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Maximum likelihood estimation for the transmuted index distribution

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Abstract

The Quadratic rank transmutation map (QRTM) has been used to introduce skewness and flexibility of the base distribution function (df) models using a single parameter known as the transmuted parameter, which has proven to be useful in controlling the length of the tail distribution. The maximum likelihood estimation method (MLE) is used to estimate the parameters of interest. Again, Kouider et al. (2025) provided a informative glimpse into the estimates of the transmuted parameter using the MLE technique for the transmuted generalized Pareto distribution (TGPD). The focus of this paper is to estimate the transmuted parameter for any original df using the classical MLE technique and the MLE via max ranked set sampling (MaxRSS) method. Additionally, we present an overview of the asymptotic distribution of the classical MLE of the unknown transmuted index for any original df. The transmuted index distribution is useful for estimating the transmuted quantile of the TGPD parameters in terms of the maximum likelihood estimators (ML-estimators). Finally, we present a simulation study with real data to assess the performance of the estimates of both the transmuted index and the transmuted quantile of the TGPD parameters.

Key words: generalized distributions, transmuted generalized Pareto distribution, maximum likelihood estimation, modified bisection algorithm, max ranked set sampling.

1. Introduction

The quality of the procedures used in statistical analysis depends largely on the model or probability distribution assumed. Likewise, modeling and analyzing lifetime data is also crucial in many applied sciences, such as medicine, engineering, finance, and others. As a result, great efforts have been made in developing large classes of standard probability distributions along with related statistical methodologies. However, there are still many real data sets for which neither classical nor standard probability models can provide a solution. Hence, in this paper, we work with a new generalization of a classical distribution called the transmuted distribution, which is referred to as the QRTM technique, using the study by Shaw and Buckley (2009). Then, a random variable (rv) X is said to have the transmuted

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distribution with the transmuted parameter λ if the probability density function (pdf) is defined as:

$$f(x,\lambda) := g(x)(1+\lambda-2\lambda G(x)), \tag{1}$$

as the cumulative distribution function (cdf) is

$$F(x,\lambda) := (1+\lambda)G(x)-\lambda G^2(x), \tag{2}$$

where $|\lambda| \leq 1$ with G is the base distribution function (df) and g is the pdf associated. Also, the upper tail of the cdf F presented in (2) can be defined as:

$$\overline{F}(x,\lambda) := (1-\lambda)\overline{G}(x)+\lambda\overline{G}^2(x). \tag{3}$$

According to relations (1), (2), and (3), we define these three special cases as follows:

$$\begin{cases} \textbf{Case1:} \text{ For } \lambda = 0, F(x,0) := G(x) \text{ with } \overline{F}(x,0) := \overline{G}(x) \text{ and } f(x,0) := g(x), \\ \textbf{Case2:} \text{ For } \lambda = -1, F(x,-1) := G^2(x) \text{ with } \overline{F}(x,-1) := \overline{G}(x)(G(x)+1) \\ \qquad\qquad \text{and } f(x,-1) := 2g(x)G(x), \\ \textbf{Case3:} \text{ For } \lambda = 1\ , F(x,1) := G(x)\left(\overline{G}(x)+1\right) \text{ with } \overline{F}(x,1) := \overline{G}^2(x) \\ \qquad\qquad \text{and } f(x,1) := 2g(x)\overline{G}(x), \end{cases}$$

with G as the base distribution function (df), with g and $\overline{G}$ as the pdf and the upper tail associated, respectively.

However, in this paper, we will focus our work on estimating the transmuted parameter $|\lambda| \leq 1$ for any base distribution G using the maximum likelihood method (MLE), while also introducing the asymptotic normality distribution. Many researchers have used this method to obtain new distributions; see Aryal et al. (2009), Aryal and Tsokos (2011), Merovci (2013), Habib et al. (2016), etc. Recently, Kouider et al. (2025) focused on estimating the parameters of the TGPD, which include the scale, the shape, and the transmuted parameters. They presented a numerical method for estimating the parameters of the TGPD using the MLE based on simple random sampling (SRS) and ranked set sampling (RSS). In this paper, in section (2), we present a numerical method for estimating the transmuted parameters for any classical df G, and we illustrate the asymptotic normality properties of the transmuted index for any cdf G. Continuing what Kouider et al (2025) also presented, we estimate the transmuted index with the MLE method based on simple max ranked set sampling (MaxRSS). To estimate the transmuted parameter using the MLE method numerically, we utilize the modified bisection algorithm (MBA) for multi-roots, which was presented by Kouider (2019a). For section (3), we present a procedure that is performed by the MLE method via MaxRSS to estimate the TGPD parameter numerically. Next, we estimate the transmuted quantile of the TGPD parameters via the MLE method for the TGPD parameters. Then, some simulations are given with three numerical examples. In the first one, we apply the MLE using SRS with a sample of 15 points of data that follows the generalized Pareto distribution (GPD) with scale and shape parameters. The second is given with real data. In the last one, we apply the MLE method with 1000 replicates using the RSS and MaxRSS. They are used to test the efficiency of the estimators of the TGPD parameters us-

ing the algorithms presented in Kouider et al. (2025) for the SRS and RSS. For MaxRSS, we use the numerical method presented in Section(3.1). The final Section is the appendix part, in which we present how the variance of the transmuted index was calculated. A conclusion is given before the appendix in this paper.

2. Estimating the transmuted index distribution

2.1. The MLE for the transmuted index

For this sub-section, we introduce the MLE for the transmuted index and provide the formula of the function through which we can estimate the index by searching for its roots using the MBA, which was developed by Kouider (2019a). Then, we define the confidence intervals of the transmuted index and present the asymptotic distribution of the MLE for the transmuted index. Recall that in this paper, we will propose the estimation of the λ transmuted parameter based on the MLE. To specify it, let $X_1,\ldots,X_n$ be independent and identically distributed (iid) random variables(rv's) with the common transmuted df F, which is defined in (2). We assume that F is absolutely continuous. Let $X_{1,n}\leq\cdots\leq X_{n,n}$ denote the order statistics corresponding to $X_1,\ldots,X_n$; then, the likelihood function of the transmuted df F can be written as

$$\ell(X_{i,n};\lambda)=\prod_{i=1}^{n}f(X_{i,n};\lambda)=\prod_{i=1}^{n}g(X_{i,n})(1+\lambda-2\lambda G(X_{i,n})) \tag{4}$$

Or

$$\ell(X_{i,n};\lambda)=\prod_{i=1}^{n}g(X_{i,n})\left(1-\lambda+2\lambda\overline{G}(x)\right)$$

where the transmuted parameter $|\lambda|\leq 1$. By accumulating, taking the logarithm of equation (4),

$$\log\ell(X_{i,n};\lambda)=\sum_{i=1}^{n}(\log g(X_{i,n})+\log(1+\lambda-2\lambda G(X_{i,n}))) \tag{5}$$

And from (5) we can extract these three cases:

$$\begin{cases}\textbf{Case1: For } \lambda=0, \log\ell(X_{i,n};\lambda)=\sum\limits_{i=1}^{n}(\log g(X_{i,n})) \text{ (the classic case)},\\ \textbf{Case2: For } \lambda=1, \log\ell(X_{i,n};\lambda)=\sum\limits_{i=1}^{n}\left(\log g(X_{i,n})+\log\left(\overline{G}(X_{i,n})\right)+\log 2\right),\\ \textbf{Case3: For } \lambda=-1, \log\ell(X_{i,n};\lambda)=\sum\limits_{i=1}^{n}(\log g(X_{i,n})+\log(G(X_{i,n}))+\log 2).\end{cases}$$

From (5), where $g(X_i)>0$, and since $G(X_{i,n})\geq 0$ is an increasing function with $-1\leq\lambda\leq 0$, we have $1+\lambda-2\lambda G(X_{i,n})\leq 1+\lambda-2\lambda G(X_{n,n})$, where $X_{n,n}=\max(X_{i,n})$ for $i=1,\ldots,n$. Since $G(X_{n,n})>0$, we find $1+\lambda-2\lambda G(X_{n,n})>1+\lambda$ with $0\leq 1+\lambda\leq 1$; then $\log(1+\lambda-2\lambda G(X_{n,n}))$ exists. Then, the MLE exists for $-1\leq\lambda\leq 0$. Hence, with $0\leq\lambda\leq 1$, $1+\lambda>0$, $0<2\lambda G(X_{i,n})\leq 2\lambda$, and $1-\lambda<1+\lambda-2\lambda G(X_{i,n})\leq 1+\lambda$ for $i=1,\ldots,n$. Since $1-\lambda>0$ and $1+\lambda>0$ for $0\leq\lambda\leq 1$, the MLE exists for $0\leq\lambda\leq 1$.

