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Inferential aspects of a M/M/1 queuing system using bivariate prior

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Abstract

This paper presents a Bayesian framework for estimating the performance measures of a single-server Markovian queue. A bivariate prior is employed to naturally incorporate the system's ergodicity condition along with the traditional likelihood function, which fundamentally differentiates this work from that of the existing literature. The Bayesian estimators of the performance measures are presented in the study, and we investigate standard errors and credible intervals of the Bayesian estimates. In addition, the predictive distributions for waiting times and the number of customers in the system at departure epoch were derived, offering deeper insights into the system dynamics. We also conducted a simulation study to assess the efficacy and reliability of the proposed methodology. Finally, a real-world scenario was presented to demonstrate the practical applicability of the developed algorithms.

Key words: Bayesian inference, bivariate prior, credible region, Markovian queuing system, performance measures, single server queuing model, standard error, traffic intensity.

Mathematics Subject Classification: 90B22, 60K25.

1. Introduction and motivation

Researchers developing probabilistic models for a wide range of queuing scenarios and deriving precise formulae to describe system performance in classical settings remark on the evolution of queueing theory over the years. However, a real-world application of theoretical models requires proper parameter estimation and data alignment to account for system complexity. These issues highlight fundamental concerns about whether statistical estimators can maintain their reliability and effectiveness in the face of real-world uncertainty and variability.

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The area of statistical inference for queueing systems features two main methodological frameworks: the frequentist approach and the Bayesian approach. Early research predominantly employed frequentist methods. Pioneering works by Clarke (1957) and Lilliefors (1966) introduced estimation procedures for the M/M/1 model. Subsequent contributions by Scruben and Kulkarni (1982) and Basawa and Prabhu (1988) revealed notable challenges, including estimator instability and the lack of standard errors for certain performance measures. To address these limitations, researchers such as Zheng and Seila (2000) and Dutta and Choudhury (2020) presented bounding strategies. Choudhury and Basak (2018a) derived the Maximum Likelihood Estimates (MLEs) of traffic intensity (ρ) for an M/M/1 queue. Subsequently, Choudhury and Basak (2018b) obtained MLEs of the average number of customers in the system (L) and the average number of customers in the queue (L_q). Despite these methodological advances, the frequentists often face difficulties in incorporating prior knowledge and accounting for model uncertainty, leading to a growing interest in Bayesian alternatives.

Bayesian inference provides more flexibility by treating parameters as random variables and integrating prior information. McGrath *et al.* (1987a, 1987b) were among the first to apply Bayesian techniques to queueing models, highlighting their advantages over their classical counterparts. Building on this foundation, Armero and Bayarri (1994a, 1994b) introduced informative priors, such as the Gauss-Hypergeometric prior, to overcome issues like the non-existence of moments and to enhance predictive accuracy. Sharma and Kumar (1999) discussed the issue of statistical inference both from a frequentist and a Bayesian perspective. Their work consisted of constructing the UMP critical region for testing the hypothesis on performance measures using a randomized testing procedure. Choudhury and Borthakur (2008) further demonstrated the relative robustness of Bayesian estimators compared to frequentist counterparts, while Choudhury and Mukherjee (2013) evaluated different priors and loss functions to refine the estimation process. Jose and Manoharan (2014) incorporated the ergodicity condition (arrival rate is less than service rate, that is, $\lambda < \mu$) through a bivariate prior, improving model fidelity. Cruz *et al.* (2017) used Sampling/Importance Resampling (SIR) to estimate ρ in M/M/s and M/M/s/K queues. Almeida and Cruz (2017) worked with Jeffreys prior in an M/M/1 queue. Later, Jose and Manoharan (2018) proposed an alternate bivariate prior to further strengthen estimation stability. Suyama *et al.* (2018) suggested estimators for λ, μ, and ρ in M/M/s queues using samples of system size and sojourn times at arbitrary points. Cruz *et al.* (2018) investigated ρ in M/M/1/K queues using maximum likelihood, Bayesian, and bootstrap techniques. Basak and Choudhury (2021) developed Bayesian estimators based on observed queue lengths. Bura and Sharma (2024) addressed queueing models with customer balking. Basak and Choudhury (2024) addressed estimation in an M/M/1/K system under known traffic intensity. Das *et al.* (2025) investigated a novel M/M/1 queueing model with balking, and consequently estimated traffic intensity and performance measures using both classical and Bayesian methods. Recently, Kushvaha *et al.* (2025) refined Bayesian inference under a variety of prior structures and loss functions, emphasizing the benefits of posterior risk analysis. Basak and Choudhury (2025) estimated λ and μ in an M/M/1 queue using McKay's bivariate gamma prior and compared them with the corresponding MLEs.

Motivated by these advancements, the present article adopts a Bayesian approach for parameter estimation in the M/M/1 queueing model, employing the notion of stochasticity in parameters. In an M/M/1 queue, time-variability refers to the evolution of parameters, such as the arrival rate or service rate, over time, and this evolution may be either deterministic or stochastic. On the other hand, stochasticity arises within Bayesian inference, where parameters are modelled as random variables. This is not because they are inherently random in reality. It is done to allow the incorporation of prior information and uncertainty about their true but fixed values. Consequently, Bayesian inference remains valid for both fixed and time-varying parameters. However, the current work adopts a stochastic treatment through the use of a bivariate prior, which serves to capture uncertainty as well as potential dependence between the arrival and service rates, without presuming that the parameters themselves are physically random processes.

The novelty and main contributions of this article are the use of a bivariate prior along with a simple and analytically convenient likelihood function, which assumes exponential distributions for both interarrival and service times. This bivariate prior explicitly incorporates the natural ergodicity condition ($\lambda < \mu$). It is worth noting that the use of such priors has been relatively underexplored in the existing literature. This is the principal research gap that the current work seeks to address. Also, explicit expressions for the predictive distributions of waiting times have not yet been thoroughly explored. Owing to the need in the existing literature, we address this limitation by formulating such predictive distributions within the M/M/1 framework. This deepens the theoretical understanding of the system dynamics and extends the scope of Bayesian analysis in queuing models.

