistical modeling (Cobelli and Distefano (1980)). A key challenge in this context is the redundancy of model parameters, where distinct values produce identical outputs, complicating the estimation process and potentially leading to inaccuracies and ambiguous conclusions.

We seek to evaluate the identifiability of the K-TIIEHL-G FD. To achieve this, we examine two sets of parameters, $\Delta_1 = (\alpha_1, c_1, \lambda_1, \varpi_1)$ and $\Delta_2 = (\alpha_2, c_2, \lambda_2, \varpi_2)$.

The likelihood ratio is defined as follows:

$$\begin{aligned} L &= \left(\frac{\alpha_1 c_1 \lambda_1}{\alpha_2 c_2 \lambda_2}\right)\left(\frac{g(x;\varpi_1)}{g(x;\varpi_1)}\right)\left(\frac{[\bar{G}(x,\varpi_1)]^{\alpha_1-1}}{[\bar{G}(x,\varpi_2)]^{\alpha_2-1}}\right)\left(\frac{[1+G(x,\varpi_1)]^{\alpha_1+1}}{[1+G(x,\varpi_2)]^{\alpha_2+1}}\right) \\ &\times \left(\frac{\left(1-[K_G(x,\varpi_1)]^{\alpha_1}\right)^{c_1-1}}{\left(1-[K_G(x,\varpi_2)]^{\alpha_2}\right)^{c_2-1}}\right)\left(\frac{\left[1-\left(1-[K_G(x,\varpi_1)]^{\alpha_1}\right)^{c_1}\right]^{\lambda_1-1}}{\left[1-\left(1-[K_G(x,\varpi_2)]^{\alpha_2}\right)^{c_2}\right]^{\lambda_2-1}}\right). \end{aligned}$$

Consider the logarithm of the likelihood, that is

$$\begin{aligned} Log(L) &= \log\left(\frac{\alpha_1 c_1 \lambda_1}{\alpha_2 c_2 \lambda_2}\right) + \log(g(x;\varpi_1)) - \log(g(x;\varpi_2)) + (\alpha_1-1)\log\left(\bar{G}(x,\varpi_1)\right) \\ &- (\alpha_2-1)\log\left(\bar{G}(x,\varpi_2)\right) + (\alpha_1+1)\log\left([1+G(x,\varpi_1)]\right) \\ &- (\alpha_2+1)\log\left([1+G(x,\varpi_2)]\right) + (c_1-1)\log\left(1-[K_G(x,\varpi_1)]^{\alpha_1}\right) \\ &- (c_2-1)\log\left(1-[K_G(x,\varpi_2)]^{\alpha_2}\right) \\ &+ (\lambda_1-1)\log\left(1-\left(1-[K_G(x,\varpi_1)]^{\alpha_1}\right)^{c_1}\right) \\ &- (\lambda_2-1)\log\left(1-\left(1-[K_G(x,\varpi_2)]^{\alpha_2}\right)^{c_2}\right). \end{aligned}$$

The derivative of the log-likelihood with respect to x is now given by

$$\begin{aligned} \frac{d\log(L)}{d(x)} &= \frac{g'(x;\varpi_1)}{g(x;\varpi_1)} - \frac{g'(x;\varpi_2)}{g(x;\varpi_2)} + \frac{(\alpha_1-1)}{\bar{G}(x,\varpi_1)}\frac{d\bar{G}(x,\varpi_1)}{dx} - \frac{(\alpha_2-1)}{\bar{G}(x,\varpi_2)}\frac{d\bar{G}(x,\varpi_2)}{dx} \\ &- \frac{(\alpha_1+1)}{1+G(x,\varpi_1)}\frac{dG(x,\varpi_1)}{dx} + \frac{(\alpha_2+1)}{1+G(x,\varpi_2)}\frac{dG(x,\varpi_2)}{dx} \\ &- (c_1-1)\frac{(1+G(x,\varpi_1))\frac{d\bar{G}(x,\varpi_1)}{dx} - \bar{G}(x,\varpi_1)\frac{dG(x,\varpi_1)}{dx}}{(1+G(x,\varpi_1))^2\,(1-[K_G(x,\varpi_1)])} \\ &+ (c_2-1)\frac{(1+G(x,\varpi_2))\frac{d\bar{G}(x,\varpi_2)}{dx} - \bar{G}(x,\varpi_2)\frac{dG(x,\varpi_2)}{dx}}{(1+G(x,\varpi_2))^2\,(1-[K_G(x,\varpi_2)])} \\ &- (\lambda_1-1)c_1 A + (\lambda_2-1)c_1 B, \end{aligned}$$

where $A = \frac{\left(1-[K_G(x,\varpi_1)]^{\alpha_1}\right)^{c_1-1}\alpha_1[K_G(x,\varpi_1)]^{\alpha_1-1}\frac{(1+G(x,\varpi_1))\frac{d\bar{G}(x,\varpi_1)}{dx}-\bar{G}(x,\varpi_1)\frac{dG(x,\varpi_1)}{dx}}{(1+G(x,\varpi_1))^2}}{1-\left(1-[K_G(x,\varpi_1)]^{\alpha_1}\right)^{c_1}}$ and

$$B = \frac{\left(1-[K_G(x,\varpi_2)]^{\alpha_2}\right)^{c_2-1}\alpha_2[K_G(x,\varpi_2)]^{\alpha_2-1}\frac{(1+G(x,\varpi_2))\frac{d\bar{G}(x,\varpi_2)}{dx}-\bar{G}(x,\varpi_2)\frac{dG(x,\varpi_2)}{dx}}{(1+G(x,\varpi_2))^2}}{1-\left(1-[K_G(x,\varpi_2)]^{\alpha_2}\right)^{c_2}}.$$

Therefore, $\frac{d\log(L)}{dx} = 0$, if $\Delta_1 = \Delta_2$, implying that the parameters can be uniquely determined.

2.3. Some Particular Models

In this subsection, specific cases of K-TIIEHL-G FDs are delineated by specifying the parent cdf and pdf in Equations (6) and (5). The Lindley, Weibull and exponential distributions serve as baseline distributions.

2.3.1 Kumaraswamy Type II Exponentiated Half logistic-Lindley (K-TIIEHL-Lin) distribution

The Lindley distribution has cdf $G(x) = 1 - \left(1 + \frac{sx}{1+s}\right) e^{-sx}$ and pdf $g(x) = \frac{s^2}{1+s}(1 + x)e^{-sx}$, for $s, x > 0$. Utilizing the Lindley as the parent distribution, we have the K-TIIEH-Lin distribution with cdf, pdf and hrf respectively given by

$$F(x) = 1 - \left[1 - \left(1 - \left[\frac{\left(1 + \frac{sx}{1+s}\right) e^{-sx}}{2 - \left(1 + \frac{sx}{1+s}\right) e^{-sx}}\right]^{\alpha}\right)^{c}\right]^{\lambda},$$

$$\begin{aligned} f(x) &= \frac{\frac{2\alpha c\lambda s^2}{(1+s)(1+x)} e^{-sx} \left(\left(1 + \frac{sx}{1+s}\right) e^{-sx}\right)^{\alpha-1}}{\left(2 - \left(1 + \frac{sx}{1+s}\right) e^{-sx}\right)^{\alpha+1}} \left(1 - \left[\frac{\left(1 + \frac{sx}{1+s}\right) e^{-sx}}{2 - \left(1 + \frac{sx}{1+s}\right) e^{-sx}}\right]^{\alpha}\right)^{c-1} \\ &\times \left[1 - \left(1 - \left[\frac{\left(1 + \frac{sx}{1+s}\right) e^{-sx}}{2 - \left(1 + \frac{sx}{1+s}\right) e^{-sx}}\right]^{\alpha}\right)^{c}\right]^{\lambda-1}, \end{aligned}$$

and

$$\begin{aligned} h(x) &= \frac{\frac{2\alpha c\lambda s^2}{(1+s)(1+x)} e^{-sx} \left(\left(1 + \frac{sx}{1+s}\right) e^{-sx}\right)^{\alpha-1}}{\left(2 - \left(1 + \frac{sx}{1+s}\right) e^{-sx}\right)^{\alpha+1}} \left(1 - \left[\frac{\left(1 + \frac{sx}{1+s}\right) e^{-sx}}{2 - \left(1 + \frac{sx}{1+s}\right) e^{-sx}}\right]^{\alpha}\right)^{c-1} \\ &\times \left[1 - \left(1 - \left[\frac{\left(1 + \frac{sx}{1+s}\right) e^{-sx}}{2 - \left(1 + \frac{sx}{1+s}\right) e^{-sx}}\right]^{\alpha}\right)^{c}\right]^{-1}, \end{aligned}$$

for $\alpha, c, \lambda, s > 0$ and $x > 0$.