This means that $1+\lambda>0$ and $1-\lambda+2\lambda G(X_{n,n})$ will always be positive for all $|\lambda|\leq 1$, and we agree that the MLE exists for $|\lambda|\leq 1$.

The likelihood equation of the transmuted parameter is given in terms of the partial derivative as:

$$\frac{\partial \log \ell(X_i;\lambda)}{\partial \lambda}=\sum_{i=1}^{n}\frac{1-2G(X_i)}{1+\lambda-2\lambda G(X_i)} \tag{6}$$

The ML estimator of λ, denoted by $\widehat{\lambda}^{ML}$, is obtained by solving the equation $\psi(\lambda)=0$ for $|\lambda|\leq 1$, which is related to the likelihood equation (6). The equation is as follows:

$$\psi(\lambda)=\sum_{i=1}^{n}\frac{1-2G(X_i)}{1+\lambda-2\lambda G(X_i)} \tag{7}$$

And from (6) we get

$$\begin{cases} \textbf{Case1:}\ \ \psi(\lambda)|_{\lambda=0}=\sum\limits_{i=1}^{n}\left(1-2G(X_i)\right), \\ \textbf{Case2:}\ \ \psi(\lambda)|_{\lambda=1}=\sum\limits_{i=1}^{n}\frac{1-2G(X_i)}{2\overline{G}(X_i)}=\psi(\lambda)=\sum\limits_{i=1}^{n}\phi(X_i), \\ \textbf{Case3:}\ \ \psi(\lambda)|_{\lambda=-1}=\sum\limits_{i=1}^{n}\frac{1-2G(X_i)}{2G(X_i)}=\psi(\lambda)=\sum\limits_{i=1}^{n}\varphi(X_i). \end{cases}$$

Clearly, we find that $\psi(\lambda)|_{\lambda=0}=\frac{1}{2}\sum\limits_{i=1}^{n}\left(\phi(X_i)^{-1}+\varphi(X_i)^{-1}\right)^{-1}$, where $\phi(X_i)=\frac{1-2G(X_i)}{2\overline{G}(X_i)}$ and $\varphi(X_i)=\frac{1-2G(X_i)}{2G(X_i)}$.

To determine $\widehat{\lambda}^{ML}$, the zero of $\psi(\lambda)$ must be checked, which is presented in (7) with $|\lambda|\leq 1$ using a numerical method. We use the MBA for multi-roots to search for a root of the function $\psi(\lambda)$ on the interval $[-1;1]$. Next, we denote the second order derivative of $\frac{\partial \log \ell(X_i;\lambda)}{\partial \lambda}$ with respect to λ as $\psi^{'}(\lambda)$, and according to the above inequalities (6)-(7), we have

$$\frac{\partial \psi(\lambda)}{\partial \lambda}=\frac{\partial^2 \log \ell(X_i;\lambda)}{\partial^2 \lambda}=-\sum_{i=1}^{n}\left(\frac{1-2G(X_i)}{1+\lambda-2\lambda G(X_i)}\right)^2 \tag{8}$$

The Fisher information for λ contained in the rv X is defined as:

$$I(\lambda)=-E_{\lambda}\left[\left(\frac{1-2G(X_i)}{1+\lambda-2\lambda G(X_i)}\right)^2\right] \tag{9}$$

It is easy to see that

$$E_{\lambda}\left[\frac{1-2G(X_i)}{1+\lambda-2\lambda G(X_i)}\right]=0 \tag{10}$$

Therefore, the definition of the Fisher information (9) can be rewritten as

$$Var\left(\frac{1-2G(X_i)}{1+\lambda-2\lambda G(X_i)}\right)=I(\lambda) \tag{11}$$

From the definition of $\psi(\lambda)$ notice that

$$E_\lambda(\psi(\lambda)) = 0 \text{ for } |\lambda| \leq 1 \text{ and } Var(\psi(\lambda)) = nI(\lambda) \tag{12}$$

For $\lambda = 0$, we consider that $Var(\psi(\lambda)) = nI(0)$, where

$$I(0) = -E_{\lambda=0}\left[(1-2G(X_i))^2\right]$$

To specify the asymptotic normality of the ML-transmuted estimator $\widehat{\lambda}^{ML}$, we use the central limit theorem (CLT) of the ML-estimator. We note $\xrightarrow{d}$ convergence in distribution . This theorem indicates the asymptotic normality of the transmuted index for any base df G.

Theorem 2.1 *Let $X_1,\ldots,X_n$ be a sample of size n from a distribution for which the pdf (1) involves λ as the unknown parameter. Consider the MLE of λ as $\widehat{\lambda}^{ML}$; then the probability distribution of $\sqrt{nI(\lambda)}\left(\widehat{\lambda}^{ML}-\lambda\right)$ tends to a standard normal distribution. In other words, the asymptotic distribution of $\widehat{\lambda}^{ML}$ is*

$$\sqrt{n}\left(\widehat{\lambda}^{ML}-\lambda\right) \xrightarrow{d} N\left(0;\frac{1}{I(\lambda)}\right)$$

where

$$I(\lambda) = \frac{1}{\lambda^2}\left(\lambda + \ln\sqrt{\frac{1-\lambda}{1+\lambda}}\right) \tag{13}$$

and $|\lambda| < 1$ with $\lambda \neq 0$.

Since for $\lambda = 0$, we will find that $I(\lambda)$ in (13) is negative, $I(\lambda) = \frac{-1}{3} < 0$, and

$$\lim_{\lambda \overset{<}{\to} 1} I(\lambda) = +\infty = \lim_{\lambda \overset{>}{\to} -1} I(\lambda) = +\infty$$

This indicates that the closer the transmuted parameter gets to the numbers 1 and -1, the greater the value of the variance, approaching infinity. And it is clear that the Fisher information $I(\lambda)$ in (13) exists for $\lambda \in]-1;0[\cup]0;1[$.

