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Moments and estimation of Burr Hatke Exponential model based on generalized order statistics with real data applications

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Abstract

This paper presents a recurrence relation for single and product moments of the Burr-Hatke exponential distribution based on generalized order statistics and record values. Moreover, we have derived the explicit and exact expressions for the single moment of generalized order statistics from the Burr-Hatke exponential distribution. Furthermore, the classical and Bayesian estimation procedures for estimating the Burr-Hatke exponential distribution parameter are discussed when the data are in the form of records and *gos*. In addition, the asymptotic confidence interval, boot-p confidence interval, and credible interval for the parameter of the Burr-Hatke exponential distribution are discussed. Monte Carlo simulations were conducted to evaluate the performance of the estimators derived from both classical and Bayesian approaches, with a particular emphasis on their mean squared error. Moreover, two real datasets are included for illustrative purposes in this study.

Keywords: Generalized order statistics, single moments, recurrence relations, Burr Hatke Exponential distribution, MLE, Bayesian estimation, credible interval.

1. Introduction

Kamps (1995) introduced the concept of generalized order statistics (*gos*). This concept incorporates various significant models related to ordered random variables (*rv's*). These models include order statistics (OS), non-integral sample size order statistics, sequential order statistics, progressive Type II censored order statistics, record values, and Pfeifer's record values, all embedded within the framework of *gos*.

Let $n \in \mathbb{N}, k \geq 1, \widetilde{m} = (m_1, m_2, \ldots, m_{n-1}) \in \mathbb{R}^{n-1}$, be the parameters such that, for all $r \in \{1, \ldots, n-1\}$, $\gamma_r = k + n - r + \sum_{j=r}^{n-1} m_j \geq 1$, then $X(1,n,\widetilde{m},k), X(2,n,\widetilde{m},k), \ldots, X(n,n,\widetilde{m},k)$ are termed as *gos*, if their joint probability density function (*pdf*) is given by

$$\begin{aligned} &f_{X(1,n,\widetilde{m},k),X(2,n,\widetilde{m},k),\ldots,X(n,n,\widetilde{m},k)}(x_1,\ldots,x_n) \\ &= k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left(\prod_{i=1}^{n-1}(\bar{F}(x_i))^{m_i}f(x_i)\right)(\bar{F}(x_n))^{k-1}f(x_n), \end{aligned} \tag{1}$$

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on the cone $F^{-1}(1) > x_n \geq \ldots \geq x_1 > F^{-1}(0)$ of $\mathbb{R}^n$, where $\bar{F}(x) = 1 - F(x)$ denotes the survival function.

By modifying the parameters $\widetilde{m}$ and k, it is possible to construct a wide variety of significant models that fall under the category of ordered *rv's*. For instance, OS can be obtained by setting $m_j = 0,\ k = 1$ *and* $\gamma_j = n - j + 1,$where $1 \leq j \leq n-1$. Similarly, order statistics with non-integral sample size are generated when $m_j = 0,\ k = \beta - n + 1$ *and* $\gamma_j = \beta - j + 1$. In the case of progressive Type II censored order statistics, $m_j = R_j \in \mathbb{N}_0,\ k = R_n + 1$ *and* $\gamma_j = n - j + 1 + \sum_{i=r}^{n} R_i$. Sequential order statistics arise when $k = \beta_n,\ m_j = (n-j+1)\alpha_j - (n-j)\alpha_{j+1} - 1$ and $\gamma_j = (n-j+1)\beta_j; \beta_1, \ldots, \beta_n > 0$, while record values are produced by letting $m_j \to -1,\ k = 1$. Extending this further, k^{th} record values are defined by $m_j \to -1,\ k \in \mathbb{N}$, whereas Pfeifer's record values are obtained by setting $m_j = \beta_j - \beta_{j+1} - 1,\ k = \beta_n$ and $\gamma_j = \beta_j$.
When γ_j's are pairwise different, i.e. $\gamma_j \neq \gamma_i\ ; \forall j \neq i = 1, 2, \ldots, n-1$. Kamps and Cramer (2001) derived the *pdf* of $X(r,n,\widetilde{m},k)$ as:

$$f_{X(r,n,\widetilde{m},k)}(x) = c_{r-1}\left[\sum_{i=1}^{r} a_i(r)[\bar{F}(x)]^{\gamma_i}\right]\frac{f(x)}{\bar{F}(x)}, \quad -\infty < x < \infty \tag{2}$$

and for $1 \leq r < s \leq n$, the joint *pdf* of $X(r,n,\widetilde{m},k)$ and $X(s,n,\widetilde{m},k)$ is given by (Kamps and Cramer, 2001)

$$\begin{aligned} f_{X(r,n,\widetilde{m},k),X(s,n,\widetilde{m},k)}(x,y) = c_{s-1}&\left[\sum_{t=r+1}^{s} a_t^{(r)}(s)\left(\frac{\bar{F}(y)}{\bar{F}(x)}\right)^{\gamma_t}\right]\left[\sum_{i=1}^{r} a_i(r)(\bar{F}(x))^{\gamma_i}\right] \\ &\times\frac{f(x)}{\bar{F}(x)}\frac{f(y)}{\bar{F}(y)}, \quad -\infty < x < y < \infty. \end{aligned} \tag{3}$$

In case of records, i.e. $m \to -1,\ k = 1$, the *pdf* of $X(r,n,-1,1)$ is given by

$$f_{X(r,n,-1,1)}(x) = \frac{1}{(r-1)!}[-\ln\bar{F}(x)]^{r-1}f(x), \quad -\infty < x < \infty. \tag{4}$$

The joint *pdf* of $X(r,n,-1,1)$ and $X(s,n,-1,1)$, $1 \leq r < s$, is given by

$$\begin{aligned} f_{X(r,n,-1,1),X(s,n,-1,1)}(x,y) = [(r-1)!(s-r-1)!]^{-1}[-\ln\bar{F}(x)]^{r-1}[\ln\bar{F}(x) - \ln\bar{F}(y)]^{s-r-1} \\ \times\frac{f(x)}{\bar{F}(x)}f(y), \quad -\infty < x < y < \infty. \end{aligned} \tag{5}$$

Further, j^{th} moment of r^{th} *gos* is defined as:

$$\mu^{j}_{r,n,\widetilde{m},k} = E[X^j(r,n,\widetilde{m},k)] = c_{r-1}\sum_{i=1}^{r} a_i(r)\int_{-\infty}^{\infty} x^j[\bar{F}(x)]^{\gamma_i}\frac{f(x)}{\bar{F}(x)}\,dx,$$

and j^{th} and l^{th} product moment of r^{th} and s^{th} *gos* is defined as:

$$\mu^{j,l}_{r,s,n,\widetilde{m},k} = E[X^j(r,n,\widetilde{m},k)Y^l(s,n,\widetilde{m},k)] = c_{s-1}\sum_{i=1}^{r}\sum_{t=r+1}^{s} a_i(r)a^r_t(s)\int_{-\infty}^{\infty}\int_{x}^{\infty} x^j y^l \times [\bar{F}(y)]^{\gamma_t}\,[\bar{F}(x)]^{\gamma_i-\gamma_t}\frac{f(x)}{\bar{F}(x)}\frac{f(y)}{\bar{F}(y)}\,dy\,dx.$$

Where, $c_{r-1}=\prod_{j=1}^{r}\gamma_j$, $a_i(r)=\prod_{j=1}^{r}(\gamma_j-\gamma_i)^{-1}, \quad \forall\ \gamma_i \neq \gamma_j \quad 1 \leq i \leq r \leq n$, and $a^{(r)}_t(s)=\prod_{j=r+1,t\neq j}^{s}(\gamma_j-\gamma_t)^{-1}, \quad \gamma_t \neq \gamma_j, \quad \forall\ \gamma_j \neq \gamma_t, \quad r<t\leq s.$