2.3.2 Kumaraswamy Type II Exponentiated Half logistic-Weibull (K-TIIEHL-W) distribution

Taking the one-parameter Weibull distribution whose cdf and pdf are given by $G(x) = 1 - \exp(-x^d)$, and $g(x) = dx^{d-1}\exp(-x^d)$, respectively for $x, d > 0$ as a parent distribution, we get the K-TIIEH-W distribution with cdf, pdf and hrf respectively given by

$$F(x) = 1 - \left(1 - \left(1 - [W(x;d)]^{\alpha}\right)^{c}\right)^{\lambda},$$

and pdf

$$\begin{aligned} f(x) &= 2c\alpha\lambda dx^{d-1} \left(e^{-x^d}\right)^{\alpha} \left(2 - e^{-x^d}\right)^{-\alpha-1} \left(1 - (W(x;d))^{\alpha}\right)^{c-1} \\ &\times \left(1 - \left(1 - (W(x;d))^{\alpha}\right)^{c}\right)^{\lambda-1}, \end{aligned}$$

$$\begin{aligned} h(x) &= 2c\alpha\lambda dx^{d-1} \left(e^{-x^d}\right)^{\alpha} \left(2 - e^{-x^d}\right)^{-\alpha-1} \left(1 - (W(x;d))^{\alpha}\right)^{c-1} \\ &\times \left(1 - \left(1 - (W(x;d))^{\alpha}\right)^{c}\right)^{-1}, \end{aligned}$$

where $W(x;d)=\frac{e^{-x^d}}{2-e^{-x^d}}$, for $\alpha,c,\lambda,d>0$ and $x>0$.

2.3.3 Kumaraswamy Type II Exponentiated Half logistic-Exponential (K-TIIEHL-E) distribution

Taking the exponential distribution with the cdf given as $G(x;a)=1-\exp(-ax)$, and pdf $g(x;a)=a\exp(-ax)$, for $x,a>0$ as a parent distribution, the K-TIIEH-E distribution is obtained with cdf, pdf and hrf respectively given by

$$F(x)=1-\left(1-\left(1-[E(x;a)]^{\alpha}\right)^{c}\right)^{\lambda},$$

and pdf

$$\begin{aligned} f(x) &= 2\alpha c\lambda a\left(e^{-ax}\right)^{\alpha}\left(2-e^{-ax}\right)^{-\alpha-1}\left(1-(E(x;a))^{\alpha}\right)^{c-1} \\ &\times \left(1-\left(1-(E(x;a))^{\alpha}\right)^{c}\right)^{\lambda-1}, \end{aligned}$$

$$\begin{aligned} h(x) &= 2\alpha c\lambda a\left(e^{-ax}\right)^{\alpha}\left(2-e^{-ax}\right)^{-\alpha-1}\left(1-(E(x;a))^{\alpha}\right)^{c-1} \\ &\times \left(1-\left(1-[E(x;a)]^{\alpha}\right)^{c}\right)^{-1}, \end{aligned}$$

where $E(x;a)=\frac{e^{-x^d}}{2-e^{-x^d}}$ for $\alpha,c,\lambda,a>0$ and $x>0$.

2.4. Geometries of the density and hazard functions

We selected the Lindley, Weibull, and exponential distributions as parent distributions for the K-TIIEHL-G model to plot the pdf and hrf of the K-TIIEHL-G model. This approach allows us to examine the diverse shapes exhibited by this FDs.

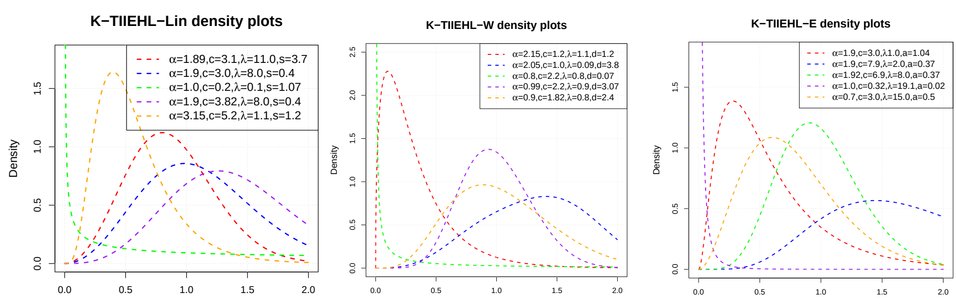

Figure 1. Density shapes for K-TIIEHL-Lin, K-TIIEHL-W and K-TIIEHL-E distributions

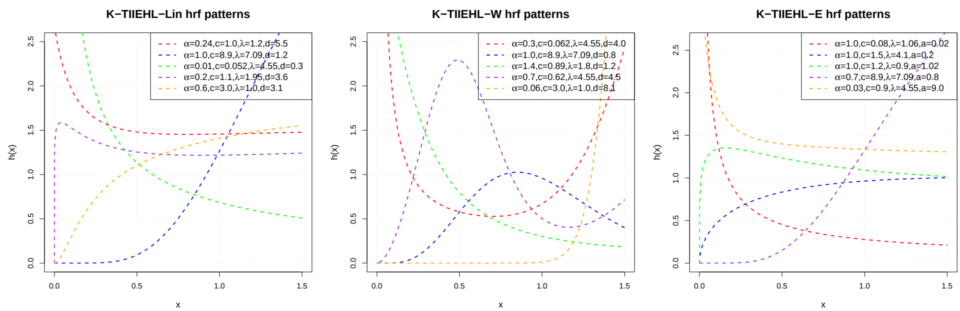

Figure 2. Hazard rate function plots of K-TIIEHL-Lin, K-TIIEHL-W and K-TIIEHL-E distributions

The plots presented in Figure [1] and Figure [2] illustrate the remarkable flexibility of this FDs. Figure [1] demonstrates a variety of density shapes, including reversed-J, positively skewed, negatively skewed, and nearly symmetric forms. Additionally, the hazard rate functions in Figure [2] encompass a range of profiles, such as bathtub, inverted bathtub, as well as both increasing and decreasing patterns. This diversity underscores the model's adaptability in capturing different statistical behaviors.

2.5. Some 3D shapes for kurtosis and skewness of special cases

This subsection showcases some 3D plots of kurtosis and skewness for the Lindley, Weibull and exponential distributions discussed Section 2.4.