Proof 2.1 *We apply the Taylor expansion of $\psi(\lambda)$ at the point λ, yielding*

$$\psi\left(\widehat{\lambda}^{ML}\right) \approx \psi(\lambda) + \left(\widehat{\lambda}^{ML}-\lambda\right)\psi'(\lambda) \tag{14}$$

Since $\psi\left(\widehat{\lambda}^{ML}\right) = 0$, therefore

$$\sqrt{n}\left(\widehat{\lambda}^{ML}-\lambda\right) \approx -\frac{\frac{1}{\sqrt{n}}\psi(\lambda)}{\frac{1}{n}\psi'(\lambda)} \tag{15}$$

Its expectation is

$$E\left(\frac{1}{\sqrt{n}}\psi(\lambda)\right) = \frac{1}{\sqrt{n}}E(\psi(\lambda)) = 0 \tag{16}$$

And its variance is

$$Var\left(\frac{1}{\sqrt{n}}\psi(\lambda)\right)=\frac{1}{n}Var(\psi(\lambda))=I(\lambda) \tag{17}$$

Next, we consider the denominator:

$$\frac{1}{n}\psi^{'}(\lambda)=\frac{1}{n}\frac{\partial^2\log\ell(x_i;\lambda)}{\partial^2\lambda}=-\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1-2G(X_i)}{1+\lambda-2\lambda G(X_i)}\right)^2 \tag{18}$$

By the strong law of large numbers, this expression converges to

$$E\left(\frac{\partial^2\log\ell(X_i;\lambda)}{\partial^2\lambda}\right)=nI(\lambda) \tag{19}$$

We thus have

$$\sqrt{n}\left(\widehat{\lambda}^{ML}-\lambda\right)\approx\frac{\frac{1}{\sqrt{n}}\psi(\lambda)}{I(\lambda)} \tag{20}$$

Therefore,

$$E\left[\sqrt{n}\left(\widehat{\lambda}^{ML}-\lambda\right)\right]=\frac{1}{I(\lambda)}E\left(\frac{1}{\sqrt{n}}\psi(\lambda)\right)=0 \tag{21}$$

And

$$Var\left[\sqrt{n}\left(\widehat{\lambda}^{ML}-\lambda\right)\right]=\frac{1}{(I(\lambda))^2}Var\left(\frac{1}{\sqrt{n}}\psi(\lambda)\right)=\frac{1}{I(\lambda)} \tag{22}$$

And as $n\to+\infty$ applying the central limit theorem, we have

$$\sqrt{n}\left(\widehat{\lambda}^{ML}-\lambda\right)\xrightarrow{d}N\left(0;\frac{1}{I(\lambda)}\right) \tag{23}$$

i.e.,

$$\sqrt{nI(\lambda)}\left(\widehat{\lambda}^{ML}-\lambda\right)\xrightarrow{d}N(0;1) \tag{24}$$

This completes the proof.

According to result (24), the distribution of $\sqrt{nI(\lambda)}\left(\widehat{\lambda}^{ML}-\lambda\right)$ is approximately the standard normal distribution. Regarding the variance calculation, we explain it in the Appendix parts. Since the value of λ is unknown, we use the value of $\widehat{\lambda}^{ML}$ to estimate the Fisher information $I(\lambda)$ as follows:

$$\widehat{I(\lambda)}=\left(\frac{1}{\widehat{\lambda}^{ML}}\right)^2\left(\widehat{\lambda}^{ML}+\ln\sqrt{\frac{1-\widehat{\lambda}^{ML}}{1+\widehat{\lambda}^{ML}}}\right) \tag{25}$$

In addition, for $t_{\kappa/2}$ denotes the $1-\kappa/2$ quantile of the standard normal distribution, which is symmetric about 0, we use $\widehat{\lambda}^{ML}\pm t_{\kappa/2}I(\lambda)$ as an approximate $100(1-\kappa)\%$ confidence interval of λ.

2.2. The MLE for the transmuted parameters using Max Ranked Set Sampling

Recently, many sampling frameworks have been developed, and one of the most popular schemes is RSS as shown by McIntyre (1951). Many modified RSS strategies have been developed to address the limitations on which they are based, which may restrict their applicability under certain conditions. Among the various RSS frameworks, there is a modified and popular RSS approach called MaxRSS. In this subsection, we estimate the unknown transmuted parameter λ of the cdf F based on MLE using the MaxRSS. The concept of RSS was initially introduced by Bhoj (2001) in his work. The following is a description of the MaxRSS scheme by Eskandarzadeh et al. (2018): First, pick a simple random sample (SRS) of r sets of size n from the group you want to study. Second, pick the max element with rank i from the j set in a cycle where $i \in \{1,2,...,n\}$ and $j \in \{1,2,...,r\}$. For example, with $r = 1$ one cycle, the MaxRSS is described as follows:

$$
\begin{array}{cccccccc}
\text{Set} & \text{Rankig} & & & & & & \text{MaxRSS} \\
1 & \mathbf{X}^{(1)}_{1,1} & & & & & \longrightarrow & \mathbf{X}^{(1)}_{1,1} \\
2 & X^{(1)}_{1,2} & \mathbf{X}^{(1)}_{2,2} & & & & \longrightarrow & \mathbf{X}^{(1)}_{2,2} \\
 & X^{(1)}_{1,3} & & \mathbf{X}^{(1)}_{3,3} & & & \longrightarrow & \mathbf{X}^{(1)}_{3,3} \\
 & \vdots & & & \ddots & & \longrightarrow & \vdots \\
 & \vdots & & & & \ddots & \longrightarrow & \vdots \\
n & X^{(1)}_{1,n} & X^{(1)}_{2,n} & X^{(1)}_{3,n} & \cdots & \mathbf{X}^{(1)}_{n,n} & \longrightarrow & \mathbf{X}^{(1)}_{n,n}
\end{array}
$$

There are $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ identical units selected from the population, which have been randomly divided into r groups of sizes $1,2,...,n$. Then we get a MaxRSS sample of size $m = r \times n$. Let $X^{(1)}, X^{(2)}, ..., X^{(r)}$ be an SRS from the cdf F with pdf f given in (1). Then the random variables $X^{(1)}_{1,1}, X^{(1)}_{2,2}, ..., X^{(1)}_{n,n}$ and $X^{(2)}_{1,1}, X^{(2)}_{2,2}, ..., X^{(2)}_{n,n}$ with $X^{(r)}_{1,1}, X^{(r)}_{2,2}, ..., X^{(r)}_{n,n}$ are independent, where $X^{(j)}_{i,i}$ denotes the j^{th} and i^{th} sample of size n for $j \in \{1,2,...,r\}$ and $i \in \{1,2,...,n\}$, where $m = r \times n$ is the size of MaxRSS. For $j = 1$ we have $r = 1$ then $m = n$. Now, let $X_1, X_2, ..., X_n$ be a random sample with the common distribution cdf F given in (2) and pdf f given in (1). The pdf $f_{n,\lambda}$ of MaxRSS for the $X_{i:i}$ order statistics is given by

$$f_{n,\lambda}(x_{i:i}) = i \times f(x_{i:i};\lambda)\left[F(x_{i:i};\lambda)\right]^{i-1} \tag{26}$$

for changeable i, which takes the value of the set, where $i = 1,2,\ldots,n$. Using equations (1) and (2), we have for $|\lambda| \leq 1$.

$$f_{n,\lambda}(x_{i:i}) = i \times g(x_{i:i})\left(1+\lambda-2\lambda G(x_{i:i})\right)\left[(1+\lambda)G(x_{i:i}) - \lambda G^2(x_{i:i})\right]^{i-1} \tag{27}$$

Let $X_1, X_2, ..., X_n$ be an iid random sample with the cdf F given in (2). Then, the likelihood function can be written as

$$\ell(x,\lambda) = \prod_{i=1}^{n}\left(i \times g(x_{i:i})\left(1+\lambda-2\lambda G(x_{i:i})\right)\left[(1+\lambda)G(x_{i:i}) - \lambda G^2(x_{i:i})\right]^{i-1}\right) \tag{28}$$

Using the logarithmic accumulation of equation (28),

$$\begin{aligned}\log \ell(x,\lambda) = \sum_{i=1}^{n} (\log(i) + \log(g(x_{i:i})) + \log(1+\lambda-2\lambda G(x_{i:i})) \\ + (i-1)\log\left((1+\lambda)G(x_{i:i}) - \lambda G^2(x_{i:i})\right))\end{aligned} \tag{29}$$

The likelihood equation (29) is then given in terms of the partial derivatives.

$$\frac{\partial \log \ell(x,\lambda)}{\partial \lambda} = \sum_{i=1}^{n} \left(\frac{1-2G(x_{i:i})}{1+\lambda-2\lambda G(x_{i:i})} + (i-1)\frac{G(x_{i:i}) - G^2(x_{i:i})}{(1+\lambda)G(x_{i:i}) - \lambda G^2(x_{i:i})} \right) \tag{30}$$

We can formulate the equation (30) as follows:

$$\frac{\partial \log \ell(x,\lambda)}{\partial \lambda} = \frac{1}{\lambda}\sum_{i=1}^{n} \left(1 - \frac{1}{1+\lambda-2\lambda G(x_{i:i})} + (i-1)\left(1 - \frac{G(x_{i:i})}{(1+\lambda)G(x_{i:i}) - \lambda G^2(x_{i:i})}\right)\right) \tag{31}$$

To obtain the estimator of the transmuted parameter of cdf F, which is defined in (2) via MLE with MaxRSS, we set the equations (31) equal to zero for $0 \leq \lambda \leq 1$ and $-1 \leq \lambda \leq 0$. Hence, to solve these nonlinear equations with iterative approaches, we use numerical methods, such as the Newton-Raphson method or the MBA, which is presented by Kouider (2019a). If there is more than one root, we take the root with the maximum log-likelihood, which is given in (5) with G as the base distribution.