Another important case of *gos* arises when the parameters $m_1 = m_2 = \ldots = m_{n-1} = m \neq -1$, termed as *m-gos*. In this case, $a_i(r)$ and $a^{(r)}_t(s)$ may further be simplified as (Khan and Khan, 2012)

$$a_i(r) = (-1)^{r-i}((r-1)!)^{-1}(m+1)^{-(r-1)}\binom{r-1}{r-i}, \tag{6}$$

$$a^{(r)}_t(s) = (-1)^{s-t}((s-r-1)!)^{-1}(m+1)^{-(s-r-1)}\binom{s-r-1}{s-t}. \tag{7}$$

With applications ranging from reliability to medicine, the one-parameter exponential distribution is frequently used to characterize the amount of time that passes between events. Furthermore, exponential distribution is a fundamental tool for the study of Poisson and Markov processes. A non-negative *rv* T is said to follow an exponential distribution if its cumulative distribution function (*cdf*) and the *pdf* are given by respectively

$$G(t;\theta) = 1-\exp(-\theta t), \quad 0<t<\infty, \quad \theta>0, \tag{8}$$

$$g(t;\theta) = \theta\exp(-\theta t), \quad 0<t<\infty, \quad \theta>0. \tag{9}$$

The Burr-Hatke exponential (BHE) distribution was introduced by Yadav et al. (2021) using the Burr-Hatke-G (BH-G) family of distributions, which is given by Yousof et al. (2018). The BHE distribution's *pdf* is right-skewed, and its hazard rate function (*hrf*) exhibits a declining pattern, as shown in Figures 1-2.

A *rv* X follows the BHE distribution if its *cdf* is of the form:

$$F(x) = 1-\frac{e^{-\theta x}}{1+\theta x}, \quad 0<x<\infty, \quad \theta>0. \tag{10}$$

Furthermore, its *pdf* is given as follows:

$$f(x) = \theta e^{-\theta x}\frac{2+\theta x}{(1+\theta x)^2}, \quad 0<x<\infty, \quad \theta>0. \tag{11}$$

The BHE distribution's *hrf* can be found as:

$$h(x) = \frac{\theta(2+\theta x)}{(1+\theta x)}. \tag{12}$$

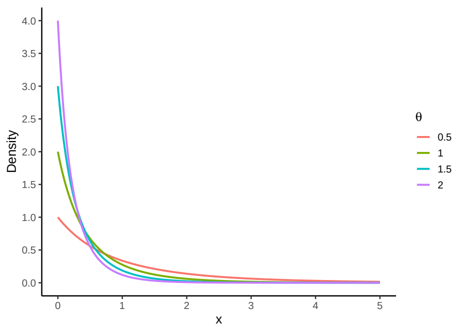

Figure 1. *pdf* plot of the BHE distribution

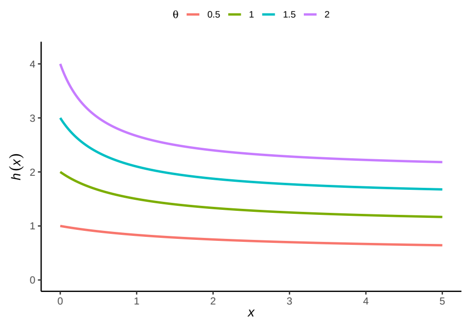

Figure 2. *hrf* plot of the BHE distribution

Quantile Function of BHE Distribution: The quantile, when it can be derived analytically, forms the basis for data generation. This facilitates the simulation of random samples, thereby enabling the determination of a model's behavior. The quantile function (QF) can be obtained by adopting the technique of Jodrá (2010) using $X = F^{-1}(U)$, where *rv* U is generated from $U(0,1)$. Notably, for certain probability distributions, the QFs can be formulated using the the Lambert W function. A review of the relevant literature, including citations such as Jodrá and Arshad (2022), Barco et al. (2017), and others, provides compelling evidence to support this claim.

For the sake of completeness, it is essential to recall the definition of the Lambert W function, which is a multivalued, complex function. This function can be expressed by the equation $w(z)e^{w(z)} = z$, where z represents the complex number. For real $z \geq -1/e$, the Lambert W function possesses two distinct real branch points, designated as W_0 (principal branch) and W_{-1} (negative branch), respectively. The function $W_0(z)$ assumes values within the interval $[-1,\infty)$ for $z \geq -1/e$, whereas $W_{-1}(z)$ assumes values within the interval $(-\infty,-1]$ for $z \in [-1/e, 0)$ (see Jodrá and Arshad (2022) and Jodrá (2010)).

Theorem 1. For any $\theta \in \mathbb{R}^+$, the BHE distribution's quantile function may be represented by

$$Q(u;\theta) = \frac{1}{\theta}\left(W_0\left(\frac{e}{1-u}\right) - 1\right), \quad 0 < u < 1. \tag{13}$$

Proof: For any $u \in (0,1)$, the function $F(x) = u$ has to be solved *w.r.t.* x. After simplification, we get

$$(1+\theta x)e^{(1+\theta x)} = \frac{e}{1-u}.$$

By combining the aforementioned equation with the definition of the Lambert W function, we can derive the following result:

$$W\left(\frac{e}{1-u}\right) = 1+\theta x, \tag{14}$$

where $\frac{e}{1-u} \geq -\frac{1}{e}$, the result is true. In light of the aforementioned properties of W_0, it can be concluded that the branch of W in Equation (14) is the principal branch. Ultimately, the proof is completed upon solving Equation (14) for x, which yields Equation (13).

In the last few years, various studies have been conducted on the extensions and properties of the Burr-Hatke Exponential distribution and its variants. Abouelmagd (2018) studied logarithmic BHE distribution and obtained its moments and maximum likelihood estimates (MLEs). Yousof et al. (2018) proposed the BH-G family of distributions by using the BH differential equation, whereas discrete forms of the BHE distribution were discussed in Morshedy et al. (2020) and Pandey et al. (2021). Afify et al. (2021) and El-Morshedy et al. (2021) discussed the classical and Bayesian estimation for the two-parameter and exponentiated BH distribution, respectively. Bhatti et al. (2021) derived the extended Chen distribution using a generalized BH differential equation and the relation between the exponential and gamma variables. The authors also obtained the MLE of an unknown parameter of the extended Chen distribution. Yadav et al. (2021) discussed Bayesian and classical estimation under Type II hybrid censoring. Khan et al. (2021) derived the moments and estimation of the inverted Topp-Leone distribution based on record values. Some other related works are Almuqrin (2023), Shirozhan et al. (2023), and Ragab et al. (2024), who discussed new BH-type models and their inferential properties.

Several authors have discussed classical and Bayesian inference under *gos*. For instance, Kumar (2017), Khan et al. (2018), Khan and Iqrar (2019), Khan and Khatoon (2020), Khan et al. (2020), established several moment relations and estimation procedures. Other studies, such as Alawady et al. (2022), Aly (2023), and Arshad et al. (2023), dealt with Bayesian and classical estimations, recurrence relations, predictive inference, and reliability estimation using *gos*.

The manuscript is structured into eight sections. In Section 2, we established the recurrence relations for the BHE distribution based on *gos* and records. In Section 3, we derive the exact moment of BHE distribution based on *gos*. In Section 4, the MLE of BHE distribution parameters is presented. In Section 5, we derive a Bayesian estimation of the scale parameter of the BHE distribution based on *gos*. In Section 6, two algorithms for generating a random sample from the BHE distribution based on *gos* are developed. Furthermore, the performance of the BHE distribution parameter is evaluated using the Markov Chain Monte Carlo (MCMC) simulation technique. To demonstrate the flexibility and applicability of the results, two real datasets are considered in Section 7. Moreover, the manuscript is concluded in Section 8.

2. Recursive relations for the moments of the BHE distribution

The recursive relations for the single and product moments of the BHE distribution derived from *gos* are presented in this section. Utilizing these recursive relations, one can compute the higher order moments.