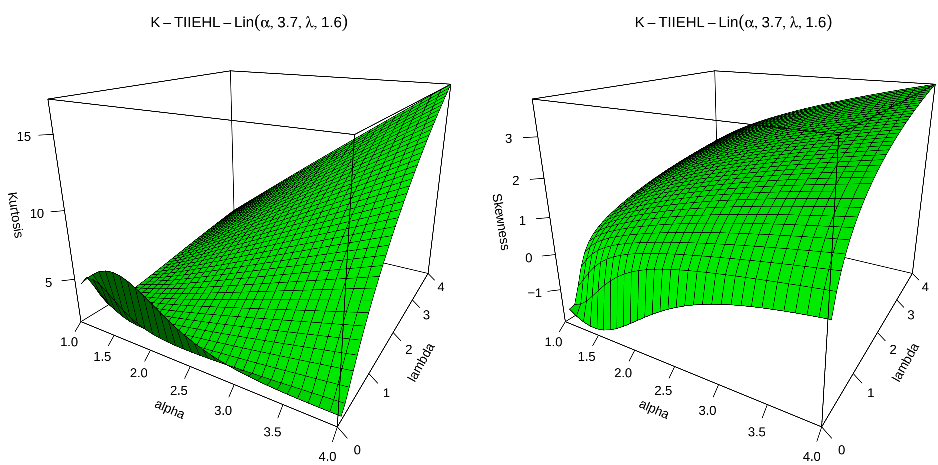

Figure 3. 3D plots of K-TIIEHL-Lin distribution

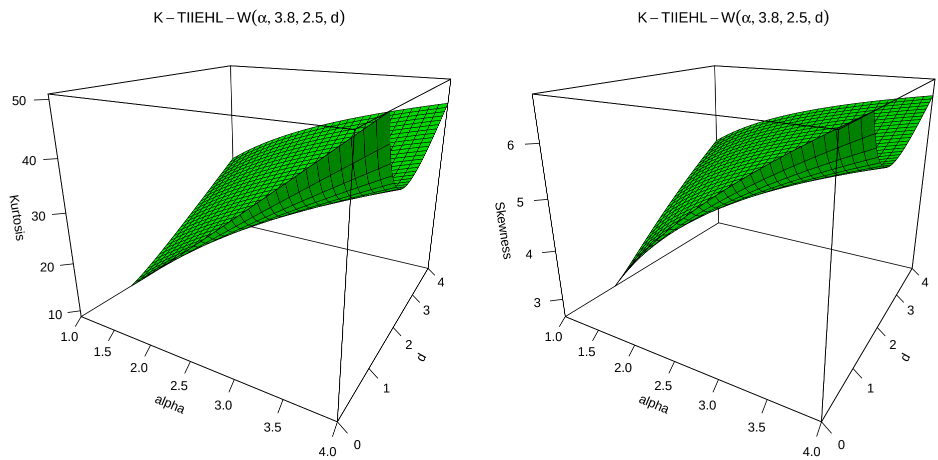

Figure 4. 3D plots of K-TIIEHL-W distribution

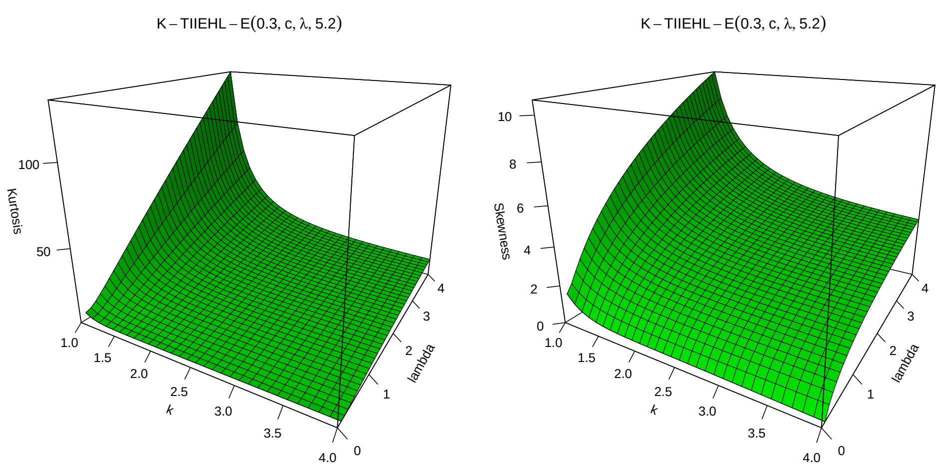

Figure 5. 3D plots of K-TIIEHL-E distribution

Figures [3], [4] and [5] shows sets of 3D plots of some kurtosis and skewness features of K-TIIEHL-G distribution for the special cases. Analyzing these plots, we observe that varying skewness and kurtosis levels are attained after fixing some selected parameter values.

3. Model properties

We seek to develop some statistical and mathematical properties associated with this novel FDs (K-TIIEHL-G) in this section.

3.1. Quantile function (Qfn)

The Qfn plays a crucial role in data transformation, underpinning techniques such as random number generation, quantile normalization, and quantile regression. Its importance lies in its ability to facilitate a variety of statistical analyses and data manipulation methods.

The Qfn of the K-TIIEHL-G FDs can be obtained by inverting the K-TIIEHL-G cdf, that is

$$u = 1 - \left[1 - \left(1 - \left[\frac{\bar{G}(Q_X(u))}{1+G(Q_X(u))}\right]^{\alpha}\right)^{c}\right]^{\lambda}$$

for $0 \leq u \leq 1$, utilizing statistical software like R, for a specified baseline cdf $G(.)$. Consequently, we obtain the quantile values by solving the non-linear function,

$$Q_X(u) = G^{-1}\left[\frac{1-\left[1-(1-u)^{\frac{1}{\lambda}}\right]^{\frac{1}{c}}}{1+\left[1-(1-u)^{\frac{1}{\lambda}}\right]^{\frac{1}{c}}}\right]. \tag{8}$$

3.2. Density expansion

The pdf of the K-TIIEHL-G FDs can be written as

$$f(x) = \sum_{m=0}^{\infty} \Upsilon_{m+1} g_{m+1}(x;\varpi),$$

where $g_{m+1}(x;\varpi) = (m+1)g(x;\varpi)G^m(x;\varpi)$ is the exponentiated-G (Expo-G) distribution with power parameter $(m+1)$ and

$$\begin{aligned}\Upsilon_{m+1} &= 2\alpha c\lambda \sum_{i,j,k,l=0}^{\infty} \frac{(-1)^{i+l+j+m}}{m+1}\binom{\lambda-1}{i}\binom{c(i+1)-1}{j}\binom{k+\alpha(j+1)}{k} \\ &\times \binom{k}{l}\binom{\alpha+l-1}{m}\end{aligned} \tag{9}$$

is the linear component. Therefore, the K-TIIEHL-G FDs is expressible as an infinite linear combination of the Expo-G densities from which we can directly derive other statistical properties. Visit the *appendix* for detailed derivations.

3.3. Order statistics

One of the valuable concepts in statistical sciences is order statistics, which possess a diverse array of applications. They are used in car races, modeling auctions, and insurance policies, as well as in optimizing production processes and estimating distribution parameters. The pdf of the i^{th} order statistics from the K-TIIEHL-G FDs is given by

$$\begin{aligned} f_{i:n}(x) &= \frac{f(x)}{B(i,n-i+1)} \sum_{j=0}^{n-i} \binom{n-i}{j} [F(x)]^{j+i-1} \\ &= \frac{1}{B(i,n-i+1)} \sum_{j=0}^{n-i} \sum_{r=0}^{\infty} \binom{n-i}{j} \Upsilon_{r+1} g_{r+1}(x;\varpi), \end{aligned}$$

where, $B(.,.)$ represents the beta function, $g_{r+1}(x,\varpi) = (r+1)g(x,\varpi)G^{r}(x,\varpi)$ is the Expo-G distribution, $(r+1)$ is the power parameter and

$$\begin{aligned} \Upsilon_{r+1} &= \alpha\beta\theta^{2} \sum_{k,l,m,q,p,r=0}^{\infty} (-1)^{k+l+m+q+r} (1-\theta)^{p} \binom{i+j-1}{k} \\ &\times \binom{\beta(k+1)-1}{l} \binom{p}{q} \binom{q(\theta-1)}{r} \left(\frac{1}{r+1}\right). \end{aligned}$$

In light of this, we draw the conclusion that the pdf of the i^{th} order statistic from the K-TIIEHL-G FDs is expressible as an infinite linear mixture of the Expo-G densities.