The paper has been structured as follows. Section 2 includes a brief description of the model. Section 3 formulates Bayesian estimators for the performance measures and derives the predictive distributions for the departure epoch and waiting times. Section 4 presents a simulation study to exemplify the proposed methodology. Section 5 contains an illustrative numerical example. Section 6 outlines the concluding remarks.

2. Model description

In this study, we consider an M/M/1 queuing model, where customer arrivals follow a Poisson process with mean $\frac{1}{\lambda}$, and the server provides exponential service with mean $\frac{1}{\mu}$. The system operates on a First-Come First-Served (FCFS) basis, and the queue is assumed to be in equilibrium. This implies that the traffic intensity ($\rho = \frac{\lambda}{\mu}$) must be less than one ($\rho < 1$), ensuring system stability. If this condition is not initially met, operational adjustments must be made to avoid an unstable system where the queue size would grow infinitely.

For an M/M/1 system, the number of customers in the system at departure epoch (M) is given by (Medhi, 2003):

$$P(M = m) = \begin{cases} \left(1 - \frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^m, m = 0,1,2,\dots \\ 0, otherwise \end{cases}. \tag{1}$$

The key performance measures for the $M/M/1$ queue, in terms of λ and μ, are:

- Average number of customers in the system, $L = \frac{\lambda}{\mu-\lambda}$;
- Average number of customers in the queue, $L_q = \frac{\lambda^2}{\mu(\mu-\lambda)}$;
- Average waiting time of a customer in the system is $W = \frac{1}{(\mu-\lambda)}$;
- Average waiting time of a customer in the queue is $W_q = \frac{\lambda}{\mu(\mu-\lambda)}$;
- The probability that the number of customers in the system is greater than or equal to s, that is, $P(N \geq s) = \left(\frac{\lambda}{\mu}\right)^s$.

The distribution of waiting time in the system, W, (Gross and Harris, 2008), is given by

$$f(t) = (\mu - \lambda)e^{-(\mu-\lambda)t}, t > 0. \tag{2}$$

The distribution of waiting time in the queue, W_q, (Gross and Harris, 2008), is

$$f(t) = \{1 - \left(\frac{\lambda}{\mu}\right),\ t = 0\ \frac{\lambda}{\mu}(\mu - \lambda)e^{-\lambda\mu(\mu-\lambda)t}, t > 0. \tag{3}$$

These expressions form the basis for evaluating system performances, and they will be used in the subsequent sections to derive Bayes estimators for key performance measures and the predictive distributions.

3. Bayesian inference and predictive distributions

This section applies Bayesian inferential technique to estimate the performance measures of an M/M/1 queueing system. We proceed by adopting a joint prior of λ and μ (Jose and Manoharan, 2014, p. 72) defined as

$$\pi(\lambda, \mu) = \alpha\beta e^{-\{(\alpha-\beta)\lambda+\beta\mu\}} \tag{4}$$

where $0 < \lambda < \mu < \infty$ and $\alpha, \beta > 0$.

Unlike independent priors commonly used in the literature, this bivariate form inherently satisfies the equilibrium constraint, making it more appropriate for modelling queueing systems and ensuring valid posterior inference.

The likelihood function is constructed from the observed data of the M/M/1 queueing system. Let $x_1, x_2, \dots, x_{n_s}$ be a random sample of service times of size n_s and $y_1, y_2, \dots, y_{n_a}$ be a random sample of interarrival times of size n_a. Since interarrival and service times follow exponential distributions with rate parameters λ and μ, respectively, the likelihood function for (λ, μ) is given by:

$$L(\lambda, \mu) = \lambda^{n_a} e^{-\lambda t_a} \mu^{n_s} e^{-\mu t_s} \tag{5}$$

where $t_a = \sum y_j$ and $t_s = \sum x_i$ are the total observed interarrival and service times, respectively. This form is computationally convenient and has been utilized in similar Bayesian analyses to ensure mathematical feasibility (e.g. Armero and Bayarri, 1994b).

Combining the prior and the likelihood, the posterior joint distribution of λ and μ is then given by

$$\Pi(\lambda, \mu | data) = \frac{\lambda^{n_a} e^{-(\alpha-\beta+t_a)\lambda} \mu^{n_s} e^{-(\beta+t_s)\mu}}{\left[\left(\frac{n_s!}{(\beta+t_s)^{n_s+1}}\right)\left(\frac{n_a!}{(\alpha+t_a+t_s)^{n_a+1}}\right)\right]\left[\sum_{i=0}^{n_s}\binom{n_a+i}{i}\left(\frac{\beta+t_s}{\alpha+t_a+t_s}\right)^i\right]}. \tag{6}$$

This posterior distribution (6) will serve as the basis for deriving Bayes estimators of the performance measures considered in this study.

Before proceeding with Bayesian estimation, we perform a sensitivity analysis with respect to the hyperparameters α and β of the bivariate prior in (4) and evaluate it against independent Gamma priors for $\lambda = 2$ and $\mu = 3$. Table 1 presents posterior means and mean squared errors (MSEs) from one hundred simulated samples. The posterior means of both λ and μ consistently approximate their respective true values across all prior specifications. Furthermore, the MSEs remain stable, showing minimal sensitivity to the choice of the prior. Together, these findings support that the inferential conclusions are robust and not overly sensitive to reasonable changes in prior specification, hence validating the proposed Bayesian framework.