Theorem 2. For any $j_1 \in \mathbb{N}_0$ and $r = 1, 2, \ldots, n$, the recurrence for single moment of the BHE distribution based on gos is given by

$$\begin{aligned}&\mu^{j_1}_{r,n,\widetilde{m},k} + \theta\left(1 - \frac{2\gamma_r}{j_1+1}\right)\mu^{j_1+1}_{r,n,\widetilde{m},k} + \left(\frac{2\theta\gamma_r}{j_1+1}\right)\mu^{j_1}_{r-1,n,\widetilde{m},k}\\ &= \left(\frac{\theta^2\gamma_r}{j_1+2}\right)\left(\mu^{j_1+2}_{r,n,\widetilde{m},k} - \mu^{j_1+2}_{r-1,n,\widetilde{m},k}\right),\end{aligned}$$

and in the case of records ($m \to -1,\ k = 1$), the recursive relation for a single moment of the BHE distribution is given by

$$\begin{aligned}&\mu^{j_1}_{r,n,-1,1} + \theta\left(1 - \frac{2}{j_1+1}\right)\mu^{j_1+1}_{r,n,-1,1} + \left(\frac{2\theta}{j_1+1}\right)\mu^{j_1}_{r-1,n,-1,1}\\ &= \left(\frac{\theta^2}{j_1+2}\right)\left(\mu^{j_1+2}_{r,n,-1,1} - \mu^{j_1+2}_{r-1,n,-1,1}\right).\end{aligned}$$

Proof : In light of (2) and (12), we get

$$\begin{aligned}\mu^{j_1}_{r,n,\widetilde{m},k} + \theta\mu^{j_1+1}_{r,n,\widetilde{m},k} &= 2\theta c_{r-1}\sum_{i=1}^{r} a_i(r)\int_0^\infty x^{j_1}[\bar{F}(x)]^{\gamma_i}\,dx\\ &+ \theta^2 c_{r-1}\sum_{i=1}^{r} a_i(r)\int_0^\infty x^{j_1+1}[\bar{F}(x)]^{\gamma_i}\,dx.\end{aligned} \tag{15}$$

Simplifying the aforementioned integration, treating $[\bar{F}(x)]^{\gamma_i}$ as the part of integrand and simplifying, we get the theorem.

Similarly, in case the of records ($m \to -1,\ k = 1$), and in view of (4) and (12), we have

$$\begin{aligned}\mu^{j_1}_{r,n,-1,1} + \theta\mu^{j_1+1}_{r,n,-1,1} &= \frac{2\theta}{r!}\int_0^\infty x^{j_1}[-\ln\bar{F}(x)]^{r-1}\bar{F}(x)\,dx\\ &+ \frac{\theta^2}{r!}\int_0^\infty x^{j_1+1}[-\ln\bar{F}(x)]^{r-1}\bar{F}(x)\,dx.\end{aligned} \tag{16}$$

Simplifying the integration (16) by parts, $[-\ln\bar{F}(x)]^{r-1}\bar{F}(x)$ is considered part of the integrand. Thus, the theorem is proven.

Theorem 3. For any $j_1, j_2 \in \mathbb{N}_0$ and $1 \le r \le s+1 \le n$, the recursive relation for the product moment of the BHE distribution is given by

$$\begin{aligned}
&\mu^{j_1,j_2}_{r,s,n,\widetilde{m},k} + \theta\left(1-\frac{2\gamma_r}{j_2+1}\right)\mu^{j_1,j_2+1}_{r,s,n,\widetilde{m},k} + \left(\frac{2\theta}{j_2+1}\right)\mu^{j_1,j_2}_{r,s-1,n,\widetilde{m},k}\\
&= \left(\frac{\theta^2}{j_2+2}\right)\left[\mu^{j_1,j_2+2}_{r,s,n,\widetilde{m},k} - \mu^{j_1,j_2+2}_{r,s-1,n,\widetilde{m},k}\right],
\end{aligned}$$

and in the case of records ($m \to -1,\ k=1$), the recursive relation for the product moment of the BHE distribution is as follows:

$$\begin{aligned}
&\mu^{j_1,j_2}_{r,s,n,-1,1} + \theta\left(1-\frac{2}{j_2+1}\right)\mu^{j_1,j_2+1}_{r,s,n,-1,1} + \left(\frac{2\theta}{j_2+1}\right)\mu^{j_1,j_2+1}_{r,s-1,n,-1,1}\\
&= \left(\frac{\theta^2}{j_2+2}\right)\left[\mu^{j_1,j_2+2}_{r,s,n,-1,1} - \mu^{j_1,j_2+2}_{r,s-1,n,-1,1}\right].
\end{aligned}$$

Proof : In view of (3) and (12), we have

$$\begin{aligned}
&\mu^{j_1,j_2}_{r,s,n,\widetilde{m},k} + \theta\mu^{j_1,j_2+1}_{r,s,n,\widetilde{m},k}\\
&= 2\theta c_{s-1}\sum_{t=r+1}^{s}\sum_{i=1}^{r} a_i(r)a_t^{(r)}(s)\int_0^\infty\int_x^\infty x^{j_1}y^{j_2}\left(\frac{\bar{F}(y)}{\bar{F}(x)}\right)^{\gamma_t}(\bar{F}(x))^{\gamma_i}\,dydx\\
&+ \theta^2 c_{s-1}\sum_{t=r+1}^{s}\sum_{i=1}^{r} a_i(r)a_t^{(r)}(s)\int_0^\infty\int_x^\infty x^{j_1}y^{j_2+1}\left(\frac{\bar{F}(y)}{\bar{F}(x)}\right)^{\gamma_t}(\bar{F}(x))^{\gamma_i}\,dydx
\end{aligned} \tag{17}$$

Simplifying the above integration (17) by parts, $\left(\frac{\bar{F}(y)}{\bar{F}(x)}\right)^{\gamma_t}$ is considered as the parts of the integrand. The theorem is thus proven.

Similarly, in the case of records ($m \to -1,\ k=1$), and in view of (5) and (12), we have

$$\begin{aligned}
\mu^{j_1,j_2}_{r,s,n,-1,1} + \theta\mu^{j_1,j_2+1}_{r,s,n,-1,1} &= \frac{2\theta}{(r-1)!(s-r-1)!}\int_0^\infty\int_x^\infty x^{j_1}y^{j_2}[-\ln\bar{F}(x)]^{r-1}\\
&\quad\times[-\ln\bar{F}(y)+\ln\bar{F}(x)]^{s-r-1}\bar{F}(y)\,dydx\\
&+ \frac{\theta^2}{(r-1)!(s-r-1)!}\int_0^\infty\int_x^\infty x^{j_1}y^{j_2+1}[-\ln\bar{F}(x)]^{r-1}\\
&\quad\times[-\ln\bar{F}(y)+\ln\bar{F}(x)]^{s-r-1}\bar{F}(y)\,dydx\,.
\end{aligned} \tag{18}$$

Simplifying the integration (18) by parts, we may consider the term $[-\ln\bar{F}(y) + \ln\bar{F}(x)]^{s-r-1}$ as representing the parts of the integrand. The theorem is thus proven.

3. Exact moment of the BHE distribution using *gos*

This section presents the closed form expressions for a single moment of the BHE distribution based on *gos*, represented in terms of an exponential integral function.

Theorem 4. For $j \in \mathbb{N}_0$, and $1 \le r \le n$. The single moment of the BHE distribution based on gos is expressed as:

$$\mu^{j}_{r,n,\widetilde{m},k} = c_{r-1} \sum_{i=1}^{r} \sum_{i_1=0}^{j} (-1)^{i_1+j} a_i(r) \frac{e^{\gamma_i}}{\theta^j} \binom{j}{i_1} \left[E_{\gamma_i - i_1 + 1}(\gamma_i) + E_{\gamma_i - i_1}(\gamma_i) \right], \tag{19}$$

provided that $\gamma_i - i_1 \in \mathbb{N}_0$ and $E_n(x) = \int_1^\infty \frac{e^{-xt}}{t^n}\, dt\,, \quad (n = 0, 1, \ldots, \quad \Re(x) > 0)$ (Gradshteyn and Ryzhik, 2014, p. xxxv).