3.4. Moments and incomplete moments

Moments are vital statistical tools that reveal insights into the shape, variability, and distributional properties of data, going beyond mere central tendency measures. Incomplete moments provide essential details often missed by aggregate statistics, making them particularly relevant for analyzing complex socio-economic issues such as inequality. If Y_{r+1} is an Expo-G distributed random variable (rv), then, the q^{th} moment of the K-TIIEHL-G FDs is

$$E(Z^{q}) = \sum_{r=0}^{\infty} \Upsilon_{r+1} E(Y_{r+1}^{q}),$$

where $E(Y_{r+1}^{q})$ represents the q^{th} moment of Y_{r+1} and, Υ_{r+1} is specified by Equation (9). We define the q^{th} incomplete moments as

$$I_{Z}(t) = \int_{0}^{t} z^{q} f(x)dx = \sum_{r=0}^{\infty} \Upsilon_{r+1} I_{r+1}(t),$$

where $I_{r+1}(t) = \int_{0}^{t} z^{q} g_{r+1}(x;\varpi)dx$. We state the moment generating function (mgf) of Z as

$$M_{Z}(t) = \sum_{r=0}^{\infty} \Upsilon_{r+1} E(e^{tY_{r+1}}),$$

where Υ_{r+1} is defined in Equation (9) and $E(e^{tY_{r+1}})$ is the mgf of the Expo-G distribution. Note that Lorenz and Bonferroni curves for the K-TIIEHL-G can be found from incomplete moments.

3.5. Likelihood ratio order

The likelihood ratio (LR) is essential for model selection using information criteria. By incorporating stochastic ordering, particularly in machine learning models, one can enhance predictive accuracy and generalization, especially in fields where the ordinal nature of the data is significant. Consider X and Y to be continuous rvs with pdfs $f(x)$ and $g(x)$, respectively. If the function $z \to \frac{g(x)}{f(x)}$ is increasing across the combined supports of X and Y, we say X is less than Y in the lr order, denoted as $X \leq_{lr} Y$ (Shaked and Shanthikumar (2007)).

Proposition: If $X \sim K-TIIEHL-G(\alpha, c, \lambda_1, \varpi)$ and $X \sim K-TIIEHL-G(\alpha, c, \lambda_2, \varpi)$, then $Y \leq_{lr} X$ provided $b_1 < b_2$ and $Y \leq_{lr} Z$ provided $b_2 < b_1$.
Proof: Let f_X and f_Y be densities to X and Y, respectively. From Equation (6),

$$\frac{f_X(\alpha, c, \lambda_1, \varpi)}{f_Y(\alpha, c, \lambda_2, \varpi)} = \frac{\lambda_1}{\lambda_2}\left[1-\left(1-[K_G(x,\varpi)]^{\alpha}\right)^{c}\right]^{\lambda_1-\lambda_2}. \tag{10}$$

Differentiating Equation (10) with respect to x, we have

$$\frac{\frac{f_X(\alpha,c,\lambda_1,\varpi)}{f_Y(\alpha,c,\lambda_2,\varpi)}}{\partial x} = \frac{\lambda_1(\lambda_1-\lambda_2)}{\lambda_2}\left[1-\left(1-[K_G(x,\varpi)]^{\alpha}\right)^{c}\right]^{\lambda_1-\lambda_2-1}$$

increasing in x for $\lambda_1 < \lambda_2$. Consequently, $Y \leq_{lr} X$. Likewise, it is proven that $X \leq_{lr} Y$ if $\lambda_2 < \lambda_1$. It follows therefore that $X \leq_{st}$Y, where (st) denotes stochastic order.

3.6. Rényi entropy

K-TIIEHL-G FDs' Rényi entropy (Rényi (1960)) can be written as

$$I_R(z) = \frac{\log\left[\sum_{j=0}^{\infty}\Lambda_l e^{[(1-\omega)I_{REG}]}\right]}{1-\omega},$$

for $\omega \neq 1, \omega > 0$, where

$$\begin{aligned}\Lambda_l &= (\alpha\beta\theta^2)^{\omega}\sum_{e,f,h,k,l=0}^{\infty}(-1)^{e+f+k+l}\binom{\omega(\lambda-1)}{e}\binom{\omega(c-1)+ce}{f}\\ &\times \binom{\omega([\alpha+1)]+\alpha f+h-1}{h}\binom{h}{k}\binom{\omega(\alpha-1)+k}{l}\left(\frac{l}{\omega}+1\right)^{-\omega},\end{aligned}$$

and $I_{REG} = \frac{1}{1-\omega}\int_0^{\infty}\left[\left(\left(\frac{l}{\omega}+1\right)[g(x,\varpi)G^{\frac{l}{\omega}}(x,\varpi)]\right)^{\omega}\right]dx$ represents the RE of the Expon-G distribution with power parameter $(\frac{l}{\omega}+1)$. As a result, the Rényi entropy RE of the K-TIIEHL-G FDs stems directly from Rényi entropy of the Expo-G distribution. For comprehensive derivations refer to the *appendix* .

4. Estimation

Consider a random sample of size n, where each observation, X_i, follows the K-TIIEHL-G distribution, and let $\Delta = (\alpha, c, \lambda, \Psi)^T$ represent the vector of model parameters. The log-likelihood function $\ell = \ell(\Delta)$ for this sample can be written as

$$\begin{aligned}
\ell(\Delta) &= n\ln(2\alpha c\lambda) + \sum_{i=1}^{n} \ln\left(g(x_i;\varpi)\right) + (\alpha-1)\sum_{i=1}^{n} \ln\left[(\bar{G}(x_i;\varpi)\right] \\
&- (\alpha+1)\sum_{i=1}^{n} \ln\left[1+G(x_i,\varpi)\right] + (c-1)\sum_{i=1}^{n} \ln\left[1-[K_G(x,\varpi)]^{\alpha}\right] \\
&+ (\lambda-1)\sum_{i=1}^{n} \ln\left[1-\left(1-[K_G(x,\varpi)]^{\alpha}\right)^{c}\right]. \qquad (11)
\end{aligned}$$

The numerical solution of the nonlinear equations $\left[\frac{\partial \ell}{\partial \alpha}, \frac{\partial \ell}{\partial c}, \frac{\partial \ell}{\partial \lambda}, \frac{\partial \ell}{\partial \varpi_k}\right]^T = \mathbf{0}$ using iterative methods in R for a specific baseline cdf $G(x;\varpi)$, yields the MLEs of α, c, λ and ϖ_k. Visit the *appendix* for individual components of the score vector.

5. Simulation

We seek to weigh the efficiency of MLEs by carrying out a simulation study. Table 1 gives simulation studies results. We simulated for n= 25, 60, 100, 250, 500, 1000 and 2000 for N=3000 from the K-TIIEHL-W distribution. Average bias (AvBIAS) and root mean square error (RMSErr) for an estimated parameter, for instance ($\hat{c}$), are computed as follows:

$$AvBIAS(\hat{c}) = \frac{\sum_{i=1}^{N} \hat{c}_i}{N} - c, \quad \text{and} \quad RMSErr(\hat{c}) = \sqrt{\frac{\sum_{i=1}^{N} (\hat{c}_i - c)^2}{N}},$$

respectively. Examining the data presented in Table 1, it is evident that the average estimated parameter values converge towards the true parameter values with an increasing sample size. Moreover, both the Root Mean Square Error (RMSErr) and Average Bias (AvBIAS) show a consistent decline towards zero across all parameters with larger sample sizes. This observation highlights that the K-TIIEHL-W distribution provides consistent and efficient parameter estimates.