Table 1. Sensitivity analysis results with posterior summaries under different prior parameters

Prior	Average posterior mean of (λ)	MSE (λ)	Average posterior mean of (μ)	MSE (μ)
Bivar ($\alpha = 1, \beta = 0.1$)	2.029	0.0397	3.033	0.0753
Bivar ($\alpha = 1, \beta = 1$)	2.064	0.0457	2.956	0.0672
Bivar ($\alpha = 5, \beta = 1$)	2.063	0.0458	2.953	0.0661
Gamma (1,1)	1.982	0.0366	2.952	0.0685
Gamma (2,1)	2.005	0.0367	2.981	0.0690

3.1. Derivation of Bayes estimators

For Bayesian estimation, we employ the Squared Error Loss Function (SELF). Under SELF, the Bayes estimator for a parameter is its posterior mean. We thus formulate a unified expression for the estimators as follows:

$$E\left\{\frac{\lambda^u}{\mu^v(\mu-\lambda)^w}\right\} = \int_0^\infty \int_\lambda^\infty \frac{\lambda^u}{\mu^v(\mu-\lambda)^w} \Pi\,(\lambda,\mu|data)d\mu d\lambda = \int_0^\infty \int_\lambda^\infty \frac{\lambda^u}{\mu^{v+w}\left(1-\frac{\lambda}{\mu}\right)^w} \Pi\,(\lambda,\mu|data)d\mu d\lambda \tag{7}$$

where u, v, and w are non-negative integers.

Applying the negative binomial series expansion and performing algebraic simplification results in the simplified expression for the estimators.

$$E\left\{\frac{\lambda^u}{\mu^v(\mu-\lambda)^w}\right\} = \frac{\sum_{k=0}^{\infty}\left[\binom{w+k-1}{k}\int_0^\infty \lambda^{n_a+u+k}e^{-(\alpha-\beta+t_a)\lambda}\left\{\int_\lambda^\infty \mu^{n_s-v-w-k}e^{-(\beta+t_s)\mu}d\mu\right\}d\lambda\right]}{\left[\left(\frac{n_s!}{(\beta+t_s)^{n_s+1}}\right)\left(\frac{n_a!}{(\alpha+t_a+t_s)^{n_a+1}}\right)\right]\left[\sum_{i=0}^{n_s}\binom{n_a+i}{i}\left(\frac{\beta+t_s}{\alpha+t_a+t_s}\right)^i\right]}. \tag{8}$$

Assuming n_s and, consequently, $n_s - v - w$ as sufficiently large, the inner integral $\int_\lambda^\infty \mu^{n_s-v-w-k}e^{-(\beta+t_s)\mu}d\mu$ can be approximated using the incomplete gamma function, provided $k < n_s - v - w - 1$. This leads to the following generalized formula:

$$E\left\{\frac{\lambda^u}{\mu^v(\mu-\lambda)^w}\right\} \approx \frac{\sum_{k=0}^{n_s-v-w-1}\left[\binom{w+k-1}{k}\left\{\frac{\binom{n_a+u+k}{u+k}}{\binom{n_s}{v+w+k}}\frac{(u+k)!}{(v+w+k)!}\right\}\left\{\sum_{i=0}^{n_s-v-w-k}\binom{n_a+u+k+i}{i}\left(\frac{\beta+t_s}{\alpha+t_a+t_s}\right)^{i+k}\right\}\right]}{\left[\sum_{i=0}^{n_s}\binom{n_a+i}{i}\left(\frac{\beta+t_s}{\alpha+t_a+t_s}\right)^i\right]}. \tag{9}$$

Assigning particular values to u, v, and w in (9) yields explicit Bayes estimators for key queueing characteristics, as demonstrated below.

Bayes Estimator of the Traffic intensity (ρ) *and its standard error*

The Bayes estimator ρ^* of ρ is obtained by putting $u = 1, v = 0, w = 0$ in (9).

$$\rho^* = \frac{(n_a+1)}{n_s}\frac{\sum_0^{n_s-1}\binom{n_a+i+1}{i}\left(\frac{\beta+t_s}{\alpha+t_a+t_s}\right)^{i+1}}{\sum_0^{n_s}\binom{n_a+i}{i}\left(\frac{\beta+t_s}{\alpha+t_a+t_s}\right)^i}. \tag{10}$$

The corresponding standard error is obtained by substituting $u = 2, v = 0, w = 0$ in (9).

$$S.E.(\rho^*) = \sqrt{\left[\frac{(n_a+2)(n_a+1)}{n_s(n_s-1)}\frac{\sum_0^{n_s-2}\binom{n_a+i+2}{i}\left(\frac{\beta+t_s}{\alpha+t_a+t_s}\right)^{i+2}}{\sum_0^{n_s}\binom{n_a+i}{i}\left(\frac{\beta+t_s}{\alpha+t_a+t_s}\right)^i}\right] - (\rho^*)^2}\,. \tag{11}$$

Bayes Estimator of the average number of customers in the system (L) and its standard error.

Putting $u = 1, v = 0, w = 1$ in (9), the Bayes estimator $L^*$of L is

$$L^* = \frac{\sum_{k=0}^{n_s-2} \frac{\binom{n_a+k+1}{k+1}}{\binom{n_s}{k+1}} \sum_{i=0}^{n_s-k-1} \binom{n_a+i+k+1}{i} \left(\frac{\beta+t_s}{\alpha+t_a+t_s}\right)^{i+k+1}}{\sum_0^{n_s} \binom{n_a+i}{i} \left(\frac{\beta+t_s}{\alpha+t_a+t_s}\right)^i}. \tag{12}$$

The standard error of this estimator is obtained by substituting $u = 2, v = 0, w = 2$ in (9).

$$S.E.(L^*) = \sqrt{\left[\frac{\sum_{k=0}^{n_s-3} (k+1) \frac{\binom{n_a+k+2}{k+2}}{\binom{n_s}{k+2}} \sum_{i=0}^{n_s-k-2} \binom{n_a+i+k+2}{i} \left(\frac{\beta+t_s}{\alpha+t_a+t_s}\right)^{i+k+2}}{\sum_0^{n_s} \binom{n_a+i}{i} \left(\frac{\beta+t_s}{\alpha+t_a+t_s}\right)^i}\right] - (L^*)^2}. \tag{13}$$