Proof : In light of (2), (10), and (11), we get

$$\mu^{j}_{r,n,\widetilde{m},k} = c_{r-1} \sum_{i=1}^{r} a_i(r) \int_0^\infty x^j \left[\frac{e^{-\theta x}}{1+\theta x} \right]^{\gamma_i - 1} \theta e^{-\theta x} \frac{(2+\theta x)}{(1+\theta x)^2}\, dx\,.$$

Substituting $1 + \theta x = t \implies \theta dx = dt$, we have

$$\begin{aligned} \mu^{j}_{r,n,\widetilde{m},k} &= c_{r-1} \sum_{i=1}^{r} a_i(r) \int_1^\infty \left(\frac{t-1}{\theta} \right)^j \left[\frac{e^{-(t-1)}}{t} \right]^{\gamma_i - 1} \frac{e^{-(t-1)}(1+t)}{t^2}\, dt \\ &= c_{r-1} \sum_{i=1}^{r} \sum_{i_1=0}^{j} (-1)^{i_1+j} a_i(r) \frac{e^{\gamma_i}}{\theta^j} \binom{j}{i_1} \left[\int_1^\infty \frac{e^{-t\gamma_i}}{t^{\gamma_i - i_1 + 1}}\, dt + \int_1^\infty \frac{e^{-t\gamma_i}}{t^{\gamma_i - i_1}}\, dt \right]. \end{aligned} \tag{20}$$

The theorem has been proved using the result of (Gradshteyn and Ryzhik, 2014, p. xxxv).

Corollary 1. In the special case of m-gos, where $m_1 = m_2 = \ldots = m \neq -1$, the single moment of the BHE distribution based on m-gos is expressed as:

$$\begin{aligned} \mu^{j}_{r,n,m,k} = \frac{c_{r-1}}{(m+1)^{r-1}(r-1)!} \sum_{i=0}^{r-1} \sum_{i_1=0}^{j} (-1)^{i+i_1+j} \frac{e^{\gamma_{r-i}}}{\theta^j} \binom{j}{i_1} \\ \times \left[E_{\gamma_{r-i} - i_1 + 1}(\gamma_{r-i}) + E_{\gamma_{r-i} - i_1}(\gamma_{r-i}) \right]. \end{aligned}$$

Remark 1. The single moment of the BHE distribution based on OS ($m = 0,\ k = 1$) is given by

$$\begin{aligned} \mu^{j}_{r,n,0,1} = \frac{c_{r-1}}{(m+1)^{r-1}(r-1)!} \sum_{i=0}^{r-1} \sum_{i_1=0}^{j} (-1)^{i+i_1+j} \frac{e^{(n-r+i+1)}}{\theta^j} \binom{j}{i_1} \\ \times \left[E_{n-r+i-i_1+2}(n-r+i+1) + E_{n-r+i-i_1+1}(n-r+i+1) \right]. \end{aligned}$$

Remark 2. Putting $m_i = R_i \in \mathbb{N}_0,\ k = R_n + 1$ *and* $\gamma_i = n - i + 1 + \sum_{i=r}^{n} R_i$ in (19), then the single moment of the BHE distribution based on progressive Type II censored OS is given by

$$\mu_{r:m:n}^{(R_1,R_2,\ldots,R_n)^{(j)}} = c_{r-1} \sum_{i=1}^{r} \sum_{i_1=0}^{j} (-1)^{i_1+j} a_i(r) \frac{e^{\gamma_i}}{\theta^j} \binom{j}{i_1} \times \left[E_{\gamma_i - i_1 + 1}(\gamma_i) + E_{\gamma_i - i_1}(\gamma_i) \right].$$

4. Maximum Likelihood Estimation

In this section, we have derived the MLE of the parameter of BHE distribution based on *gos*.

Let $X(1,n,\widetilde{m},k), X(2,n,\widetilde{m},k), \ldots, X(n,n,\widetilde{m},k)$ be the n *gos* from the BHE distribution having *pdf* given in (9). To simplify, consider the realization of these n *gos* as $\underline{\mathbf{x}} = (x_1, x_2, \ldots, x_n)$. The likelihood function of the BHE distribution is then expressed as follows:

$$L(\theta|\underline{\mathbf{x}}) = k\theta^n \left(\prod_{j=1}^{n-1} \gamma_j \right) \left[\prod_{i=1}^{n-1} \left\{ \frac{e^{-\theta x_i}}{1+\theta x_i} \right\}^{m_i+1} \right] \left[\prod_{i=1}^{n} \frac{2+\theta x_i}{(1+\theta x_i)} \right] \left[\frac{e^{-\theta x_n}}{(1+\theta x_n)} \right]^k. \tag{21}$$

Thus, the log-likelihood function of the BHE distribution derived from *gos* can be expressed as follows:

$$\begin{aligned} l(\theta|\underline{\mathbf{x}}) = \ln k + n \ln \theta + \sum_{j=1}^{n-1} \ln \gamma_j + \sum_{i=1}^{n-1} (m_i+1) \ln \left\{ \frac{e^{-\theta x_i}}{1+\theta x_i} \right\} \\ + \sum_{i=1}^{n} \ln \left\{ \frac{2+\theta x_i}{(1+\theta x_i)} \right\} + k \ln \left\{ \frac{e^{-\theta x_n}}{1+\theta x_n} \right\}. \end{aligned} \tag{22}$$

Differentiating (22) *w.r.t.* θ and setting it equal to zero, we have

$$\frac{n}{\theta} - \sum_{i=1}^{n-1} \frac{(m_i+1)(2x_i + \theta x_i^2)}{(1+\theta x_i)} - \sum_{i=1}^{n} \frac{x_i}{(2+\theta x_i)(1+\theta x_i)} - \frac{k(2x_n + \theta x_n^2)}{(1+\theta x_n)} = 0. \tag{23}$$

Further, in case of records ($m \to -1,\ k = 1$), the log-likelihood function based on first n records, $\underline{\mathbf{r}} = (r_1, \ldots, r_n)$ is given by

$$l(\theta|\underline{\mathbf{r}}) = n \ln \theta - \theta r_n - \ln(1+\theta r_n) + \sum_{i=1}^{n} \ln \left(\frac{2+\theta r_i}{1+\theta r_i} \right). \tag{24}$$

Differentiating (24) with respect to θ and setting it equal to zero, we get

$$\frac{n}{\theta}-r_n-\frac{r_n}{1+\theta r_n}-\sum_{i=1}^{n}\frac{r_i}{(1+\theta r_i)(1+2\theta r_i)}=0 \tag{25}$$

Since Equations (23) and (25) are nonlinear, a direct solution for θ cannot be achieved. Consequently, the R software (R Core Team et al., 2013) and the ***nleqslv*** (Hasselman and Hasselman, 2018) package are employed to compute the MLE of θ. The MLE's convergence is examined using Newton-Raphson technique.

4.1. Asymptotic Confidence Interval

In this subsection, we have derived the asymptotic confidence interval (ACI) for the parameter $\hat{\theta}$ of the BHE distribution based on *gos*.