Table 1. Results of Monte Carlo simulations for the K-TIIEHL-W distribution: Mean, RMSErr, and AvBIAS

		$\alpha = 1.1, c = 1.11, \lambda = 0.82, d = 1.1$			$\alpha = 1.1, c = 1.1, \lambda = 1.1, d = 0.62$		
	n	Mean	RMSErr	AvBIAS	Mean	RMSErr	AvBIAS
	25	1.6801	1.3076	0.5801	1.7864	1.6681	0.6864
	60	1.5704	0.9813	0.4704	1.5948	1.1871	0.4948
	100	1.4687	0.8097	0.3687	1.3970	0.8084	0.2971
α	250	1.4275	0.7076	0.3275	1.3330	0.6859	0.2330
	500	1.3194	0.5797	0.2194	1.1870	0.3480	0.0870
	1000	1.2261	0.4213	0.1261	1.1331	0.1550	0.0331
	2000	1.1815	0.2744	0.0815	1.1283	0.0997	0.0283
	25	1.2442	0.5341	0.1442	2.9936	4.9795	1.8936
	60	1.1575	0.3272	0.0925	1.6388	1.9090	0.5388
	100	1.1234	0.2425	0.0766	1.3158	1.0450	0.2158
c	250	1.0998	0.1852	0.0602	1.2267	0.7108	0.1267
	500	1.0697	0.1534	0.0338	1.1426	0.4061	0.0426
	1000	1.0162	0.1346	0.0203	1.1361	0.2702	0.0416
	2000	1.0085	0.0126	0.0094	1.1237	0.2202	0.0337
	25	1.0492	1.3722	0.2492	1.6964	2.5982	0.5964
	60	0.8763	0.6451	0.0763	1.2657	1.1870	0.1657
	100	0.8516	0.4362	0.0516	1.2080	0.6763	0.1081
λ	250	0.8445	0.3220	0.0286	1.2051	0.4791	0.1051
	500	0.8344	0.2444	0.0195	1.1989	0.3057	0.0989
	1000	0.8286	0.1935	0.0144	1.1862	0.2183	0.0862
	2000	0.8081	0.1536	0.0081	1.1373	0.1973	0.0273
	25	2.2212	4.3650	1.1212	1.4258	3.1007	0.8258
	60	1.7284	1.1997	0.6284	1.0248	0.9991	0.4248
	100	1.5857	1.0656	0.4857	0.8108	0.6889	0.2108
d	250	1.5393	0.9524	0.4393	0.7112	0.4017	0.1112
	500	1.3852	0.7719	0.2852	0.6548	0.1919	0.0548
	1000	1.2565	0.5620	0.1565	0.6265	0.0944	0.0265
	2000	1.1972	0.3739	0.0972	0.6245	0.0713	0.0245

		$\alpha = 0.83, c = 1.1, \lambda = 1.1, d = 0.63$			$\alpha = 0.9, c = 1.0, \lambda = 0.8, d = 1.2$		
	n	Mean	RMSErr	AvBIAS	Mean	RMSErr	AvBIAS
	25	1.4503	1.6864	0.6503	1.4685	1.1435	0.5685
	60	1.2697	1.0268	0.4697	1.3473	1.0018	0.4473
	100	1.0496	0.6277	0.2496	1.2106	0.9649	0.3106
α	250	0.9591	0.4094	0.1591	1.1390	0.7247	0.2390
	500	0.8828	0.2049	0.0828	1.0796	0.4094	0.1796
	1000	0.8551	0.1012	0.0551	0.9789	0.1716	0.0789
	2000	0.8510	0.0814	0.0510	0.9176	0.0380	0.0176
	25	2.0024	2.2259	1.4024	1.1118	0.2362	0.2118
	60	1.9122	1.6588	0.9122	1.0757	0.1140	0.1757
	100	1.7680	1.5527	0.6680	1.0566	0.0965	0.1466
c	250	1.6235	1.1358	0.5235	1.0419	0.0604	0.1219
	500	1.5244	0.9048	0.4244	1.0285	0.0372	0.1085
	1000	1.3954	0.6954	0.2954	1.0227	0.0166	0.0927
	2000	1.2982	0.4760	0.1982	1.0205	0.0078	0.0605
	25	1.3164	1.5108	0.2164	1.0165	0.8932	0.2069
	60	1.2170	0.4721	0.1170	0.9508	0.5928	0.2052
	100	1.2118	0.3613	0.1118	0.9332	0.4495	0.2027
λ	250	1.2012	0.2484	0.1012	0.9235	0.3788	0.1765
	500	1.1893	0.2168	0.0893	0.8573	0.3268	0.1668
	1000	1.1691	0.1764	0.0691	0.8348	0.2596	0.1492
	2000	1.1634	0.1280	0.0634	0.8131	0.2013	0.0835
	25	1.5605	2.6207	0.9605	2.1491	3.2233	0.9491
	60	1.2535	1.2710	0.6535	1.9198	1.6246	0.7198
	100	0.9593	0.8515	0.3593	1.7307	1.2020	0.5307
d	250	0.8355	0.5619	0.2355	1.6073	1.0181	0.4073
	500	0.7250	0.2926	0.1250	1.5180	0.6697	0.3180
	1000	0.6815	0.1475	0.0815	1.3500	0.4386	0.1500
	2000	0.6737	0.1162	0.0737	1.2286	0.2114	0.0486

6. Applications

This section aims to illustrate the practicality and adaptability of the K-TIIEHL-G FDs, with a specific focus on the K-TIIEHL-W distribution, which is distinguished for its superiority over various non-nested models. To assess model performance, a range of statistical measures were employed, including: $-2\log L$, Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (CAIC), Bayesian Information Criterion (BIC), Cramér-von Mises (W^*), Anderson-Darling (A^*), and Kolmogorov-Smirnov (K-S) statistics, along with its corresponding p-values. The model with the lowest statistics and the highest K-S p-value is deemed most effective in capturing the data's characteristics. Parameter estimation for the K-TIIEHL-W model was conducted using the non-linear minimization function (nlm) in R software.

In our analysis, we conducted a comparative analysis of the K-TIIEHL-W distribution against five alternative non-nested models including Kumaraswamy and exponentiated half logistic extensions. This comprehensive approach highlights the robustness of the K-TIIEHL-W distribution in capturing various underlying data patterns.

The contending models (whose pdfs are provided in the *appendix*) are: the exponentiated half logistic Weibull Poisson (EHLWP) by Chipepa et al. (2022), Kumaraswamy Weibull (KW) by Cordeiro, Ortega and Nadarajah (2010), exponentiated half logistic odd Burr III-log-logistic Poisson (EHL-OBIII-LLOG) by Oluyede et al. (2022), odd exponentiated half logistic Burr XII (OEHLBXII) distribution by Aldahlan and Afify (2018), and the Topp-Leone odd Burr III log-logistic (TL-OBIII-LLOG) introduced by Moakofi, Oluyede and Gabanakgosi (2022). Parameter estimates accompanied by their respective standard errors (SEs), in parentheses, are presented in Tables 2 and 3. To ensure the uniqueness of the parameter estimates, profile plots were provided in all datasets. Furthermore, various other visual representations such as fitted densities, empirical cumulative distribution function (ECDF), Kaplan-Meier (KM) survival curves, and scaled total time on test (TTT) plots were also included.

6.1. Blood cancer data

The initial dataset includes the lifetimes (in days) of 43 blood cancer patients from a health hospital in Saudi Arabia. This data was also analyzed by Abouammoh and Abdulghani (1994) and by Elangovan and Arunkumar (2022). The data values are: 115, 181, 255, 418, 441, 461, 516, 739, 743, 789, 807, 865, 924, 983, 1025, 1062, 1063, 1165, 1191, 1222, 1222, 1251, 1277, 1290, 1357, 1369, 1408, 1455, 1478, 1519, 1578, 1578, 1599, 1603, 1605, 1696, 1735, 1799, 1815 ,1852, 1899, 1925, 1965.

The maximum likelihood estimates along with their corresponding SEs (shown in parentheses), and GoF statisstics are displayed in Table 2. The GoF results show that the K-TIIEHTL-W model surpasses its competitors.