Bayes Estimator of the average number of customers in the queue (L_q)
Putting $u = 2, v = 1, w = 1$ in (9), the Bayes estimator $L^*$of L is

$$L_q^* = \frac{\sum_{k=0}^{n_s-3} \frac{\binom{n_a+k+2}{k+2}}{\binom{n_s}{k+2}} \sum_{i=0}^{n_s-k-2} \binom{n_a+i+k+2}{i} \left(\frac{\beta+t_s}{\alpha+t_a+t_s}\right)^{i+k+2}}{\sum_0^{n_s} \binom{n_a+i}{i} \left(\frac{\beta+t_s}{\alpha+t_a+t_s}\right)^i}. \tag{14}$$

The standard error of this estimator is obtained by substituting $u = 4, v = 2, w = 2$ in (9).

$$\mathrm{S.E.}\left(L_q^*\right) = \sqrt{\left[\frac{\sum_{k=0}^{n_s-5} (k+1) \frac{\binom{n_a+k+4}{k+4}}{\binom{n_s}{k+4}} \sum_{i=0}^{n_s-k-4} \binom{n_a+i+k+4}{i} \left(\frac{\beta+t_s}{\alpha+t_a+t_s}\right)^{i+k+4}}{\sum_0^{n_s} \binom{n_a+i}{i} \left(\frac{\beta+t_s}{\alpha+t_a+t_s}\right)^i}\right] - (L_q^*)^2}. \tag{15}$$

Bayes Estimator of the average waiting time of a customer in the system (W)
Putting $u = 0, v = 0, w = 1$ in (9), the Bayes estimator $W^*$of W is

$$W^* = \frac{\sum_{k=0}^{n_s-2} \frac{\binom{n_a+k}{k}}{\binom{n_s}{k+1}} \left(\frac{\beta+t_s}{k+1}\right) \sum_{i=0}^{n_s-k-1} \binom{n_a+i+k}{i} \left(\frac{\beta+t_s}{\alpha+t_a+t_s}\right)^{i+k}}{\sum_0^{n_s} \binom{n_a+i}{i} \left(\frac{\beta+t_s}{\alpha+t_a+t_s}\right)^i}. \tag{16}$$

The standard error of this estimator is obtained by substituting $u = 0$, $v = 0$, $w = 2$ in (9).

$$S.E.(W^*) = \sqrt{\left[\frac{\sum_{k=0}^{n_s-3} (k+1) \frac{\binom{n_a+k}{k}}{\binom{n_s}{k+2}} \frac{(\beta+t_s)^2}{(k+1)(k+2)} \sum_{i=0}^{n_s-k-2} \binom{n_a+i+k}{i} \left(\frac{\beta+t_s}{\alpha+t_a+t_s}\right)^{i+k}}{\sum_0^{n_s} \binom{n_a+i}{i} \left(\frac{\beta+t_s}{\alpha+t_a+t_s}\right)^i}\right] - (W^*)^2}. \tag{17}$$

Bayes Estimator of the average waiting time of a customer in the queue (W_q)

Putting $u = 1, v = 1, w = 1$ in (9), the Bayes estimator W_q^* of W_q is

$$W_q^* = \frac{\sum_{k=0}^{n_s-3} \frac{\binom{n_a+k+1}{k+1}}{\binom{n_s}{k+2}} \left(\frac{\beta+t_s}{k+2}\right) \sum_{i=0}^{n_s-k-2} \binom{n_a+i+k+1}{i} \left(\frac{\beta+t_s}{\alpha+t_a+t_s}\right)^{i+k+1}}{\sum_0^{n_s} \binom{n_a+i}{i} \left(\frac{\beta+t_s}{\alpha+t_a+t_s}\right)^i}. \tag{18}$$

The standard error of this estimator is obtained by substituting $u = 2, v = 2, w = 2$ in (9).

$$S.E.\left(W_q^*\right) = \sqrt{\left[\frac{\sum_{k=0}^{n_s-5} (k+1) \frac{\binom{n_a+k+2}{k+2}}{\binom{n_s}{k+4}} \frac{(\beta+t_s)^2}{(k+3)(k+4)} \sum_{i=0}^{n_s-k-4} \binom{n_a+i+k+2}{i} \left(\frac{\beta+t_s}{\alpha+t_a+t_s}\right)^{i+k}}{\sum_0^{n_s} \binom{n_a+i}{i} \left(\frac{\beta+t_s}{\alpha+t_a+t_s}\right)^i}\right] - (W_q^*)^2}. \tag{19}$$

Bayes Estimator of Probability of the system state being greater than s

We have, $p = Pr(N \geq s) = \left(\frac{\lambda}{\mu}\right)^s$, where N is the state of the system.

Putting $u = s, v = s, w = 0$ in (9), the Bayes estimator p^* of p is-

$$p^* = \frac{\binom{n_a+s}{s}}{\binom{n_s}{s}} \frac{\sum_{i=0}^{n_s-s} \binom{n_a+s+k+1}{i} \left(\frac{\beta+t_s}{\alpha+t_a+t_s}\right)^{i+k+s}}{\sum_{i=0}^{n_s} \binom{n_a+i}{i} \left(\frac{\beta+t_s}{\alpha+t_a+t_s}\right)^i}. \tag{20}$$

The standard error of this estimator is obtained by substituting $u = 2s$, $v = 2s$, $w = 0$ in (9).

$$S.E.(p^*) = \sqrt{\left[\frac{\binom{n_a+2s}{2s}}{\binom{n_s}{2s}} \frac{\sum_{i=0}^{n_s-2s} \binom{n_a+2s+k+1}{i} \left(\frac{\beta+t_s}{\alpha+t_a+t_s}\right)^{i+k+2s}}{\sum_{i=0}^{n_s} \binom{n_a+i}{i} \left(\frac{\beta+t_s}{\alpha+t_a+t_s}\right)^i}\right] - (p^*)^2}. \tag{21}$$

3.2. Predictive distributions and their moments

Predictive distributions in the Bayesian paradigm are crucial for forecasting queuing system metrics, including queue lengths and waiting times. By quantifying uncertainty, these distributions optimize resource allocation and staffing, thereby enhancing system efficiency. This subsection outlines the methodology for deriving predictive distributions for the number of customers at departure epoch, waiting time in the system, and waiting time in the queue.