The observed Fisher information for the BHE distribution is specified as follows:

$$I_0(\hat{\theta})=-\frac{\partial^2 l(\theta|\underline{\mathbf{x}})}{\partial\theta^2}|_{\theta=\hat{\theta}},$$

The second-order derivative of the log-likelihood function *w.r.t.* the parameter θ can be expressed as

$$\frac{\partial^2 l(\theta|\underline{\mathbf{x}})}{\partial\theta^2}=-\frac{n}{\theta^2}+\sum_{i=1}^{n-1}\frac{(m_i+1)x_i^2}{(1+\theta x_i)^2}+\sum_{i=1}^{n}\frac{(3x_i^2+2\theta x_i^3)}{(1+\theta x_i)^2(2+\theta x_i)^2}+\frac{kx_n^2}{(1+\theta x_n)^2}.$$

By utilizing the Fisher information, the asymptotic variance of the MLE of the parameter for the BHE distribution can be obtained as

$$Var_0(\hat{\theta})=\frac{1}{I_0(\hat{\theta})}.$$

Under certain regularity conditions, the sampling distribution of $\frac{(\hat{\theta}-\theta)}{\sqrt{Var_0(\hat{\theta})}}$ can be approximated utilizing a standard normal distribution. The large sample CI also referred to as the ACI for θ, is given by $[\hat{\theta}_L,\hat{\theta}_U]=\hat{\theta}\ \pm z_{\frac{\alpha}{2}}\sqrt{Var_0(\hat{\theta})}$.

In the case of the lower bound, $\hat{\theta}_L$ can be negative. In such cases, $\hat{\theta}_L$ is determined by $\hat{\theta}_L=max\ (0,\hat{\theta}_L)$. We can also use the simulation to obtain the average length (AL) and the coverage probability (CP).

5. Bayesian Estimation

The prior information for the parameter of the BHE distribution is taken into account. This informative prior has a two-parameter gamma distribution. Therefore, the *pdf* of the prior distribution of the parameter is given by

$$\pi(\theta)=\frac{b_1^{a_1}}{\Gamma a_1}\theta^{a_1-1}e^{-b_1\theta},\quad a_1,b_1,\theta>0. \tag{26}$$

We have used two loss functions: the squared error (SE) loss function (symmetric) and the linear exponential (LINEX) loss function (asymmetric). The LINEX loss function combines linear and exponential components to model asymmetric penalties for over- and underestimation in the statistical decision theory and risk analysis. $L_2(\delta,\Theta)=\exp(v(\delta-\Theta))-v(\delta-\Theta)-1$, where δ is the true value, Θ is the estimate, and v controls for asymmetry, which penalizes overestimation more heavily as v increases. This function is useful in scenarios with unequal consequences for over- and underestimation, such as inventory management and financial risk assessment, and has applications in Bayesian estimation, forecasting, and decision-making under uncertainty. These loss functions add flexibility to our results and provide applicability in many real-world scenarios. The symmetric loss function, or SE loss function, is selected for its ability to penalize both underestimation and overestimation equally, which is beneficial in most cases. Nevertheless, in several scenarios, a positive loss can be more serious than a negative loss and vice versa. In such cases, the use of an asymmetric loss function is recommended. The present study examined two distinct loss functions: symmetrical (SE) and asymmetrical (LINEX).

The mathematical representation of these loss functions and the associated Bayesian estimators is as follows:

$$L_1(\delta,\theta)=(\delta-\theta)^2, \quad \theta>0.$$

In the case of the SE loss function, the Bayes estimator is equal to the posterior mean represented by (δ_{SE}). The definition of the LINEX loss function is as follows:

$$L_2(\delta,\theta)=\exp(v(\delta-\theta))-v(\delta-\theta)-1, \quad v\neq 0$$

and the corresponding Bayes estimator as

$$\delta_{LINEX}=-\frac{1}{v}\ln(E(\exp(-v\theta))).$$

In view of (21) and (26), the posterior density function of θ given $\underline{\mathbf{x}}$ is given by

$$\pi(\theta|\underline{\mathbf{x}})=\frac{1}{A}P^{\star}(\theta;\underline{\mathbf{x}}),$$

where

$$P^{\star}(\theta;\underline{\mathbf{x}})=\theta^{n+a_1-1}e^{-b_1\theta}\left[\prod_{i=1}^{n-1}\left\{\frac{e^{-\theta x_i}}{1+\theta x_i}\right\}^{m_i+1}\right]\left[\prod_{i=1}^{n}\frac{2+\theta x_i}{(1+\theta x_i)}\right]\left[\frac{e^{-\theta x_n}}{(1+\theta x_n)}\right]^k.$$

In case of records $(m\to -1,\ k=1)$,

$$P^{\star}(\theta;\underline{\mathbf{r}})=\theta^{n+a_1-1}\frac{e^{-\theta(b_1+r_n)}}{(1+\theta r_n)}\prod_{i=1}^{n}\left(\frac{2+\theta r_i}{1+\theta r_i}\right)$$

and $A=\int_0^{\infty}P^{\star}(\theta;\underline{\mathbf{x}})\,d\theta$. Then, under the SE loss function, the Bayes estimate of any function of the parameter, say $g(\theta)$, becomes

$$\hat{g}_{SE} = E(g(\theta)|\underline{\mathbf{x}}) = \frac{1}{A}\int_0^\infty g(\theta)P^\star(\theta;\underline{\mathbf{x}})\,d\theta\ . \tag{27}$$

The aforesaid equation is the ratio of two integrals, and its explicit solution is not available. Therefore, we adopt the Metropolis- Hastings (M-H) algorithm to compute the approximate Bayes estimate of the parameter of the BHE distribution.

5.1. Metropolis- Hastings Algorithm

In this subsection, random samples are generated from the posterior density using the M-H algorithm. We used these generated samples to get the parameter's Bayes estimator and Bayesian interval estimates. The modified M-H algorithm is used here. For example, see Dey and Pradhan (2014). Let $N(\theta,\sigma_\theta)$ denote the normal distribution. The expression for M-H algorithm is presented as follows:

1. Set an initial value of θ_0.
2. For $j = 1,\ldots,N$, repeat the following steps:

- Set $\theta = \theta_{j-1}$.
- Generate a candidate value η from $N(\theta^\star,\sigma_\theta)$, where $\theta^\star = \ln(\theta)$ and generate u from *U(0,1)*.
- Set $\theta^{'} = \exp(\eta)$.
- Calculate $\delta = min\left\{1, \frac{\pi(\theta^{'}|\underline{\mathbf{x}})}{\pi(\theta|\underline{\mathbf{x}})}\right\}$.
- If $u \le \delta$, then set $\theta_j = \theta^{'}$ otherwise set $\theta_j = \theta_{j-1}$.

The σ_θ selection is a significant aspect of the acceptance rate. The initial B (burn-in period) generated samples may be discarded.

Let us now assume that the set of samples generated by the M-H mentioned above algorithm is $\{\theta_l, \text{ł} = 1,\ldots,L\}$, where $L = N^{'} - B$). Then, under the SE and LINEX loss functions, the estimated Bayes estimate of parameter of the BHE distribution can be determined by $\hat{\theta}_{SE} = \frac{1}{L}\sum_{l=1}^{L}\theta_l$, and $\hat{\theta}_{SE} = \frac{1}{L}\left(\sum_{l=1}^{L}\exp(-\nu\theta_l)\right)$, respectively.

6. Generation of Random Samples

In this section, we have discussed two methods of generating random samples from BHE distribution based on *gos*.

Algorithm 1: Generation algorithm using the result of Cramer (2003, p. 30). The *gos* can alternatively be defined as
$X(r,n,\widetilde{m},k) = F^{-1}\left(1-\prod_{j=1}^{r}B_j\right) = \bar{F}^{-1}\left(\prod_{j=1}^{r}B_j\right)$, where $B_1,B_2,\ldots,B_r$ are independent power function distributed *rv's* with distribution function $P(B_j \le x) = x^{\gamma_j},\ x \in (0,1)$, and $1 \le j < n$.

The result above can be utilized to develop an efficient algorithm for the generation of *gos* random samples based on the BHE distribution as follows:

1. Generate n independent and identically distributed (*iid*) *rv's* uniform random samples, $U_1, U_2, , \ldots, U_n$.

2. Transform $Y_j = U_j^{\frac{1}{\gamma_j}}$; $j = 1, 2, \ldots, n$. This implies that Y_j follows $B(\gamma_j, 1)$ or power function distribution with exponent γ_j.

   Since
   $$x = \frac{1}{\theta}\left(W_0\left(\frac{e}{1-u}\right) - 1\right),$$
   where, W_0 denotes the principal branch of lambert W function.