Table 2. Estimates and GoF statistics for K-TIIEHL-W and other distributions on blood cancer data

	Estimates				GoF statistics							
Model	α	c	λ	d	$-2\log(L)$	AIC	AICC	BIC	W^*	A^*	$K-S$	$p-value$
$K-TIIEHL-W$	0.0128	6.3945	1.18×10^3	0.4766	660.4967	668.4967	669.5494	675.5415	0.1584	1.0159	0.1197	0.6684
	(5.13×10^{-3})	(2.94×10^{-3})	(6.27×10^{-7})	(0.0563)								
	α	β	δ	θ								
$EHL-WP$	0.5602	0.0079	1.0316	3.7292	695.6587	703.6587	704.7114	710.7114	0.1745	1.1327	0.1888	0.1014
	(0.0.0523)	(0.0.0025)	(0.1794)	(0.6659)								
	a	b	α	β								
KW	26.8610	238.2508	8.57×10^{-3}	0.2171	664.2811	672.2811	673.3338	679.3260	0.2048	1.2840	0.1395	0.3729
	(7.49×10^{-5})	(1.89×10^{-6})	(2.54×10^{-3})	(0.0253)								
	a	b	α	c								
$EHL-OBIII-LLoG$	546.0500	1.55×10^3	2.4683	2.45×10^{-3}	699.6970	707.6970	708.7496	714.7417	0.6931	3.8696	0.2552	0.0738
	(2.15×10^{-3})	(8.65×10^{-12})	(5.77×10^{-9})	(4.79×10^{-5})								
	α	λ	a	b								
$OEHL-BXII$	0.8791	3.76×10^{-5}	0.8240	1.7557	700.8792	708.9792	710.0312	716.0234	0.1666	1.0261	0.1787	0.1283
	(1.05×10^{-10})	(5.41×10^{-6})	(2.53×10^{-9})	(1.19×10^{-9})								
	α	β	b	λ								
$TL-OBIII-LLoG$	1.4708	135.4324	3.08645	0.5315	697.4568	705.4568	706.5094	712.5016	0.6580	3.6943	0.2563	0.0705
	(0.3493)	(0.0326)	(1.4764)	(0.1262)								

Figure [6] clearly shows that the K-TIIEHL-W parameters on blood cancer data can be uniquely determined. The 95% asymptotic confidence intervals (ACI) for the model parameters of the K-TIIEHTL-W distribution are $\alpha \in [0.0128 \pm 0.0101]$, $c \in [6.3945 \pm 0.0058]$, $\lambda \in [1.18\times10^3 \pm 1.23\times10^{-6}]$, and $d \in [0.4766 \pm 0.1103]$, respectively.

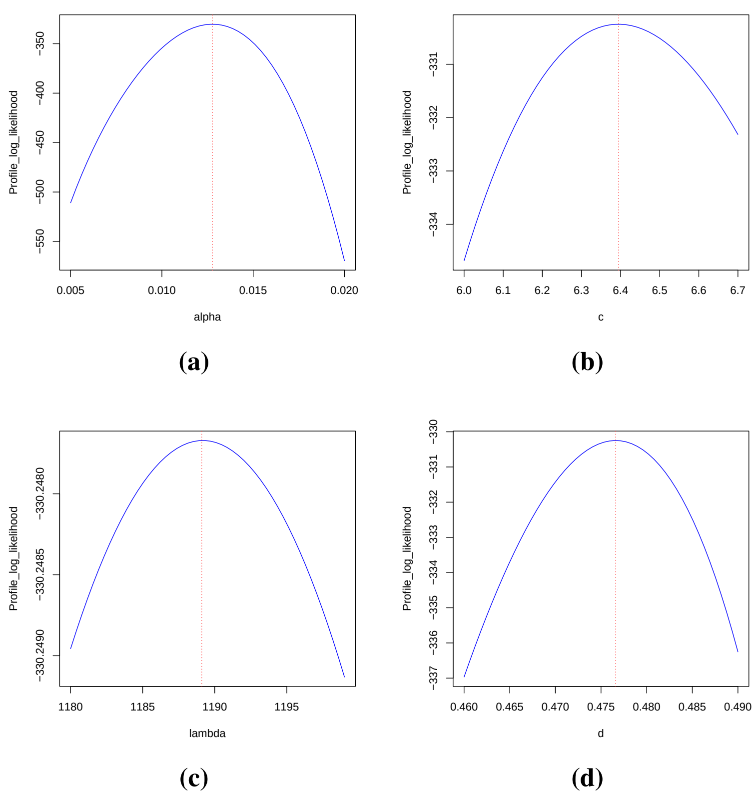

Figure 6. Profile log-likelihood plots of the K-TIIEHL-W: Blood cancer data

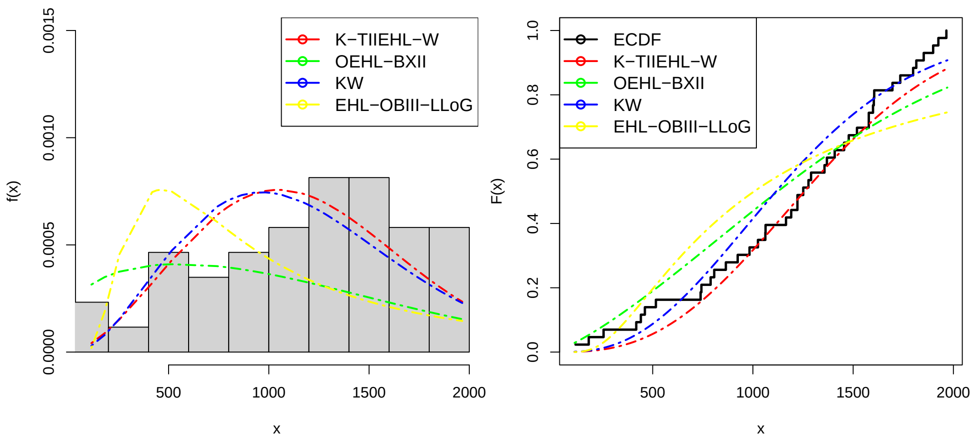

Figure 7. Plots of the observed histogram with superimposed estimated pdfs and observed ogives and estimated cdfs for blood cancer data

Figure [7] displays the histogram and ogive of the observed data, along with the fitted densities and the ECDFs for visual assessment of the fitting accuracy. These plots suggest that the K-TIIEHL-W distribution offers a better fit for the blood cancer data compared to the competing models.

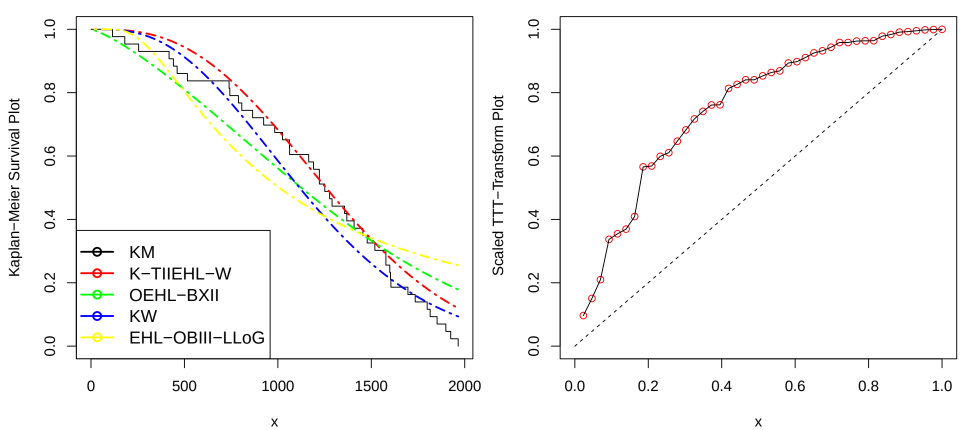

Figure 8. Plots of the KM survival curve along with the estimated KM survival curves, and the TTT-scaled plot of the blood cancer data

The KM survival curve depicted in Figure [8] demonstrates a strong alignment with the K-TIIEHL-W distribution when compared to alternative distributions on blood cancer data. Additionally, the TTT-scaled plot indicates that the hrf exhibits an increasing hazard rate pattern.