Next for $\lambda \neq 0$ we can write equation (31) for estimating the transmuted parameter via the MLE with MaxRSS in the following form:

$$\frac{1}{n}\sum_{i=1}^{n} \left(\frac{g(x_{i:i})}{f(x_{i:i};\lambda)} - (1-i)\frac{G(x_{i:i})}{F(x_{i:i};\lambda)} \right) - 1 = 0$$

where F and f are given in (2) and (1) respectively, with $G(x)$ as the base df and $g(x)$ as the pdf associated. The MLE of the transmuted parameter through RSS is denoted by $\widehat{\lambda}_{MaxRSS}$.

3. Application

3.1. The MLE for the TGPD parameters using MaxRSS

Lately, according to the QRTM approach, Kouider et al. (2025) defined the TGPD (γ,σ,λ) which depends on three parameters which are the shape parameter, the scale parameter, and the transmuted parameter. They are starting from the MLE based on simple random sampling (SRS) and ranked set sampling (RSS). In addition, in this paper we will also estimate the TGPD (γ,σ,λ) parameters based on the MLE using MaxRSS.

An rv X is said to have the TGPD with parameter (γ,σ,λ) if the pdf is defined as:

$$f_{\gamma,\sigma,\lambda}(x) := g_{\gamma,\sigma}(x)\left(1+\lambda-2\lambda G_{\gamma,\sigma}(x)\right), \tag{32}$$

and the cdf is

$$F_{\gamma,\sigma,\lambda}(x) := (1+\lambda) G_{\gamma,\sigma}(x) - \lambda G^2_{\gamma,\sigma}(x), \tag{33}$$

where $\gamma \in \mathbb{R}$ is the shape parameter, $\sigma > 0$ the scale parameter, and $|\lambda| \leq 1$ the transmuted parameter. The upper tail of TGPD $F_{\gamma,\sigma,\lambda}(x)$ can be defined as follows:

$$\overline{F}_{\gamma,\sigma,\lambda}(x) := (1-\lambda) \overline{G}_{\gamma,\sigma}(x) + \lambda \overline{G}^2_{\gamma,\sigma}(x) \tag{34}$$

where $G_{\gamma,\sigma}(x)$ is the cdf of the GPD parameters, which is defined as follows:

$$P(X \leq x) = G_{\gamma,\sigma}(x) := \begin{cases} 1 - \left(1 + \frac{\gamma}{\sigma} x\right)^{-1/\gamma} & \text{for } \gamma \neq 0 \\ 1 - \exp\left(-\frac{x}{\sigma}\right) & \text{for } \gamma = 0 \end{cases}, \tag{35}$$

and $g_{\gamma,\sigma}(x)$ is the pdf of GPD, which is given as follows:

$$g_{\gamma,\sigma}(x) := \begin{cases} \frac{1}{\sigma}\left(1 + \frac{\gamma}{\sigma} x\right)^{-1/\gamma - 1}, & \text{for } \gamma \neq 0 \\ \frac{1}{\sigma} \exp\left(-\frac{x}{\sigma}\right), & \text{for } \gamma = 0 \end{cases}, \tag{36}$$

where $\sigma > 0$ for $\gamma > 0$ and $\sigma > 0$ for $0 < x < \sigma/|\gamma|$ if $\gamma < 0$. Then, let $X_1, X_2, ..., X_n$ be a random sample with cdf $F_{\gamma,\sigma,\lambda}$ given in (11) and pdf $f_{\gamma,\sigma,\lambda}$ given in (10). The pdf $f_{i,\gamma,\sigma,\lambda}(x)$ of MaxRSS for the i^{th} order statistics is shown as

$$f_{i,\gamma,\sigma,\lambda}(x) = i f_{\gamma,\sigma,\lambda}(x) \left[F_{\gamma,\sigma,\lambda}(x)\right]^{i-1} \tag{37}$$

for $\gamma \neq 0$.

$$\begin{aligned} f_{i,\gamma,\sigma,\lambda}(x) &= i\left(\frac{1}{\sigma}\left(1 + \frac{\gamma}{\sigma} x\right)^{\frac{-1}{\gamma} - 1}\left(1 + \lambda - 2\lambda\left(1 + \frac{\gamma}{\sigma} x\right)^{-1/\gamma}\right)\right) \\ &\times \left[(1+\lambda)\left(1 - \left(1 + \frac{\gamma}{\sigma} x\right)^{-1/\gamma}\right) - \lambda\left(1 - \left(1 + \frac{\gamma}{\sigma} x\right)^{-1/\gamma}\right)^2\right]^{i-1} \end{aligned} \tag{38}$$

and with $\gamma = 0$ the pdf of MaxRSS is

$$\begin{aligned} f_{i,\gamma,\sigma,\lambda}(x) &= i\left(\frac{1}{\sigma}\exp\left(-\frac{x}{\sigma}\right)\right)\left(1 + \lambda - 2\lambda\exp\left(-\frac{x}{\sigma}\right)\right) \\ &\times \left[(1+\lambda)\left(1 - \exp\left(-\frac{x}{\sigma}\right)\right) - \lambda\left(1 - \exp\left(-\frac{x}{\sigma}\right)\right)^2\right]^{i-1} \end{aligned} \tag{39}$$

With df the TGPD, let $X_1, X_2, ..., X_n$ be an iid random sample drawn from this df. The probability function for $\gamma \neq 0$ can be expressed as

$$\begin{aligned} \ell(x; \gamma, \sigma, \lambda) = \prod_{i=1}^{n} [& i\left(\frac{1}{\sigma}\left(1 + \frac{\gamma}{\sigma} x_i\right)^{\frac{-1}{\gamma} - 1}\left(1 + \lambda - 2\lambda\left(1 + \frac{\gamma}{\sigma} x_i\right)^{-1/\gamma}\right)\right) \\ & \times \left[(1+\lambda)\left(1 - \left(1 + \frac{\gamma}{\sigma} x_i\right)^{-1/\gamma}\right) - \lambda\left(1 - \left(1 + \frac{\gamma}{\sigma} x_i\right)^{-1/\gamma}\right)^2\right]^{i-1}] \end{aligned} \tag{40}$$

Additionally, the probability function for $\gamma = 0$ can be expressed as

$$\begin{aligned}\ell(x;\gamma,\sigma,\lambda) = \prod_{i=1}^{n}[i\left(\tfrac{1}{\sigma}\exp\left(-\tfrac{x}{\sigma}\right)\right)\left(1+\lambda-2\lambda\exp\left(-\tfrac{x}{\sigma}\right)\right)\\ \times\left[(1+\lambda)\left(1-\exp\left(-\tfrac{x}{\sigma}\right)\right)-\lambda\left(1-\exp\left(-\tfrac{x}{\sigma}\right)\right)^2\right]^{i-1}]\end{aligned} \tag{41}$$