3. Compute $U(r, n, \widetilde{m}, k) = 1 - \prod_{i=1}^{r} Y_i$; $r = 1, 2, \ldots, n$.
   Since $X(r, n, \widetilde{m}, k) = F^{-1}\left(1 - \prod_{i=1}^{r} Y_i\right)$, using the quantile function of the BHE distribution, we get
   $$X(r, n, \widetilde{m}, k) = \frac{1}{\theta}\left(W_0\left(\frac{e}{\prod_{i=1}^{r} Y_i}\right) - 1\right).$$

Algorithm 2: Let $X^{\star}(r, n, \widetilde{m}, k)$; $r = 1, 2, \ldots, n$ be the *gos* based on the *cdf* $F(x) = 1 - e^{-x}$ i.e., $exp(1)$. Using the normalized spacing of *gos*, (Kamps, 1995, p. 81). $X^{\star}(r, n, \widetilde{m}, k) = \sum_{i=1}^{r} \frac{Y_i}{\gamma_i}$, where $Y_1, Y_2, \ldots, Y_n$ are *iid rv's* follow exp(1). Now adopting the technique of Arnold et al. (1992), which is as follows: Let $X^{\star}(r, n, \widetilde{m}, k)$; $r = 1, 2, \ldots, n$ be the *gos* based on continuous *cdf* F then $H(X) = -\ln[1 - F(X)]$ has standard exponential distribution and $H(X)$ is monotonic function of X and $X^{\star}(r, n, \widetilde{m}, k)$ be the *gos* based on exp(1) then

$$\begin{aligned} X(r, n, \widetilde{m}, k) &= F^{-1}\left(1 - e^{-X^{\star}(r,n,\widetilde{m},k)}\right) \\ &= F^{-1}\left(1 - e^{-\sum_{i=1}^{r} \frac{Y_i}{\gamma_i}}\right) \\ &= \frac{1}{\theta}\left[W_0\left(e^{1+\sum_{i=1}^{n} \frac{Y_i}{\gamma_i}}\right) - 1\right]. \end{aligned} \tag{28}$$

Putting $m_i = 0$ and $k = 1$ i.e., $\gamma_i = n - i + 1$ in the aforesaid algorithm, we shall get a random sample of n OS based on the BHE distribution. Similarly, if we put $m \to -1$, $k \in \mathbb{N}$ i.e., $\gamma_i = k$ in the above-discussed algorithm, we shall generate the first n k^{th} records of BHE distribution. Put $m_i = R_i \in \mathbb{N}_0$, $k = R_n + 1$ *and* $\gamma_i = n - i + 1 + \sum_{i=r}^{n} R_i$ in the above algorithm described above, we shall get a random sample for progressive Type II censored OS. Similarly, if we put $k = \beta_n$, $m_i = (n - i + 1)\alpha_i - (n - i)\alpha_{i+1} - 1$ *and* $\gamma_i = (n - i + 1)\beta_i; \beta_1, \ldots, \beta_n > 0$ in the above-discussed algorithm, we shall get a random sample of sequential OS.

6.1. Bootstrap Method

In this subsection we have discussed the percentile bootstrap method, denoted boot-p, which is used to construct an approximate confidence interval (CI) for θ using the following algorithm.

Step 1: First generate *n-gos* from the BHE distribution, then calculate the MLE of the parameter of the BHE distribution, say $\hat{\theta}$.

Step 2: Utilizing $\hat{\theta}$ derived from Step 1, generate a random sample of *n-gos* drawn from the BHE distribution, referred to as a bootstrap sample.

Step 3: Based on the bootstrap sample that is obtained in Step 2, compute the corresponding MLE $\hat{\theta}^{\star}$ of θ.

Step 4: Repeat Steps (2) and (3) B-times in order to obtain $\{\theta_1^{\star}, \theta_2^{\star}, \dots, \theta_B^{\star}\}$.

Step 5 Arrange $\{\theta_1^{\star}, \theta_2^{\star}, \dots, \theta_B^{\star}\}$ in ascending order and obtain $\{\theta_{(1)}^{\star}, \theta_{(2)}^{\star}, \dots, \theta_{(B)}^{\star}\}$.

Step 6 The approximate $100(1-\alpha)\%$ boot-p CI for θ is defined as:
$\left(\theta^{\star}_{(B\frac{\alpha}{2})}, \theta^{\star}_{(B(1-\frac{\alpha}{2}))}\right)$.

6.2. Simulation Study Based on MLEs

The performance of the estimator can be measured by the mean squared error (MSE) of the MLEs of the scale parameter θ of the BHE distribution using OS (records). Here are the steps to calculate the MSEs:

1. Here, first generate n OS (records) from the BHE distribution.

2. The MLEs of the BHE distribution's parameters are calculated using the OS (records) generated in Step 1.

3. Total 6000 iterations, are used to calculate the MLEs of the distribution parameters. Each iteration of the BHE distribution yields a new set of OS (records) from which the MLEs are calculated.

The MSE values are calculated considering numerous n values for OS (records) configurations. Furthermore, the BHE distribution's parameter value is specified for fixed value of θ. The MSEs can be seen in Figures 3-4 and shown in the graph. The most significant conclusion from Figures 3-4 is that the MSE values decrease as the sample size (n) increases. This pattern signifies that higher OS (records) improve the accuracy and precision of the estimators. The MSE decreases with increase in the sample size, suggesting that the ML estimators are getting closer to the true values of the parameter θ.

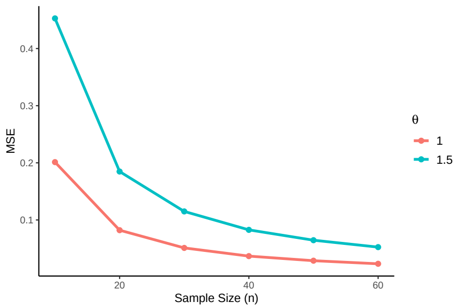

Figure 3. The MSE plot for $\theta = 1$ and $\theta = 1.5$ based on OS

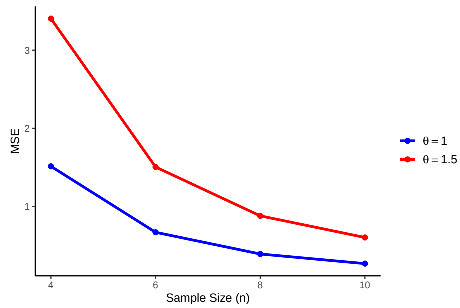

Figure 4. The MSE plot for $\theta = 1$ and $\theta = 1.5$ based on record values

6.3. Simulation Study Based on MCMC

The following subsection presents the results of a simulation study that demonstrates the performance of the estimators obtained in this manuscript. To achieve this, the method proposed by Wang and Shi (2013) is used to generate several sample sizes of upper record values from the BHE distribution. To evaluate the performance of the estimator, the MSE of the MLE of the parameter of the BHE distribution is assessed in the context of classical estimation. We conducted a Monte Carlo study for 6000 replications in the context of Bayesian analysis and present the results in Table 5. We employ the following algorithm for the calculation, as illustrated in the accompanying tables.

1. First we generate samples of upper record values from the BHE distribution discussed in section 6 (algorithm 1-2).

2. Evaluate the Bayes estimators of the unknown parameter of the BHE distribution under a symmetric loss function, i.e., the SE loss function, and a symmetric loss function, i.e., the LINEX loss function, using the generated samples and the method discussed in Section 6.

3. Repeat the above steps $N = 6000$ times and $K = 1000$ times (burn-in samples) to obtain the Bayes estimators of the parameter of the BHE distribution.

4. Now use the following expression to calculate the MSE
   $MSE(\hat{\theta}) = \frac{1}{N-K} \sum_{i=1}^{N-K} (\hat{\theta}_i - \theta)^2$, where θ is the true value of the parameter.