6.2. Time until failure of viscose/polyester yarn data

The following right-skewed dataset, as discussed by Arshad, Iqbal and Mutairi (2021), pertains to the temporal duration preceding the failure of a viscose/polyester yarn within a textile experiment aimed at assessing the tensile fatigue properties of the yarn. The data comprises a sample of 100 cm of yarn subjected to a strain level of 2.3%. The data was also analyzed by Shanker, Fesshaye and Selvaraj (2015). The values are:

86, 146, 251, 653, 98, 249, 400, 292, 131, 169, 175, 176, 76, 264, 15, 364, 195, 262, 88, 264, 157, 220, 42, 321, 180, 198, 38, 20, 61, 121, 282, 224, 149, 180, 325, 250, 196, 90, 229, 166, 38, 337, 65, 151, 341, 40, 40, 135, 597, 246, 211, 180, 93, 315, 353, 571, 124, 279, 81, 186, 497, 182, 423, 185, 229, 400, 338, 290, 398, 71, 246, 185, 188, 568, 55, 55, 61, 244, 20, 284, 393, 396, 203, 829, 239, 236, 286, 194, 277, 143, 198, 264, 105, 203, 124, 137, 135, 350, 193, 188.

Table 3. Estimates and GoF statistics for K-TIIEHL-W and other distributions on failure of polyester/viscose data

	Estimates				GoF statistics							
Model	α	c	λ	d	$-2\log(L)$	AIC	AICC	BIC	W^*	A^*	$K-S$	$p-value$
$K-TIIEHL-W$	9.00×10^{-3}	2.8380	18.4990	0.7042	1249.7020	1257.7020	1258.1230	1268.1230	0.0803	0.5169	0.0724	0.7706
	(2.48×10^{-3})	(8.16×10^{-3})	(1.23×10^{-7})	(0.0492)								
	α	β	δ	θ								
$EHL-WP$	0.3984	0.1083	4.7245	44.1292	1249.8360	1258.8360	1258.957	1268.9560	0.0929	0.5201	0.0813	0.5227
	(0.1121)	(0.1051)	(1.8259)	(0.0164)								
	a	b	α	β								
KW	67.1040	1.95×10^{3}	38.3790	0.0882	1249.9780	1257.9780	1258.399	1268.3980	0.1105	0.6073	0.0879	0.4222
	(4.07×10^{-7})	(1.95×10^{-9})	(1.51×10^{-7})	(6.17×10^{-4})								
	a	b	α	c								
$EHL-OBIII-LLoG$	0.5281	132.9400	1.8537	2.4061	1295.1610	1303.1610	1303.5820	1313.5820	0.8184	4.4807	0.1672	0.0949
	(0.0238)	(3.60×10^{-3})	(0.4569)	(5.21×10^{-3})								
	α	λ	a	b								
$OEHL-BXII$	0.5379	3.07×10^{-5}	0.9549	1.9233	1300.8180	1308.8180	1309.2390	1319.2390	0.1341	0.8553	0.1162	0.1340
	(1.15×10^{-10})	(3.69×10^{-6})	(1.29×10^{-9})	(6.42×10^{-10})								
	α	β	b	λ								
$TL-OBIII-LLoG$	0.1007	203.9300	0.5567	9.1446	1286.1350	1294.1350	1294.5560	1304.5560	0.7040	3.8564	0.1680	0.0714
	(3.89×10^{-3})	(2.34×10^{-4})	(0.1191)	(4.30×10^{-5})								

Table 3 presents the maximum likelihood estimates along with their corresponding SEs (in parentheses). The GoF results indicate that the K-TIIEHTL-W model outperforms its competitors.

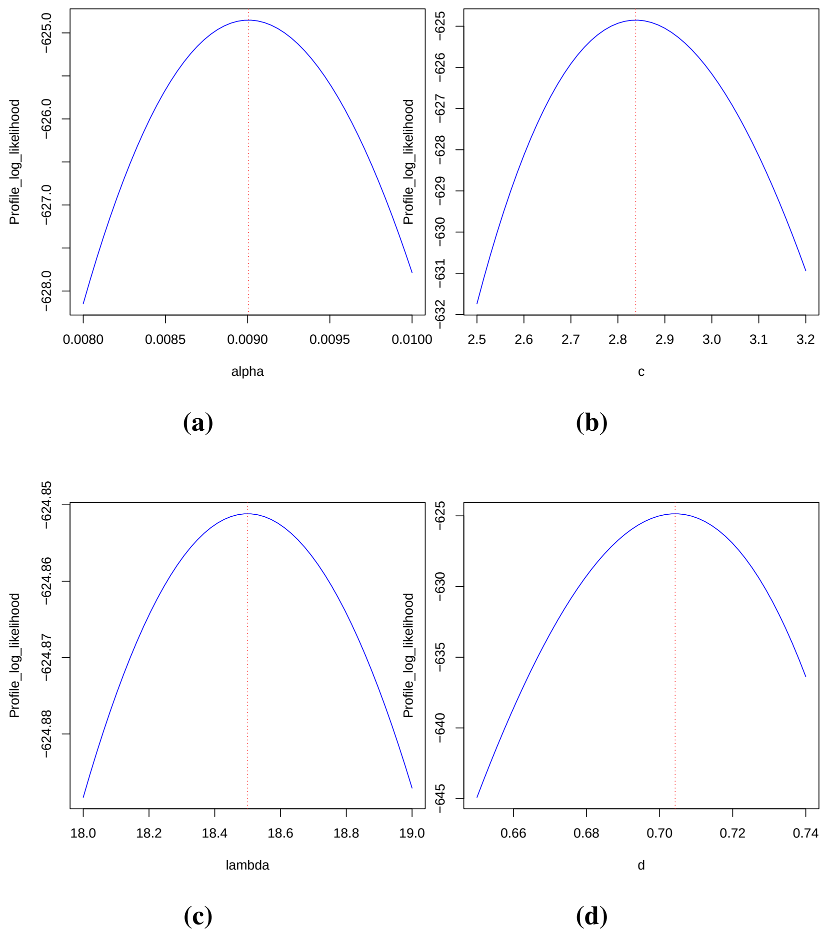

Figure 9. Profile log-likelihood plots of the K-TIIEHL-W: Viscose yarn data

Figure [9] clearly shows that the K-TIIEHL-W parameters on viscose yarn data can be uniquely determined. The 95% ACI for the parameters of the K-TIIEHTL-W distribution are as follows: $\alpha \in [9.00\times10^{-3} \pm 0.0049]$, $c \in [2.8380 \pm 0.0160]$, $\lambda \in [18.4990 \pm 0.0002]$, and $d \in [0.7042 \pm 0.0965]$.

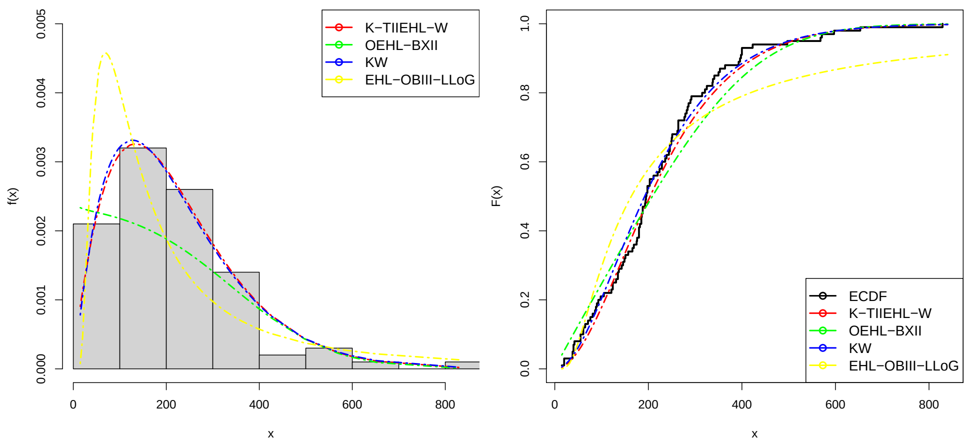

Figure 10. Plots of the observed KM survival curve alongside the estimated KM survival curves, and the TTT scaled plot for polyester/viscose yarn dataa

Figure [10] present the histogram and ogive of the observed data, accompanied by the fitted density functions and ECDF curves for an evaluative comparison of the fitting quality on failure of polyester/viscose data. The visual evidence indicates that the K-TIIEHL-W yields the most accurate representation of the data.