Let $x = (x_1, x_2, x_3, \dots, x_n)$ be a random sample drawn from a distribution characterized by the probability mass function (p.m.f.) or probability density function (p.d.f.) $f(x|\theta)$, where θ belongs to the parameter space Ω. Given a prior distribution $\pi(\theta)$ on the parameter θ, and the corresponding posterior distribution $\Pi(\theta|x)$, the posterior predictive distribution for a future observation m, conditioned on the observed data x, is defined as (Chen *et al.*, 2012):

$$p(m|x) = \int_{\Omega} f(x|\theta)\, \Pi(\theta|x), d\theta. \tag{22}$$

3.2.1. Predictive distribution of the number in the system at departure epoch

Building on the distribution of M given in equation (1), we now derive the predictive distribution for the number of customers in the system at a departure epoch.

$$P(M=m|data) = \int_0^\infty \int_\lambda^\infty \left(1-\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^m \Pi\,(\lambda,\mu|data)d\mu d\lambda$$

This integral simplifies to the following expression:

$$P(M=m|data) = \frac{\left[\frac{\binom{n_a+m}{m}}{\binom{n_s}{m}}\sum_{i=0}^{n_s-m}\left\{\binom{n_a+m+i}{i}\left(\frac{\beta+t_s}{\alpha+t_a+t_s}\right)^{i+m}\right\}\right]-\left[\frac{\binom{n_a+m+1}{m+1}}{\binom{n_s}{m+1}}\sum_{i=0}^{n_s-m-1}\left\{\binom{n_a+m+1+i}{i}\left(\frac{\beta+t_s}{\alpha+t_a+t_s}\right)^{i+m+1}\right\}\right]}{\left[\sum_{i=0}^{n_s}\left\{\binom{n_a+i}{i}\left(\frac{\beta+t_s}{\alpha+t_a+t_s}\right)^{i}\right\}\right]}. \tag{23}$$

Both raw and central factorial moments of this distribution exist and can be calculated as given below.

$$E[M(M-1)(M-2)\dots(M-r+1)|data,\lambda,\mu] = \int_0^\infty \int_\lambda^\infty r!\frac{\lambda^r}{(\mu-\lambda)^r}\Pi(\lambda,\mu|data)d\mu d\lambda$$

which simplifies to

$$\mu'_{(r)} = r!\frac{\sum_{k=0}^{n_s-r-1}\left[\frac{\binom{r+k-1}{k}\binom{n_a+r+k}{r+k}}{\binom{n_s}{r+k}}\sum_{i=0}^{n_s-r-k}\left\{\binom{n_a+r+k+i}{i}\left(\frac{\beta+t_s}{\alpha+t_a+t_s}\right)^{i+k}\right\}\right]}{\left[\sum_{i=0}^{n_s}\left\{\binom{n_a+i}{i}\left(\frac{\beta+t_s}{\alpha+t_a+t_s}\right)^{i}\right\}\right]}. \tag{24}$$

Thus, the mean number of customers in the system at departure epoch is

$$\mu'_{(1)} = \mu'_1 = \frac{\sum_{k=0}^{n_s-2}\left[\frac{\binom{n_a+k+1}{k+1}}{\binom{n_s}{r+k}}\sum_{i=0}^{n_s-k-1}\left\{\binom{n_a+k+i+1}{i}\left(\frac{\beta+t_s}{\alpha+t_a+t_s}\right)^{i+k}\right\}\right]}{\left[\sum_{i=0}^{n_s}\left\{\binom{n_a+i}{i}\left(\frac{\beta+t_s}{\alpha+t_a+t_s}\right)^{i}\right\}\right]}$$

3.2.2. Predictive distribution of waiting time in the system

Proceeding from the distribution of the system waiting time, as given in (2), we now derive its predictive distribution as follows:

$$P(T=t|data) = \int_0^\infty \int_\lambda^\infty (\mu-\lambda)\exp[-(\mu-\lambda)t]\,\Pi\,(\lambda,\mu|data)d\mu d\lambda$$

Similarly, by integrating and using the incomplete gamma integral, the predictive distribution for waiting time is obtained as

$$P(T=t|data) = \frac{\left[\left(\frac{n_s+1}{\beta+t+t_s}\right)\sum_{i=0}^{n_s-m}\left\{\binom{n_a+i}{i}\left(\frac{\beta+t_s+t}{\alpha+t_a+t_s}\right)^{i}\right\}\right]-\left[\left(\frac{n_a+1}{\alpha+t_a+t_s}\right)\sum_{i=0}^{n_s}\left\{\binom{n_a+1+i}{i}\left(\frac{\beta+t_s+t}{\alpha+t_a+t_s}\right)^{i}\right\}\right]}{\left[\sum_{i=0}^{n_s}\left\{\binom{n_a+i}{i}\left(\frac{\beta+t_s}{\alpha+t_a+t_s}\right)^{i}\right\}\right]}. \tag{25}$$