Using the logarithmic accumulation of equation (40), for $\gamma \neq 0$,

$$\begin{aligned}\log\ell(x;\gamma,\sigma,\lambda) = \log[\prod_{i=1}^{n} i] + \sum_{i=1}^{n}[-\log\sigma - \left(1+\tfrac{1}{\gamma}\right)\log\left(1+\tfrac{\gamma}{\sigma}x_i\right)\\ +\log\left(1+\lambda-2\lambda\left(1+\tfrac{\gamma}{\sigma}x_i\right)^{-1/\gamma}\right)\\ +(i-1)\log\left((1+\lambda)\left(1-\left(1+\tfrac{\gamma}{\sigma}x\right)^{-1/\gamma}\right)-\lambda\left(1-\left(1+\tfrac{\gamma}{\sigma}x\right)^{-1/\gamma}\right)^2\right)\end{aligned} \tag{42}$$

Using equation (41)'s logarithmic accumulation, for $\gamma = 0$,

$$\begin{aligned}\log\ell(x;\gamma,\sigma,\lambda) = \log[\prod_{i=1}^{n} i] + \sum_{i=1}^{n}[-\log\sigma - \left(\tfrac{x_i}{\sigma}\right) + \log\left(1+\lambda-2\lambda\exp\left(\tfrac{x_i}{\sigma}\right)\right)\\ +(i-1)\log\left[(1+\lambda)\left(1-\exp\left(\tfrac{x_i}{\sigma}\right)\right)-\lambda\left(1-\exp\left(\tfrac{x_i}{\sigma}\right)\right)^2\right]\end{aligned} \tag{43}$$

The partial derivatives are then used to provide the likelihood equations from (40) with $\gamma \neq 0$.

$$\begin{aligned}\tfrac{\partial\log\ell(x;\gamma,\sigma,\lambda)}{\partial\gamma} = \sum_{i=1}^{n}[\tfrac{1}{\gamma^2}\log\left(1+\tfrac{\gamma}{\sigma}x_i\right) - \left(1+\tfrac{1}{\gamma}\right)\tfrac{\frac{x_i}{\sigma}}{\left(1+\frac{\gamma}{\sigma}x\right)}\\ +(i-1)\tfrac{(1+\lambda)\left(1+\frac{\gamma}{\sigma}x_i\right)^{-1/\gamma}\left(-\log\left(1+\frac{\gamma}{\sigma}x_i\right)-\frac{\frac{x_i}{\gamma\sigma}}{1+\frac{\gamma}{\sigma}x_i}\right)}{(1+\lambda)\left(1+\frac{\gamma}{\sigma}x_i\right)^{-1/\gamma}-\lambda\left(1-\left(1+\frac{\gamma}{\sigma}x_i\right)^{-1/\gamma}\right)^2}\\ -(i-1)\tfrac{2\lambda\left(1-\left(1+\frac{\gamma}{\sigma}x\right)^{-1/\gamma}\right)\left(1+\frac{\gamma}{\sigma}x_i\right)^{-1/\gamma}\left(\log\left(1+\frac{\gamma}{\sigma}x\right)+\frac{\frac{x_i}{\gamma\sigma}}{1+\frac{\gamma}{\sigma}x_i}\right)}{(1+\lambda)\left(1+\frac{\gamma}{\sigma}x_i\right)^{-1/\gamma}-\lambda\left(1-\left(1+\frac{\gamma}{\sigma}x_i\right)^{-1/\gamma}\right)^2}\\ -(n-i)\tfrac{(1+\lambda)\left(1+\frac{\gamma}{\sigma}x_i\right)^{-1/\gamma}\left(-\log\left(1+\frac{\gamma}{\sigma}x_i\right)-\frac{\frac{x_i}{\gamma\sigma}}{1+\frac{\gamma}{\sigma}x_i}\right)}{1-\left((1+\lambda)\left(1+\frac{\gamma}{\sigma}x_i\right)^{-1/\gamma}-\lambda\left(1-\left(1+\frac{\gamma}{\sigma}x_i\right)^{-1/\gamma}\right)^2\right)} = 0\end{aligned} \tag{44}$$

and

$$\begin{aligned}\tfrac{\partial\log\ell(x;\gamma,\sigma,\lambda)}{\partial\sigma} = \sum_{i=1}^{k}[\tfrac{-1}{\sigma} - \left(1+\tfrac{1}{\gamma}\right)\tfrac{\frac{-\gamma}{\sigma^2}}{\left(1+\frac{\gamma}{\sigma}x_i\right)} + 2\left(1-\zeta_i\right)\left(1+\tfrac{\gamma}{\sigma}x_i\right)^{-\frac{1}{\gamma}-1}\tfrac{x_i}{\sigma^2}\\ +(i-1)\tfrac{\left((1+\lambda)\left(1+\frac{\gamma}{\sigma}x_i\right)^{-\frac{1}{\gamma}-1}\frac{x_i}{\sigma^2}\right)+2\lambda\left(1-\left(1+\frac{\gamma}{\sigma}x_i\right)^{-\frac{1}{\gamma}}\right)\left(1+\frac{\gamma}{\sigma}x_i\right)^{-\frac{1}{\gamma}-1}\frac{x_i}{\sigma^2}}{(1+\lambda)\left(1+\frac{\gamma}{\sigma}x_i\right)^{-\frac{1}{\gamma}}-\lambda\left(1-\left(1+\frac{\gamma}{\sigma}x_i\right)^{-\frac{1}{\gamma}}\right)^2} = 0\end{aligned} \tag{45}$$

$$\tfrac{\partial\log\ell(x;\gamma,\sigma,\lambda)}{\partial\lambda} = \sum_{i=1}^{n}[\tfrac{-1+2\left(1+\frac{\gamma}{\sigma}x_i\right)^{-1/\gamma}}{1-\lambda+2\lambda\left(1+\frac{\gamma}{\sigma}x_i\right)^{-1/\gamma}} + (i-1)\tfrac{\left(1+\frac{\gamma}{\sigma}x_i\right)^{-1/\gamma}-\left(1-\left(1+\frac{\gamma}{\sigma}x_i\right)^{-1/\gamma}\right)}{(1+\lambda)\left(1+\frac{\gamma}{\sigma}x_i\right)^{-1/\gamma}-\lambda\left(1-\left(1+\frac{\gamma}{\sigma}x_i\right)^{-1/\gamma}\right)^2} = 0 \tag{46}$$

Also, under (43) for $\gamma = 0$, we have

$$\begin{aligned}\frac{\partial \log \ell(x;\gamma,\sigma,\lambda)}{\partial \sigma}\big|_{\gamma=0} &= \sum_{i=1}^{n}\big[\frac{-1}{\sigma}+\frac{x_i}{\sigma^2}+2\left(1-\varepsilon_i\right)\exp\left(-\frac{x_i}{\sigma}\right)\frac{x_i}{\sigma^2} \\ &-\left(i-1\right)\frac{(1+\lambda)\exp\left(-\frac{x_i}{\sigma}\right)\frac{x_i}{\sigma^2}+2\lambda\left(1-\exp\left(-\frac{x_i}{\sigma}\right)\right)\exp\left(-\frac{x_i}{\sigma}\right)\frac{x_i}{\sigma^2}}{(1+\lambda)\left(1-\exp\left(-\frac{x_i}{\sigma}\right)\right)-\lambda\left(1-\exp\left(-\frac{x_i}{\sigma}\right)\right)^2}=0\end{aligned} \tag{47}$$

and

$$\frac{\partial \log \ell(x;\gamma,\sigma,\lambda)}{\partial \lambda}\big|_{\gamma=0} = \sum_{i=1}^{n}\big[\frac{-1+2\exp\left(-\frac{x_i}{\sigma}\right)}{1-\lambda+2\lambda\exp\left(-\frac{x_i}{\sigma}\right)}+\left(i-1\right)\frac{\left(1-\exp\left(-\frac{x_i}{\sigma}\right)\right)-\left(1-\exp\left(-\frac{x_i}{\sigma}\right)\right)^2}{(1+\lambda)\left(1-\exp\left(-\frac{x_i}{\sigma}\right)\right)-\lambda\left(1-\exp\left(-\frac{x_i}{\sigma}\right)\right)^2}=0 \tag{48}$$

To use MLE with the MaxRSS to obtain the estimators of the TGPD parameters (γ,σ,λ), the two equations (47)-(48) and the three equations (44)-(46) are set to zero. We employ numerical methods, such as the Newton-Raphson method or the MBA for multi-roots, as presented by Kouider(2019), to solve these nonlinear equations with iterative approaches. $\left(\hat{\gamma}_{MaxRSS},\hat{\sigma}_{MaxRSS},\hat{\lambda}_{MaxRSS}.\right)$ represents the MLE of TGPD (γ,σ,λ) parameters via the MaxRSS. If there is more than one root, we take the root with the maximum log-likelihood, which is given in (42) for $\gamma \neq 0$ or (43) for $\gamma = 0$.