The simulation studies are carried out using ***R-software*** with 6000 iterations. The proposed MLE and Bayesian estimates of the MSE are calculated. The results of the point estimation are summarized in Table 3 with true parameter values set at $\theta = 1$ and $\theta = 1.5$, respectively, for sample sizes $n = 4, 6$, and 10. Additionally, the performances of the proposed classical CIs and Bayesian CrIs were assessed based on the AL and CP. The ALs and CPs for the 90%, 95%, and 99% ACIs, boot-p CIs, and CrIs for θ are presented in Table 4, for sample sizes $n = 4, 6, 8$, and 10.

The Bayes estimates perform better than the MLE in terms of MSE, as shown in Table 3. Furthermore, it can be observed that the informative Bayes estimates under both the SE loss and LINEX loss functions perform better than the MLE in terms of MSEs. Clearly, the MSEs of the proposed estimates decrease as n increases.

Regarding interval estimation, Table 4 present the ALs and CPs of the ACIs, boot-p CIs, and Bayesian CrIs for $\theta = 1.5$. Evidence suggests that informative Bayesian CrIs outperform both ACI and boot-p CIs in terms of the AL and CP. Notably, the CPs of Bayesian CrIs typically align well with their nominal levels, thus offering a dependable and robust inference method. The data generally indicate that informative CrIs are the most valid approach, as they yield the highest simulated CPs compared with CIs derived from classical methods.

Table 1. Means and variances of *m-gos* and OS from the BHE distribution ($\theta = 3$, $k = 2$, $m = 1$)

		m-gos									
n		$r=1$	$r=2$	$r=3$	$r=4$	$r=5$	$r=6$	$r=7$	$r=8$	$r=9$	$r=10$
1	Mean	0.0924									
	Variance	0.0102									
2	Mean	0.0441	0.1408								
	Variance	0.0022	0.0134								
3	Mean	0.0289	0.0745	0.1739							
	Variance	0.0009	0.0032	0.0152							
4	Mean	0.0215	0.0512	0.0979	0.1992						
	Variance	0.0004	0.0015	0.0040	0.0163						
5	Mean	0.0171	0.0390	0.0693	0.1170	0.2198					
	Variance	0.0003	0.0009	0.0019	0.0045	0.0172					
6	Mean	0.0142	0.0316	0.0539	0.0848	0.1331	0.2371				
	Variance	0.0002	0.0005	0.0011	0.0022	0.0049	0.0179				
7	Mean	0.0121	0.0265	0.0442	0.0669	0.0981	0.1471	0.2522			
	Variance	0.0002	0.0004	0.0007	0.0013	0.0024	0.0053	0.0183			
8	Mean	0.0106	0.0229	0.0375	0.0554	0.0784	0.1100	0.1595	0.2654		
	Variance	0.0001	0.0003	0.0005	0.0008	0.0015	0.0026	0.0055	0.0188		
9	Mean	0.0094	0.0201	0.0326	0.0473	0.0655	0.0887	0.1206	0.1706	0.2773	
	Variance	0.0001	0.0002	0.0003	0.0007	0.0010	0.0016	0.0028	0.0057	0.0191	
10	Mean	0.0084	0.0179	0.0288	0.0414	0.0563	0.0746	0.0981	0.1303	0.1806	0.2880
	Variance	0.0000	0.0002	0.0003	0.0005	0.0007	0.0011	0.0017	0.0029	0.0059	0.0195

		OS									
n		$r=1$	$r=2$	$r=3$	$r=4$	$r=5$	$r=6$	$r=7$	$r=8$	$r=9$	$r=10$
1	Mean	0.1988									
	Variance	0.0502									
2	Mean	0.0924	0.3051								
	Variance	0.0102	0.0676								
3	Mean	0.0598	0.1578	0.3788							
	Variance	0.0040	0.0158	0.0772							
4	Mean	0.0441	0.1069	0.2087	0.4355						
	Variance	0.0022	0.0069	0.0196	0.0836						
5	Mean	0.0349	0.0809	0.1458	0.2505	0.4818					
	Variance	0.0013	0.0038	0.0089	0.0224	0.0882					
6	Mean	0.0289	0.0651	0.1125	0.1792	0.2862	0.5209				
	Variance	0.0009	0.0024	0.0050	0.0105	0.0246	0.0918				
7	Mean	0.0246	0.0544	0.0916	0.1402	0.2084	0.3173	0.5548			
	Variance	0.0007	0.0016	0.0033	0.0061	0.0118	0.0263	0.0946			
8	Mean	0.0215	0.0468	0.0774	0.1154	0.1651	0.2345	0.3449	0.5848		
	Variance	0.0004	0.0012	0.0023	0.0040	0.0069	0.0128	0.0277	0.0969		
9	Mean	0.0190	0.0410	0.0669	0.0982	0.1370	0.1875	0.2580	0.3697	0.6117	
	Variance	0.0003	0.0009	0.0017	0.0029	0.0046	0.0077	0.0137	0.0289	0.0989	
10	Mean	0.0171	0.0365	0.0590	0.0855	0.1173	0.1568	0.2080	0.2794	0.3923	0.6360
	Variance	0.0003	0.0008	0.0013	0.0021	0.0032	0.0051	0.0083	0.0145	0.0300	0.1007

Table 2. Means and variances of the BHE distribution based on progressive Type II censored OS ($\theta = 3$)

N	n	$(R_1,\ldots,R_n)$	$r=1$		$r=2$		$r=3$		$r=4$		$r=5$		$r=6$	
			$E(X)$	$Var(X)$	$E(X)$	$Var(X)$	$E(X)$	$Var(X)$	$E(X)$	$Var(X)$	$E(X)$	$Var(X)$	$E(X)$	$Var(X)$
8	3	(0,1,4)	0.6452	0.0005	0.6199	0.0012	0.5830	0.0027	-	-	-	-	-	-
		(3,0,2)	0.6452	0.0005	0.5999	0.0028	0.5357	0.0077	-	-	-	-	-	-
		(4,0,1)	0.6452	0.0005	0.5839	0.0048	0.4839	0.0170	-	-	-	-	-	-
		(2,0,3)	0.6452	0.0005	0.6093	0.0020	0.5623	0.0045	-	-	-	-	-	-
		(2,1,2)	0.6452	0.0005	0.6093	0.0020	0.5457	0.0067	-	-	-	-	-	-
		(1,2,2)	0.6452	0.0005	0.6155	0.0015	0.5522	0.0062	-	-	-	-	-	-
12	4	(3,1,1,3)	0.6525	0.0002	0.6306	0.0007	0.6004	0.0017	0.5529	0.0044	-	-	-	-
		(4,2,1,1)	0.6525	0.0002	0.6274	0.0009	0.5812	0.0033	0.4809	0.0153	-	-	-	-
		(1,2,1,4)	0.6525	0.0002	0.6351	0.0005	0.6095	0.0012	0.5721	0.0029	-	-	-	-
		(5,3,0,0)	0.6525	0.0002	0.6231	0.0011	0.5263	0.0122	0.3071	0.0723	-	-	-	-
		(2,3,1,2)	0.6525	0.0002	0.6331	0.0006	0.5967	0.0021	0.5323	0.0070	-	-	-	-
		(1,2,2,3)	0.6525	0.0002	0.6351	0.0005	0.6095	0.0012	0.5624	0.0037	-	-	-	-
16	6	(0,1,7,0,1,1)	0.6554	0.0001	0.6446	0.0003	0.6312	0.0004	0.5947	0.0019	0.5469	0.0046	0.4439	0.0173
		(0,2,0,1,4,3)	0.6554	0.0001	0.6446	0.0003	0.6300	0.0005	0.6138	0.0007	0.5935	0.0012	0.5456	0.0038
		(3,2,1,3,0,1)	0.6554	0.0001	0.6417	0.0003	0.6221	0.0007	0.5960	0.0016	0.5315	0.0063	0.4274	0.0194
		(1,2,0,5,1,1)	0.6554	0.0001	0.6438	0.0002	0.6278	0.0006	0.6099	0.0009	0.5628	0.0034	0.4609	0.0158
		(1,3,4,0,1,1)	0.6554	0.0001	0.6438	0.0002	0.6262	0.0006	0.5895	0.0021	0.5415	0.0048	0.4381	0.0176
		(0,5,1,2,2,0)	0.6554	0.0001	0.6446	0.0003	0.6250	0.0007	0.5991	0.0014	0.5515	0.0040	0.3347	0.0615
		(1,3,4,0,2,0)	0.6554	0.0001	0.6438	0.0002	0.6262	0.0006	0.5895	0.0021	0.5415	0.0048	0.3234	0.0629

In this aspect, the means and variances for a sample size of $n = 1:1:10$ are figured out in Table 1 for both OSs and *m-gos*, respectively. For progressive Type II censored OS and different choices of $(R_1, R_2, \ldots, R_n)$, a pre-specified fixed number of failures (n) is taken to be $3, 4,$ *and* 6. For $r = 1:1:6$ and $N = 8:4:16$, we obtained the means of the progressive Type II censored OS as in Table 2.