The r[th] factorial moment is obtained as

$$E[T(T-1)\ \dots(T-r+1)\ |data,\rho,\mu] = \int_0^\infty \int_\lambda^\infty \frac{r!}{(\mu-\lambda)^r} \Pi(\lambda,\mu|data)d\mu d\lambda$$

which, on simplification, gives

$$\mu'_{(r)} = r! \frac{\left[\sum_{k=0}^{n_s-r-1}\left\{\frac{\binom{r+k-1}{k}\binom{n_a+k}{k}}{\binom{n_s}{r+k}}\frac{k!(\beta+t_s)^r}{(r+k)!}\sum_{i=0}^{n_s-r-k}\left\{\binom{n_a+k+i}{i}\left(\frac{\beta+t_s}{\alpha+t_a+t_s}\right)^{i+k}\right\}\right\}\right]}{\left[\sum_{i=0}^{n_s}\left\{\binom{n_a+i}{i}\left(\frac{\beta+t_s}{\alpha+t_a+t_s}\right)^{i}\right\}\right]}. \quad (26)$$

3.2.3. Predictive distribution of waiting time in the queue

Similarly, using (3), the predictive distribution for waiting time in the queue is obtained for two cases.

Case I: For $t = 0$, the predictive distribution of waiting time in the queue is given by

$$P(T = t|data) = \int_0^\infty \int_\lambda^\infty \left(1-\frac{\lambda}{\mu}\right) \Pi\ (\lambda,\mu|data)d\mu d\lambda$$

On simplification, we have

$$P(T = t|data) = 1-\left(\frac{n_a+1}{n_s}\right)\left[\frac{\sum_{i=0}^{n_s-1}\left\{\binom{n_a+i+1}{i}\left(\frac{\beta+t_s}{\alpha+t_a+t_s}\right)^{i+1}\right\}}{\sum_{i=0}^{n_s}\left\{\binom{n_a+i}{i}\left(\frac{\beta+t_s}{\alpha+t_a+t_s}\right)^{i}\right\}}\right]. \quad (27)$$

Case II: For $t > 0$, the predictive distribution of waiting time in the queue is given by

$$P(T = t|data) = \int_0^\infty \int_\lambda^\infty \frac{\lambda}{\mu}(\mu-\lambda)\ \Pi\ (\lambda,\mu|data)d\mu d\lambda$$

which simplifies to

$$P(T = t|data) = \frac{\left[\left(\frac{n_a+1}{\alpha+t_a+t_s}\right)\sum_{i=0}^{n_s}\left\{\binom{n_a+i+1}{i}\left(\frac{\beta+t_s}{\alpha+t_a+t_s}\right)^{i}\right\}\right]-\left[\frac{(n_a+1)(n_a+2)}{2n_s}\frac{(\beta+t_s)}{(\alpha+t_a+t_s)^2}\sum_{i=0}^{n_s-1}\left\{\binom{n_a+i+2}{i}\left(\frac{\beta+t_s}{\alpha+t_a+t_s}\right)^{i}\right\}\right]}{\left[\sum_{i=0}^{n_s}\left\{\binom{n_a+i}{i}\left(\frac{\beta+t_s}{\alpha+t_a+t_s}\right)^{i}\right\}\right]}. \quad (28)$$

The r[th] factorial moment of this distribution is

$$E[T(T-1)\dots(T-r+1)|data,\lambda,\mu] = \int_0^\infty \int_\lambda^\infty \frac{r!\lambda}{\mu(\mu-\lambda)^r}\Pi(\lambda,\mu|data)d\mu d\lambda \quad (29)$$

Or,

$$\mu'_{(r)} = r! \frac{\left[\sum_{k=0}^{n_s-r-2}\left\{\frac{\binom{r+k-1}{k}\binom{n_a+k+1}{k+1}}{\binom{n_s}{r+k+1}}\frac{(k+1)!\ (\beta+t_s)^{r+1}}{(r+k+1)!(\alpha+t_a+t_s)}\sum_{i=0}^{n_s-r-k-1}\left\{\binom{n_a+k+i+1}{i}\left(\frac{\beta+t_s}{\alpha+t_a+t_s}\right)^{i+k}\right\}\right\}\right]}{\left[\sum_{i=0}^{n_s}\left\{\binom{n_a+i}{i}\left(\frac{\beta+t_s}{\alpha+t_a+t_s}\right)^{i}\right\}\right]} \quad (30)$$

The formulae (27) and (28) provide the predictive distributions for the waiting time in the queue for two scenarios: $t = 0$ and $t > 0$.

4. Simulation

This section presents a simulation study to assess the numerical performance of the Bayes estimators derived in Section 3, with particular emphasis on ρ, a critical parameter that influences system stability and efficiency. Also, the predictive distribution of the number of customers at the departure epoch is evaluated.

To examine the reliability of the estimators, simulations are conducted across varying sample sizes $n = (50, 100, 200, 500)$. Based on these settings, individual interarrival and service times are generated using inverse transform sampling from exponential distributions for three parameter configurations: $(\lambda, \mu) = \{(12,15), (18, 20), (70, 100)\}$. From these, the aggregated interarrival and service times are obtained. Finally, the simulated data produce the Bayes estimates and other performance metrics. The results are detailed in the subsequent tables.

For the different sample sizes and varying pairs of (λ, μ), Bayes estimates of ρ and their MSEs are reported in Table 2. Table 3 presents the predictive distribution of the number of customers in the system at departure epoch. The MSEs serve as frequentist evaluation metrics in the simulation context. Given the Bayesian nature of our analysis, we additionally examine the standard errors of the estimates reported in Table 4. The 95% and 99% credible regions of ρ are presented in Table 5 and Table 6 respectively.