3.2. Estimating the transmuted quantile of the TGPD

In this part, we will present a formula that allows us to estimate the transmuted quantile of the TGPD parameters (γ,σ,λ) based on estimating the parameter (γ,σ) of the GPD (35). For this, we state that there are several numerical methods to calculate the ML estimator of the GPD parameters. By using the MLE for GPD parameters via the numerical method of Kouider (2019b) to maximize the log-likelihood equation of the GPD given in (37) when $\gamma \geq -1$, we can estimate the unknown parameters of the GPD without encountering complicated calculations, as noted $\left(\widehat{\gamma}^{ML},\widehat{\sigma}^{ML}\right)$. Nevertheless, Kouider et al. (2025) worked to estimate the transmuted index of the TGPD given in (33) and referred to it as the root of the following function:

$$\widetilde{\psi}(\lambda) = \sum_{i=1}^{n}\frac{1-2G_{\widehat{\gamma}^{ML},\widehat{\sigma}^{ML}}(X_i)}{1+\lambda-2\lambda G_{\widehat{\gamma}^{ML},\widehat{\sigma}^{ML}}(X_i)} \tag{49}$$

Otherwise, we keep the same shape and scale parameters for the TGPD of the GPD (γ,σ), and the addition is the transmuted index λ. Then, we offer a formula for estimating the transmuted quantile of the TGPD parameters with certain characteristics. As agreed upon, the quantile is the inverse of the df; thus, the quantile $x_{GPD,q}$ of rv X follows the GPD given in (37) with $\gamma \neq 0$ given by

$$X_{GPD,q_{GPD}} = \frac{\sigma}{\gamma}\left(\left(q_{GPD}\right)^{-\gamma}-1\right) \tag{50}$$

where $q_{GPD} = 1-p_{GPD}$ with $p_{GPD} = G_{\gamma,\sigma}(x)$, which is the GPD. Also, the quantile $x_{TGPD,q}$

of r.v X having the cdf of the TGPD from (33) with $\gamma \neq 0$ is,

$$X_{TGPD,q_{TGPD}} = \frac{\sigma}{\gamma}\left(\left(\frac{\sqrt{4\lambda q_{TGPD}+(1-\lambda)^2}-(1-\lambda)}{2\lambda}\right)^{-\gamma}-1\right) \tag{51}$$

where $q_{TGPD} = 1 - p_{TGPD}$ with $p_{TGPD} = F_{\gamma,\sigma,\lambda}(x)$.

Remark 3.1 *Let X's rv follow the TGPD (33) with the GPD (37) as the base distribution. Then,*

1. *If $\lambda = 0$, we have $F_{\gamma,\sigma,\lambda}(x) = G_{\gamma,\sigma}(x)$ and $X_{TGPD,q_{TGPD}} = X_{GPD,q_{GPD}}$*
2. *If $\lambda = 1$, we find $\overline{F}_{\gamma,\sigma,\lambda}(x) = \overline{G}^2_{\gamma,\sigma}(x)$ and $X_{TGPD,q_{TGPD}} = X_{GPD,\sqrt{q_{TGPD}}}$*
3. *If $\lambda = -1$, we get $F_{\gamma,\sigma,\lambda}(x) = G^2_{\gamma,\sigma}(x)$ and $X_{TGPD,q_{TGPD}} = X_{GPD,1-\sqrt{q_{TGPD}}}$*
4. *In general, we can define $X_{TGPD,q_{TGPD}}$ with respect to $X_{GPD,q_{GPD}}$ by*

$$X_{TGPD,q_{TGPD}} = X_{GPD,\widetilde{q}} = \frac{\sigma}{\gamma}\left((\widetilde{q})^{-\gamma}-1\right) \tag{52}$$

where

$$\widetilde{q} = \frac{\sqrt{4\lambda q_{TGPD}+(1-\lambda)^2}-(1-\lambda)}{2\lambda} \tag{53}$$

and $q_{TGPD} = 1 - F_{\gamma,\sigma,\lambda}(x)$.

Kouider et al. (2025) also pointed to $X_{TGPD,q_{TGPD}}$ the transmuted quantile of the TGPD parameters without explicitly estimating this quantity. Consequently, the present paper constitutes a continuation and further development of their work, building upon their findings and extending the investigation of the aforementioned quantity. Since, under the relations (50) and (52), we have

$$\frac{\gamma}{\sigma}\left(X_{TGPD,q_{TGPD}} - X_{GPD,q_{GPD}}\right) = (\widetilde{q})^{-\gamma} - (q_{GPD})^{-\gamma}$$

where q_{GPD} is the queue distribution function of the GPD and $\widetilde{q}$ is given in (53).

3.3. Numerical example

First example: Random samples of size $n = 15$ of the random variable follow the GPD with the parameters (γ, σ). These parameters are $\gamma = -0.7, \sigma = 2$. Before we generate a random sample following the uniform distribution for size $n = 15$ using the R Core Team software, as

1. $p_i = runif(15)$.
2. Define the queue as $q_i = 1 - p_i$.

3. Generate a random samples of size $n = 15$ from the GPD using the method of inversion with the quantiles by

$$X_{GPD,qi} = \frac{\sigma}{\gamma}\left((q_i)^{-\gamma} - 1\right) \tag{54}$$

4. Use the Kouider estimation to determine the ML-estimators of γ and σ.

5. Solve the equation $\widetilde{\psi}(\lambda) = 0$ given in (49) using the MBA method in $[0;1]$ and $[-1;0]$ to determine the ML-estimator of λ. If there is more than one root, we take the root with the largest value.of the log-likelihood, which is given in (5) with the GPD as the base distribution.

6. Estimate $\widehat{p}_{GPD,i}$, the GPD estimator with the parameters $\widehat{\gamma}^{ML}$ and $\widehat{\sigma}^{ML}$ by

$$\widehat{p}_{GPD,i} = 1 - \left(1 + \frac{\widehat{\gamma}^{ML}}{\widehat{\sigma}^{ML}} X_{GPD,qi}\right)^{-1/\widehat{\gamma}^{ML}} \tag{55}$$

7. Estimate the quantiles of samples of size $n = 15$ that follow the TGPD, given by

$$X_{TGPD,q_{TGPD,i}} = \frac{\widehat{\sigma}^{ML}}{\widehat{\gamma}^{ML}}\left(\left(\frac{\sqrt{4\widehat{\lambda}^{ML}\widehat{q}_{TGPD,i} + \left(1-\widehat{\lambda}^{ML}\right)^2} - \left(1-\widehat{\lambda}^{ML}\right)}{2\widehat{\lambda}^{ML}}\right)^{-\widehat{\gamma}^{ML}} - 1\right) \tag{56}$$

where

$$\widehat{q}_{TGPD,i} = 1 - \left(1+\widehat{\lambda}^{ML}\right)\widehat{p}_{GPD,i} + \widehat{\lambda}^{ML}\widehat{p}^2_{GPD,i} \tag{57}$$

Then we generate a sample X_i of size 15 that follows the TGPD with parameters $\gamma = -0.7, \sigma = 2$. The 15 values are listed in an increasing order in the following Table 1.