For fixed sample size n, increasing r increases the mean and the variance. For any specific value of r, the increase in n causes a decrease in the variance; this is true universally. Similarly, for the progressive censored OS of Type II, the increase in r increases the mean for fixed N and n. However, there is no definite pattern for a fixed r and varying N and n. The means depend on the selection of $(R_1, R_2, \ldots, R_n)$. Similarly, we may interpret in case of the product moments of *m-gos*, and OS.

Table 3. MSEs of MLE and Bayes estimators of θ for different θ values (rep = 6000, burn-in = 1000, $a_1 = 1$, $b_1 = 1.5$)

n	$\theta = 1$ (MCMC)					$\theta = 1.5$ (MCMC)				
	MSE(MLE)	SE loss	LINEX loss			MSE(MLE)	SE loss	LINEX loss		
			$v = -0.5$	$v = 1$	$v = 1.5$			$v = -0.5$	$v = 1$	$v = 1.5$
4	1.5128	0.3006	0.2198	0.2656	0.2792	3.4034	0.7605	0.5830	0.7265	0.7667
6	0.6685	0.1960	0.1064	0.1455	0.1578	1.5038	0.5106	0.3194	0.4473	0.4846
8	0.3906	0.1306	0.0301	0.0573	0.0666	0.8788	0.3165	0.1233	0.2179	0.2484
10	0.2677	0.0964	0.0129	0.0292	0.0354	0.6022	0.2513	0.0567	0.1305	0.1564

Table 4. Summary of the ALs and CPs of 90%, 95%, and 99% ACIs, boot-p CIs, and Bayesian CrIs of $\theta = 1.5$ ($a_1 = 1$, $b_1 = 1.5$)

n	**90%**						**95%**						**99%**					
	ACI		boot-p		CrI		ACI		boot-p		CrI		ACI		boot-p		CrI	
	AL	CP	AL	CP	AL	CP	AL	CP	AL	CP	AL	CP	AL	CP	AL	CP	AL	CP
4	4.2268	0.93	6.1820	0.85	1.9780	0.91	5.0365	0.96	8.7556	0.89	2.3832	0.96	6.6191	0.98	14.2162	0.96	3.1999	0.99
6	2.9870	0.92	3.9340	0.87	1.8223	0.91	3.5593	0.95	5.2419	0.91	2.1868	0.96	4.6776	0.99	8.5819	0.95	2.9102	0.99
8	2.4275	0.91	3.0280	0.86	1.6832	0.91	2.8926	0.96	3.8772	0.90	2.0153	0.96	3.8015	0.99	5.8542	0.97	2.6703	0.99
10	2.0623	0.91	2.5026	0.88	1.5573	0.90	2.4573	0.96	3.0749	0.93	1.8587	0.96	3.2295	0.99	4.5861	0.97	2.4445	0.99

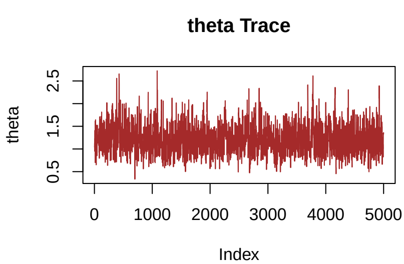

Figure 5. Trace plot for θ

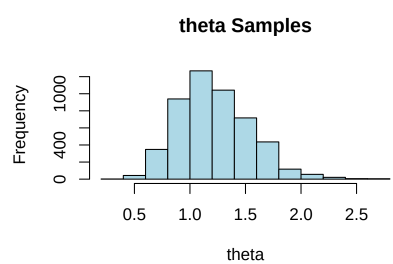

Figure 6. Histogram for θ

7. Application to the Real Data

In this section, we analyze two real data sets to illustrate the application of our model. The first data set was originally reported by Caramanis et al. (1983) and later analyzed by Altun and Hamedani (2018) and Alqasem et al. (2014), while the second data set was studied by Kabdwal et al. (2024). The first data set consists of 23 sample values measuring unit capacity factors using the "SC16" algorithm. The second data set comprises failure times of 23 secondary reactor pumps. The data sets are as follows:

Data set I: 0.853, 0.759, 0.866, 0.809, 0.717, 0.544, 0.492, 0.403, 0.344, 0.213,0.116, 0.116, 0.092, 0.070, 0.059, 0.048, 0.036, 0.029, 0.021, 0.014, 0.011, 0.008, 0.006.

Data set II: 2.160, 0.150, 4.082, 0.746, 0.358, 0.199, 0.402, 0.101, 0.605, 0.954, 1.359, 0.273, 0.491, 3.465, 0.070, 6.560, 1.060, 0.062, 4.992, 0.614, 5.320, 0.347, 1.921.

The BHE distribution is fitted to both data sets. For data set I, the Kolmogorov-Smirnov (K-S) distance and associated p-value at a fixed value of $\theta = 2.5$ were 0.26087 and 0.4218, respectively. For data set II, the K-S distance and p-value at a fixed value of $\theta = 1$ were 0.17391 and 0.888, respectively. These results suggest that the BHE distribution provides an appropriate fit for both data sets. Furthermore, three upper record values were identified in data set-2: 2.160, 4.082, and 6.560.

Using these records, the MLEs, 95% ACIs, Bayes estimates, and their corresponding 95% Bayesian CrIs were calculated for parameter θ. Table 5 show the sensitivity of the Bayesian estimate, providing a summary of the point and interval estimates derived from both classical and Bayesian methodologies, respectively.

Table 5. Estimates and 95% Confidence/Credible Intervals of θ based on the real datasets

Dataset	MLE	Bayes Estimates (MCMC)				95% Confidence Intervals for θ	
		SE	LINEX			ACI	CrI
			$v = -0.5$	$v = 1$	$v = 1.5$		
Data set I	1.3056	0.9588	1.0577	0.8217	0.7710	(-0.4981, 3.1092)	(0.1793, 2.4419)
Data set II	0.2843	0.3570	0.3666	0.3393	0.3312	(-0.0964, 0.6651)	(0.0844, 0.8162)

8. Conclusions

The manuscript presents the recursive relations for the single and product moments of the BHE distribution based on *gos* and records have been derived. Further, the closed form expression for the single moment of the BHE distribution based on *gos* is derived. Moreover, the specific case of *gos*, including OS and progressive Type II censored OS, is addressed to derive the explicit expression for the single moment of the BHE distribution. Based on *gos* and their submodels like OS, and progressive Type II censored OS, the BHE distribution's mean and variance are computed. The MLE of the BHE distribution's parameter are derived based on *gos*. The Bayes estimator of the parameter of the BHE distribution is derived using the Metropolis-Hastings algorithm. Two methods are discussed for the generation of the random sample from the BHE distribution based on *gos*. The simulation study demonstrates the validity of these findings. Finally, the results are verified using two real data sets.

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