Table 2. Bayes estimates of ρ along with their MSEs for different pairs of arrival and service rates and for different sample sizes of interarrival and service times

Sample size	Pair 1 (12, 15)		Pair 2 (18, 20)		Pair 3 (70, 100)	
	Estimate of ρ	MSE	Estimate of ρ	MSE	Estimate of ρ	MSE
$n_a = 50;\ n_s = 50$	0.80901	0.00541	0.92465	0.00670	0.69730	0.00893
$n_a = 100;\ n_s = 100$	0.87290	0.00508	0.90795	0.00415	0.76627	0.02101
$n_a = 200;\ n_s = 200$	0.80164	0.00425	0.90175	0.00261	0.76858	0.01272
$n_a = 500;\ n_s = 500$	0.84318	0.00427	0.85272	0.00169	0.76319	0.00391

From Table 2, we can observe that the estimated values are very close to the true values and the MSEs are minimal. Pair 2 (18, 20) yields the most substantial gain, where MSE decreases from 0.00670 to 0.00261 as the sample size increases from 50 to 200. This represents a 61% reduction in the estimation error. Similarly, Pair 1 (12, 15) demonstrates a steady reduction in MSE from 0.00541 to 0.00425.

Table 3. Predictive distributions of number of customers at departure epoch for different pairs of arrival and service rates and for different sample sizes of interarrival and service times

Sample size	m	Predictive distribution		
		Pair 1 (12, 15)	**Pair 2 (18, 20)**	**Pair 3 (70, 100)**
$n_a = 100; n_s = 100$	0	0.11692	0.25864	0.12216
	1	0.09707	0.18197	0.10076
	2	0.08130	0.13003	0.07044
	3	0.06867	0.09434	0.05964
	4	0.05846	0.06945	0.05088
	5	0.04329	0.05187	0.04373
$n_a = 200; n_s = 200$	0	0.19628	0.21151	0.25163
	1	0.15191	0.16096	0.18281
	2	0.11851	0.12350	0.13398
	3	0.09317	0.09552	0.09905
	4	0.07282	0.07448	0.07385
	5	0.05893	0.05853	0.05553
$n_a = 500; n_s = 500$	0	0.22895	0.09320	0.39798
	1	0.17415	0.08211	0.23814
	2	0.13297	0.07532	0.14306
	3	0.10190	0.06425	0.08628
	4	0.07839	0.05706	0.05224
	5	0.06053	0.05080	0.03175

The predictive distributions in Table 3 exhibit theoretically consistent behavior with probability mass concentrating at lower queue states and following the expected exponential decay pattern of M/M/1 systems. Notably, Pair 3 (70, 100) achieves convergence with the probability of an empty system increasing from 0.12216 to 0.39798 as the sample size grows from 100 to 500. Pair 1 (12, 15) demonstrates steady improvement with $P(m = 0)$ rising from 0.11692 to 0.22895. Pair 2 (18, 20) exhibits high-utilization behavior, with the distribution of probabilities across queue states aligning with expected steady-state characteristics. The results indicate the effectiveness of this predictive mechanism in predicting queue performance under varying configurations.

Since different methods of estimating ρ have been previously studied in the literature, here a comparison is carried out against some of the earlier established techniques and our approach. Singh and Acharya (2019) estimated ρ using both Maximum Likelihood (ML) and Bayesian procedures. Their estimates yield higher Mean Squared Errors (MSEs) of ρ using MLE (0.0112 for a sample size of 100 and 0.0413 for a sample size of 50). The corresponding MSEs for the Bayes estimates are 0.0053 and 0.0167, respectively. In contrast, this article reports an MSE of ρ for 100 samples at 0.00425 and that for a sample size 50 at 0.00541. This illustrates that both ML and Bayesian estimates result in higher MSEs compared to the findings of the current paper. Singh *et al.* (2021)

reported Root Mean Squared Errors (RMSEs) using ML and Bayesian estimation. For $n = 50$, the RMSEs are 0.1407 and 0.1133, respectively, indicating higher errors. For $n = 100$, the RMSE using MLE was 0.1004, also demonstrating higher errors. Several simulation designs by Sohn (1996) show the MSEs of Bayesian estimates of ρ. For a sample size 100, the reported MSEs of various estimators for ρ exceed those obtained in this study: ad-hoc Bayes estimator using lognormal prior with covariates (0.088, 0.0109), gamma prior with covariates (0.0900, 0.0099), model-based regression estimator (0.6280, 0.0137), and data-based raw estimator (0.1070, 0.0400) all demonstrate higher MSEs compared to the results presented here for equivalent sample size.

Table 4 shows that the standard errors of these Bayes estimates are low and decrease with an increasing sample size, thereby confirming the precision and reliability improvements inherent in Bayesian estimation.

Table 4. Standard error (SE) of the estimates of traffic intensity (ρ) for different pairs of arrival and service rates and for different sample sizes of interarrival and service times

Sample Size	**Pair of (λ, μ)**		
	(12, 15)	**(18, 20)**	**(70, 100)**
$n_a = 50, n_s = 50$	0.1070	0.0864	0.0874
$n_a = 100, n_s = 100$	0.0742	0.0752	0.0735
$n_a = 200, n_s = 200$	0.0727	0.0709	0.0646
$n_a = 500, n_s = 500$	0.0486	0.0510	0.0490

Table 5. 95% credible intervals of traffic intensity (ρ) for different pairs of arrival and service rates (λ, μ)

Sample Size	**Pair of (λ, μ)**		
	(12, 15)	**(18, 20)**	**(70, 100)**
$n_a = 50, n_s = 50$	(0.5040, 0.9704)	(0.5565, 0.9847)	(0.5549, 0.9844)
$n_a = 100, n_s = 100$	(0.6567, 0.9883)	(0.7147, 0.9939)	(0.6461, 0.9865)
$n_a = 200, n_s = 200$	(0.6611, 0.9589)	(0.7357, 0.9903)	(0.6366, 0.9345)
$n_a = 500, n_s = 500$	(0.6816, 0.8733)	(0.7653, 0.9703)	(0.6268, 0.8032)