Table 1. A random sample of size 15 is generated from the GPD with $\gamma = -0.7$ and $\sigma = 2$

0.0323	0.0853	0.1299	0.1424	0.1673
0.5108	0.6023	0.6307	0.8414	0.8437
1.2599	1.6980	2.4380	2.5098	2.6371

Using the Kouider estimation, we find $\widehat{\gamma}^{ML} = -0.707$, $\widehat{\sigma}^{ML} = 1.964$. Then, we apply the MBA method to determine the root of $\psi(\lambda)$ on (7) with $\lambda \in [0;1]$ and $\lambda \in [-1;0]$. We find one parameter that is $\widehat{\lambda}^{ML} = 0.351$. Next , we represent the transmuted quantile from the GPD with (γ, σ) to the quantile from the TGPD with $(\gamma, \sigma, \lambda)$. Under (56), we obtain a sample $\widetilde{X}_i$ of size 15 which follows the TGPD with parameters $\widehat{\gamma}^{ML} = -0.707$, $\widehat{\sigma}^{ML} = 1.964$, and $\widehat{\lambda}^{ML} = 0.351$. These values are arranged in increasing order and presented in the following Table 2, as follows:

Table 2. A set of 15 data points is generated from the TGPD with $(-0.7, 2, 0.351)$

2.15	2.17	2.18	2.18	2.18
2.26	2.28	2.29	2.33	2.33
2.43	2.53	2.70	2.72	2.75

Second example: We use tensile strength fiber data studied by Kouider et al. (2025). Data were counted from threshold exceedances in a tensile strength test of a random value of nylon carpet fibers, listed in an ascending order in Table 3.

Table 3. Random sample of $n = 15$ nylon carpet fibers (in kg/mm2)

0.011	0.030	0.051	0.056	0.092
0.100	0.140	0.184	0.200	0.2860
0.338	0.365	0.518	0.561	0.876

In his study, he found that the estimators of the GPD parameters by the MLE method were $\widehat{\gamma}^{ML} = -0.1176979$, $\widehat{\sigma}^{ML} = 0.283040$, and $\widehat{\lambda}^{ML} = 0.05885239$. Then we can define the transmuted quantile of the tensile-strength testing under (56) in Table 4 by

Table 4. Random sample of $n = 15$ transmuted nylon carpet fibers (in kg/mm2)

1.18	1.19	1.20	1.20	1.22
1.22	1.24	1.26	1.27	1.31
1.34	1.36	1.44	1.46	1.62

Third example: We apply the MLE using the RSS; MaxRSS is used to estimate the TGPD parameters given in (11). Tables from (5) to (13) show the estimator, mean square error (MSE), and bias for the unknown parameters of TGPD using values $\gamma = -0.7$, $\sigma = 2$, and $\lambda = 0.9$ based on the SRS, RSS, and MaxRSS techniques using the MLE method with 1000 replicates. Below the tables is the MSE histogram utilizing the SRS, RSS, and MaxRSS techniques.

From Tables (5) to (13) it can be noticed that:

1. In all cases, the biases are negligible.

2. All of the values of the estimators v are extremely close to the actual values of the parameters.

3. Compared to estimators that use SRS, those that use RSS and MaxRSS have lower MSEs.

4. Compared to estimators using RSS, those employing MaxRSS have lower MSEs.

5. With very little bias, the RSS outperforms the SRS and yields results that are extremely close to the actual values of the parameters.

6. The MaxRSS performs better than the RSS and yields results that are extremely close to the actual values of the parameters, with negligible biases.

Table 5. The estimators of TGPD parameters via MLE with SRS technique

	SRS		
n	$\widehat{\gamma}_{SRS}$	$\widehat{\sigma}_{SRS}$	$\widehat{\lambda}_{SRS}$
50	-0.6562	1.7378	0.6870
100	-0.6720	1.7530	0.6858
150	-0.6980	1.7551	0.7089
200	-0.6925	1.7346	0.7003

Table 6. The estimators of TGPD parameters via MLE with RSS technique

	RSS		
n	$\widehat{\gamma}_{RSS}$	$\widehat{\sigma}_{RSS}$	$\widehat{\lambda}_{RSS}$
50	-0.7163	1.8757	0.6921
100	-0.6713	1.9155	0.6590
150	-0.6680	1.9138	0.6561
200	-0.7480	1.9726	0.7121

Table 7. The estimators of TGPD parameters via MLE with MaxRSS technique

	MaxRSS		
n	$\widehat{\gamma}_{MaxRSS}$	$\widehat{\sigma}_{MaxRSS}$	$\widehat{\lambda}_{MaxRSS}$
50	-0.7468	1.8594	1.0347
100	-1.2754	2.5659	0.6597
150	-0.9277	1.9844	1.1572
200	-1.0453	2.3718	0.7325

Table 8. MSEs of the SRS technique generated with the TGPD parameters

	SRS		
n	$\widehat{\gamma}_{SRS}$	$\widehat{\sigma}_{SRS}$	$\widehat{\lambda}_{SRS}$
50	0.0178	0.0236	0.0863
100	0.0053	0.0124	0.0661
150	0.0026	0.0074	0.0588
200	0.0020	0.0037	0.0227

Table 9. MSEs of the RSS technique generated with the TGPD parameters

	RSS		
n	$\widehat{\gamma}_{RSS}$	$\widehat{\sigma}_{RSS}$	$\widehat{\lambda}_{RSS}$
50	0.0058	0.0035	0.0531
100	0.0022	0.0019	0.0577
150	0.0021	0.0010	0.0583
200	0.0008	0.0026	0.0210

Table 10. MSEs of the MaxRSS technique generated with the TGPD parameters

	MaxRSS		
n	$\widehat{\gamma}_{MaxRSS}$	$\widehat{\sigma}_{MaxRSS}$	$\widehat{\lambda}_{MaxRSS}$
50	0.00386	0.0034	0.0399
100	0.0010	0.0033	0.0334
150	0.0018	0.0009	0.0049
200	0.0001	0.0020	0.0098

Table 11. Biases for SRS technique for the TGPD parameters

	SRS		
n	$\widehat{\gamma}_{SRS}$	$\widehat{\sigma}_{SRS}$	$\widehat{\lambda}_{SRS}$
50	-0.0437	0.2621	0.2129
100	-0.0279	0.2469	0.2141
150	-0.0019	0.2448	0.1910
200	-0.0074	0.2654	0.1996

Table 12. Biases for RSS technique for the TGPD parameters

	RSS		
n	$\widehat{\gamma}_{RSS}$	$\widehat{\sigma}_{RSS}$	$\widehat{\lambda}_{RSS}$
50	0.0163	0.1242	0.2078
100	-0.0286	0.0844	0.2409
150	-0.0319	0.0861	0.2438
200	0.0480	0.1878	0.0273

Table 13. Biases for MaxRSS technique for the TGPD parameters

	MaxRSS		
n	$\widehat{\gamma}_{MaxRSS}$	$\widehat{\sigma}_{MaxRSS}$	$\widehat{\lambda}_{MaxRSS}$
50	0.0468	0.1405	-0.1347
100	0.5754	-0.5659	0.2402
150	0.2277	0.0155	-0.2572
200	0.3453	-0.3718	0.1674