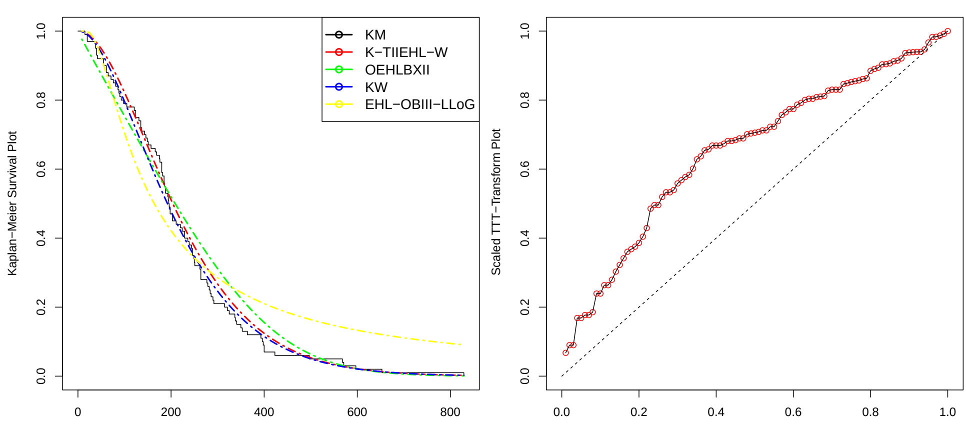

Figure 11. Plots of the observed Kaplan-Meier survival curve and the estimated KM survival curves, and the TTT-scaled plot for blood cancer data

The KM survival curve illustrated in Figure [11] reveals that the K-TIIEHL-W distribution closely aligns with the polyester/viscose yarn data in comparison to other competing distributions. Furthermore, the TTT-scaled plot suggests that the hrf demonstrates a pattern of increasing hazard rates.

7. Results and discussions

In this study, we introduced the K-TIIEHL-G FDs and explored their mathematical and statistical properties, along with specific cases. By employing maximum likelihood estimation techniques, we effectively estimated the unknown parameters and assessed their consistency through simulation studies. The results underscored the efficacy of the K-TIIEHL-W distribution, a member of the K-TIIEHL-G family. Furthermore, our analysis of the flexibility and practical applications of the K-TIIEHL-G FDs using empirical datasets demonstrated that the K-TIIEHL-W distribution significantly outperformed several existing equi-parameter models.

As we look to the future, we propose two exciting avenues for further research: first, integrating this new family of distributions into bivariate extensions through the copula approach; and second, investigating Bayesian estimation techniques to analyze these distributions, leveraging their adaptability to improve modeling accuracy and facilitate informed decision-making. Exploring these directions could greatly enhance our understanding of the theoretical properties and practical applications of the K-TIIEHL-G distributions, thereby expanding their relevance in engineering reliability and lifetime analysis, with meaningful implications for the design, maintenance, and optimization of complex engineering systems and components.

To access the appendix, please click on the link:
`https://drive.google.com/file/d/1QfzFva31__XXGN0UZrQ4IQcTvafiQjTW/view?usp=sharing`

References

Abouammoh, A. M., Abdulghani, S. A., (1994). On Partial Orderings and Testing of New Better Than Renewal Used Classes . *Reliability Engineering and System Safety*, 43(2), pp. 37–41.

Aldahlan, M., Afify, A. Z., (2018). The Odd Exponentiated Half-Logistic Burr XII Distribution. *Pakistan Journal of Statistics and Operation Research*, 14(2), pp. 305–317.

Al-Mofleh, H., Elgarhy, M., Afify, A. Z., Zannon, M., (2021). Type II Exponentiated Half Logistic Generated Family of Distributions with Applications. *Electronic Journal of Applied Statistical Analysis*, 13(2), pp. 536–561.

Arshad, M. Z. and Iqbal, M., Mutairi, A., (2021. A Comprehensive Review of Datasets for Statistical Research in Probability and Quality Control. *Journal of Mathematics and Computer Science*, 11(3), pp. 663–638.

Bashiru, S. O., (2024). A Study on Topp-Leone Kumaraswamy Fréchet Distribution with Applications: Methodological Study. *Turkiye Klinikleri Journal of Biostatistics*, 16(1), pp. 1–15.

Chipepa, F., Oluyede, B., Chamunorwa S., Makubate B., Zidana, C., (2022). The Exponentiated Half Logistic-Generalized-G Power Series Class of Distributions: Properties and Applications. *Journal of Probability and Statistical Science*, 20(1), pp. 21–40.

Chipepa, F., Oluyede, B., Makubate, B., (2019). A New Generalized Family of Odd Lindley-G Distributions with Application. *International Journal of Statistics and Probability*, 8(6), pp. 1–22.

Cobelli, C., Distefano 3rd, J. J., (1980). Parameter and Structural Identifiability Concepts and Ambiguities: A Critical Review and Analysis. *American Journal of Physiology-Regulatory, Integrative and Comparative Physiology*, 239(1), pp. 7-–24.

Cordeiro, G. M., de Castro, J., (2011). A New Family of Generalized Distributions. *Journal of Statistical Computation and Simulations*, 81, pp. 883-–893.

Cordeiro, G. M., Ortega, E. M., Nadarajah, S., (2010). The Kumaraswamy Weibull Distribution with Application to Failure Data. *Journal of the Franklin Institute*, 7(8), pp. 1399-–1429.

Elangovan, R., Arunkumar, S., (2022). A New Extension of Quasi Sujatha Distribution with Properties and Applications using Time Dependent Covariates. *Indian Journal of Natural Sciences*, 13(74), pp. 0976–0997.

Ibrahim, S., Doguwa, S. I., Isah, A., Haruna, J. M., (2020). The Topp Leone Kumaraswamy-G Family of Distributions with Applications to Cancer Disease Data. *Journal of Biostatistics and Epidemiology*, 6(1), pp. 40–51.

Moakofi, T., Oluyede, B., Gabanakgosi, M., (2022). The Topp-Leone Odd Burr III-G Family of Distributions: Model, Properties and Applications. *Statistics, Optimization and Information Computing* 1, pp. 1–27.

Nagarjunai, V. B. V., Vardhan, R. V. and Vardhan, C. (2021). Kumaraswamy generalized power Lomax distribution and its applications. *Stats* 4(1), pp. 28–45.

Oluyede, B., Peter, O. P., Nkumbuludzi, N., Huybrechts, B., (2022). The Exponentiated Half-Logistic Odd Burr III-G: Model, Properties and Applications. *Pakistan Journal of Statistics and Operation Research*, 18(1), pp. 33–57.

Shaked, M., Santhikumar, J. G., (2007). Stochastic Orders. *Springer Series in Statistics, Springer New York, NY.*

Shanker, R., Fesshaye, S., Selvaraj, S., (2015). On Modeling of Lifetimes Data Using Exponential and Lindley Distributions. *Biometrics & Biostatistics International Journal*, 2(5), pp. 883–893.

Usman R. M., Ahsan-ul-Haq, M., Talib, J., (2017). Kumaraswamy Half-Logistic Distribution: Properties and Applications. *Journal of Statistics Applications and Probability*, 6(7), pp. 597–609.

Rényi, A., (1960). On Measures of Entropy and Information. *Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability*, 25, pp. 547–561.