Table 6. 99% Credible intervals of traffic intensity (ρ) for different pairs of arrival and service rates (λ, μ)

Sample Size	**Pair of (λ, μ)**		
	(12, 15)	**(18, 20)**	**(70, 100)**
$n_a = 50, n_s = 50$	(0.4457, 0.9927)	(0.4931, 0.9960)	(0.4916, 0.9959)
$n_a = 100, n_s = 100$	(0.6030, 0.9968)	(0.6586, 0.9979)	(0.5930, 0.9964)
$n_a = 200, n_s = 200$	(0.6216, 0.9878)	(0.6926, 0.9972)	(0.5930, 0.9964)
$n_a = 500, n_s = 500$	(0.6554, 0.9079)	(0.7261, 0.9906)	(0.6029, 0.8352)

In both Table 5 and Table 6, a notable decrease in the width of the credible intervals is observed as the sample size increases from 50 to 500, thereby enhancing the precision of the posterior estimates. Under $(\lambda, \mu) = (18, 20)$, the 95% credible interval narrows from (0.5565, 0.9847) at $n = 50$ to (0.7653, 0.9703) at $n = 500$. Similarly, the 99% credible interval under the same rate pair reduces from (0.4931, 0.9960) to (0.7261, 0.9906). This pattern remains consistent across all three rate pairs, confirming that the posterior distribution concentrates around the true value of ρ as the sample size increases. Despite the differing nominal values of ρ, all three rate pairs produce credible intervals that broadly cover the true value of traffic intensity. For $(\lambda, \mu) = (70, 100)$, the intervals are generally narrower and converge more quickly. This indicates enhanced estimation performance at lower traffic intensity levels.

Comparing the 95% (Table 5) and 99% (Table 6) credible intervals reveals the expected widening of intervals at higher confidence levels, reflecting the inverse relationship between precision and certainty. The difference in bounds between the two tables is evident at lower sample sizes, showing greater uncertainty. At $n = 50$, the 99% interval for the pair (12, 15) is (0.4457, 0.9927), compared to the 95% interval (0.5040, 0.9704). At $n = 500$, this expands only slightly from (0.6816, 0.8733) at 95% to (0.6554, 0.9079) at 99%. This behavior reaffirms that Bayesian credible intervals appropriately reflect posterior uncertainty, with interval width decreasing as data volume increases, and interval coverage increasing as the confidence level rises.

Implementation of the proposed estimators:

The following steps outline the simulation-based procedure used to implement the Bayesian estimators for arrival and service rates, as well as associated performance measures:

Step 1: Select appropriate values for the hyperparameters α and β based on prior knowledge.

Step 2: Choose multiple pairs of (λ, μ) such that $\lambda < \mu$ to ensure system stability.

Step 3: Fix the sample sizes for the interarrival and service times.

Step 4: Generate random samples of interarrival and service times from exponential distributions with rates λ and μ.

Step 5: Calculate the total interarrival time $(t_a)$and the total service time (t_s) from the simulated data.

Step 6: Substitute the values of α, β, n_a, n_s, t_a , and t_s into the Bayesian estimators derived in Section 3. The output yields the Bayes estimates along with their associated standard errors for the desired performance measures.

5. Real-life queueing problem

To demonstrate the practical applicability of the proposed Bayesian framework, we apply the derived estimators to empirical data obtained from (Czech University of Life Sciences Prague, n.d., § 9.6.6)

The empirical analysis is based on observational data from a service system characterized by stochastic customer arrivals following a Markovian process. The dataset, encompasses 30 customer transactions recorded over an 8-hour operational period. The data collection includes timestamped measurements of arrival times, service initiation times, service completion times, and derived performance metrics, including queue length, interarrival intervals, service durations, and customer waiting times.

From the given data, interarrival times were calculated as the time difference between consecutive customer arrivals. The dataset yields:

- **Interarrival times**: $[t_1, t_2, \ldots, t_{30}]$ where t_i represents the time between arrivals $i-1$and i, $i = 0,1,2, \ldots, 30$
- **Service times**: $[s_1, s_2, \ldots, s_{30}]$ representing individual service durations.

These values are subsequently used to compute: $t_a = \Sigma t_\text{i} = 387$ and $t_s = \Sigma s_\text{i} = 362$

The processed data serve as input for the Bayesian procedure to obtain posterior estimates of the traffic intensity ρ and its standard error. The results obtained are $\rho^* = 0.80503$ and $S.E.(\rho^*) = 0.10253$

6. Conclusion

This study presents a Bayesian approach to parameter estimation of an M/M/1 queuing model using a bivariate prior and a simple Markovian likelihood of arrival and service rate. Bayes estimates of ρ and predictive distributions at departure epoch are simulated using inverse transform sampling. The results demonstrate good accuracy with low standard errors and MSEs as low as 0.00169 for $n = 500$. Also, the predictive distributions exhibit strong convergence, with probability mass concentrating around key values. Moreover, the credible intervals computed encompass true parameter values. As the sample size increases from 50 to 500, the credible intervals become narrower, thereby indicating improved precision. The 99% intervals are wider than the 95% intervals, particularly at smaller sample sizes, confirming the statistical validity of the Bayesian procedure. This enables users to assess uncertainty at different confidence levels depending on the context of the application. Therefore, the findings establish that the developed methodology is reliable for uncertainty assessment of traffic intensity, with improved performance at larger sample sizes and lower utilization levels.

While the proposed Bayesian model demonstrates strong predictive capabilities and practical relevance, several considerations may guide future research. First, the use of the M/M/1 queueing model, although widely studied, may not fully reflect the dynamics of more complex or non-Markovian service systems encountered in real-world scenarios. Second, it introduces a specific bivariate prior distribution for parameter inference. Exploring alternative bivariate priors and higher-order Markovian and non-Markovian queues could provide further insights into estimation accuracy and model robustness.

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