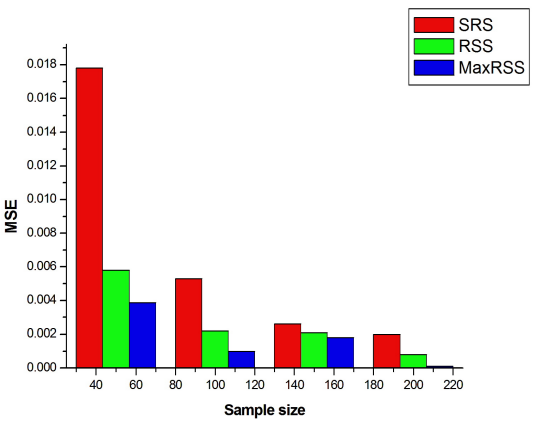


Figure 1. The MSE of $\left(\widehat{\gamma}^{ML}\right)$ for different sample sizes

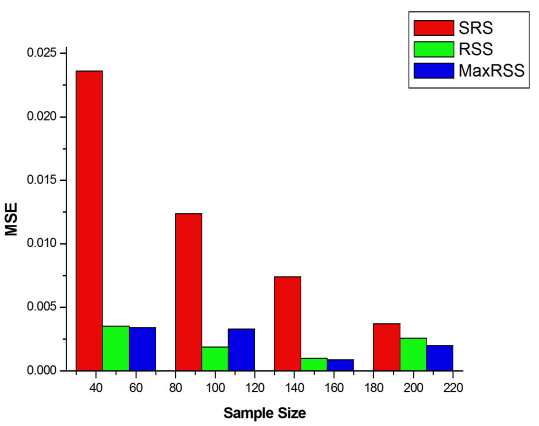


Figure 2. The MSE of $\left(\widehat{\sigma}^{ML}\right)$ for different sample sizes

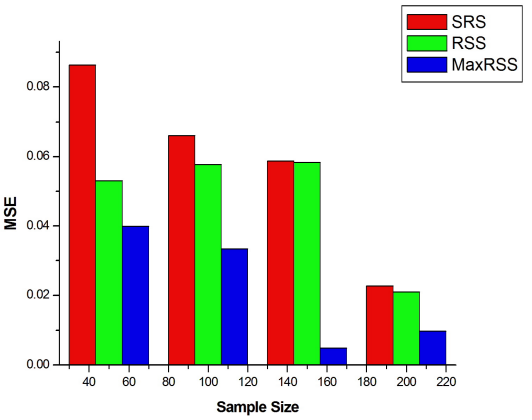


Figure 3. The MSE of $\left(\widehat{\lambda}^{ML}\right)$ for different sample sizes.

4. Conclusions

The obvious reason for generalizing the standard distribution is that the generalized form provides greater flexibility in modeling real data. The estimation of parameters is approached by the method of maximum likelihood. Since Kouider et al. (2025) studied the ML-estimator of the unknown transmuted parameter for the TGPD based on simple random sampling (SRS) and ranked set sampling (RSS), where the GPD parameters are the base distribution, our work is a continuation of what they completed and discusses the asymptotic distribution of the MLE of the unknown transmuted index with any original function using the QRTM technique. We prove that the Fisher information $I(\lambda)$ in (13) exists for $\lambda \in]-1;0[\cup]0;1[$. In addition to the two previous methods, in this article we estimate the TGPD parameters via the MaxRSS. Furthermore, we propose a procedure to transmute the quantile sample of the GPD to a new sample following the TGPD with the same shape and scale parameters using the steps shown in the first example of part 3.3. Additionally, through this work, we can find a sample that follows a certain distribution function and determine the transmuted index. Therefore, it must be estimated using the MLE methods with the relationship (7), which we discussed previously. We use this work as a reference to help advance future research in this field.

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Appendix

First, for $0 \leq \lambda < 1$ and from the definition of the Fisher information, which is given by (9), we get

$$I(\lambda) = -E_\lambda \left[\frac{\partial^2 \log \ell(x;\lambda)}{\partial^2 \lambda} \right] = -\int_{-\infty}^{+\infty} \frac{\partial^2 \log \ell(x;\lambda)}{\partial^2 \lambda} f(x,\lambda)\, dx.$$

Then

$$I(\lambda) = -\int_{-\infty}^{+\infty} \left(\frac{1-2G(x)}{1+\lambda-2\lambda G(x)} \right)^2 (g(x)(1+\lambda-2\lambda G(x)))\, dx.$$

So

$$I(\lambda) = -\int_{-\infty}^{+\infty} \left(\frac{1-2G(x)}{1+\lambda-2\lambda G(x)} \right) (1-2G(x))\, g(x)\, dx.$$

Next, with $\lambda \neq 0$, we get

$$I(\lambda) = -\frac{1}{\lambda} \left(\int_{-\infty}^{+\infty} (1-2G(x))\, g(x)\, dx - \frac{1}{\lambda} \int_{-\infty}^{+\infty} \left(\frac{\lambda-2\lambda G(x)}{1+\lambda-2\lambda G(x)} \right) g(x)\, dx \right).$$

Thus

$$\begin{aligned} I(\lambda) &= -\frac{1}{\lambda} \left(\int_{-\infty}^{+\infty} g(x)\, dx - 2\int_{-\infty}^{+\infty} g(x)\, G(x)\, dx \right. \\ &\quad \left. -\frac{1}{\lambda} \int_{-\infty}^{+\infty} \left(1 - \frac{1}{1+\lambda-2\lambda G(x)} \right) g(x)\, dx \right). \end{aligned}$$

Therefore, we found that

$$\begin{aligned} I(\lambda) &= -\frac{1}{\lambda} \left(\left(1-\frac{1}{\lambda}\right) \int_{-\infty}^{+\infty} g(x)\, dx - 2\int_{-\infty}^{+\infty} g(x)\, G(x)\, dx \right. \\ &\quad \left. +\frac{1}{2\lambda^2} \int_{-\infty}^{+\infty} \left(\frac{2\lambda g(x)}{1+\lambda-2\lambda G(x)} \right) dx \right) \\ &= -\frac{1}{\lambda} \left(\left(1-\frac{1}{\lambda}\right) I_1 - 2I_2 + \frac{1}{2\lambda^2} I_3 \right), \end{aligned}$$

where

$$\begin{aligned} I_1 &= \int_{-\infty}^{+\infty} g(x)\,dx, \\ I_2 &= \int_{-\infty}^{+\infty} g(x)\,G(x)\,dx, \text{ and} \\ I_3 &= \int_{-\infty}^{+\infty} \left(\frac{2\lambda g(x)}{1+\lambda-2\lambda G(x)}\right) dx. \end{aligned}$$

And after mathematical operations, we find

$$\begin{aligned} I_1 &= 1, \\ I_2 &= \frac{1}{2}\left[G^2(x)\right]_{-\infty}^{+\infty} = \frac{1}{2}, \text{ and} \\ I_3 &= -\left[\ln(1+\lambda-2\lambda G(x))\right]_{-\infty}^{+\infty} = -\ln\left(\frac{1-\lambda}{1+\lambda}\right). \end{aligned}$$

Then

$$I(\lambda) = -\frac{1}{\lambda}\left(1-\frac{1}{\lambda}-1-\frac{1}{2\lambda^2}\ln\left(\frac{1-\lambda}{1+\lambda}\right)\right).$$

Finally, we get

$$I(\lambda) = \frac{1}{\lambda^2}\left(\lambda-\ln\sqrt{\frac{1+\lambda}{1-\lambda}}\right),$$

where $\lambda \in]-1;0[\cup]0;1[$. And for $\lambda = 0$ the definition of the Fisher information (9) is given by:

$$I(0) = -\int_{-\infty}^{+\infty} (1-2G(x))^2 f(x,0)\,dx = -\int_{-\infty}^{+\infty} (1-2G(x))^2 g(x)\,dx.$$

Therefore,

$$I(0) = \frac{1}{6}\left[(1-2G(x))^3\right]_{-\infty}^{+\infty} = \frac{-1}{3}.$$

This contradicts the definition of variance, which must be strictly positive.