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Area-biased one-parameter exponential distribution with financial applications

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Abstract

Area-biased distributions are special cases of size-biased distributions. We have used the idea of area-biased distributions in this paper to propose a generalisation of a one-parameter linear exponential distribution. The concept is called the area-biased one-parameter linear exponential distribution. Its various characteristics are deduced and thoroughly explored. Some numerical studies are implemented which demonstrate that the distribution is skewed to the right with heavier tail than the normal distribution. The mean waiting and residual life time are also studied. Six methods of estimation are employed to estimate the parameters distribution. A simulation study is conducted which shows that the estimators are approximately unbiased and consistent. Three financial real data sets are applied. They represent the earning per share in the financial, industry and service sectors at the Amman Stock Exchange. The study shows that the suggested distribution has the best fit for these data sets compared to some competence distributions.

Key words: one-parameter linear exponential distribution, area-biased, methods of estimation, earning per share.

1. Introduction

In statistics, modelling lifetime data is an important issue in many fields, including biomedical sciences, economics, finance, engineering, and many others. A lot of continuous distributions have been introduced for modelling such data, because they can tend to be more efficient than the base distributions. Many methods have been used to propose new models such as the combination of two or more distributions.

Weighted distributions, involving a variety of sampling surveys, have been widely applied to model data in nature, offering more insights and adequacy in the modelling. The theory of weighted distributions ensures a collective access to the problems of model specification and data interpretation. It provides a procedure for fitting models to unknown weight functions when samples can be taken from both the original distribution and the developed distribution. They take into account the method of ascertainment by adjusting the probabilities of the actual occurrence of events in order to arrive at a specification of the probabilities of those events as observed and recorded.

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The idea of weighted distributions was initially introduced by Fisher (1934) and developed by Rao (1965). Recently, this concept has been employed in a lot of researches related to reliability, ecology, family data analysis, bio-medicine, and some other fields for the improved performance of appropriate statistical models. It is defined by

$$f_w(x) = \frac{w(x)f(x)}{E(w(x))}, \; x > 0, \tag{1}$$

where $w(x)$ is a non-negative weight function.

Let X be a random variable with probability density function (pdf) $f(x)$, then the size-biased distribution can be produced by using the weight function $w(x) = X^m$. It was first studied by Patil and Ord (1976). For $m = 2$, we get the area-biased distribution, which was first employed by Cox (1968) and Zelen (1974). Thus, the resulting pdf takes the form of

$$f_1(x) = \frac{x^2 f(x)}{E(X^2)}. \tag{2}$$

Area-biased distributions, as mentioned above, are special types of weighted distributions. In recent times, many authors have been interested in studying these types of distributions, such as Sharma et al. (2018) who introduced the length- and area-biased Maxwell distribution. Al-Omari et al. (2019a) suggested the power length-biased Suja distribution as a new extension of the length-biased Suja distributions. Saghir et. al. (2017) studied a new class of Maxwell length-biased distributions. Shen et al. (2009) used semi-parametric transformations to model the length-biased data. Al-Omari and Alanzi (2021) suggested and studied the properties of the one-parameter inverse length-biased Maxwell distribution. Das and Roy (2011) suggested the length-biased form of the weighted Weibull distribution. The weighted distributions based on the mixture of two distributions based on weights p_1 and p_2, with $p_1 + p_2 = 1$ are used by many authors, such as: Alzoubi et al. (2022), Benrabia and Alzoubi (2022a), Benrabia and Alzoubi (2022b) and Alzoubi et al. (2022).

In this paper, we propose a new distribution. This distribution is applied to financial data extracted from the Amman Stock Exchange (ASE). We have used Earnings per share (EPS) data to compare the suggested distribution with other distributions. EPS is one way of measuring a company's success. An increase in the EPS indicates higher investor prosperity (Ferniawan et al. 2024). Some researchers concluded that EPS has a significant and positive impact on company value (Kristanti et al. 2024), others indicated a substantial and positive impact of EPS on the stock prices (Taubah et al. 2024 and Dang et al., 2024). The ASE was established in March 1999 as a non-profit independent institution, authorised to function as a regulated market for trading securities in Jordan. ASE aims to operate, manage and develop the operations and activities of security, commodity, and derivatives markets inside and outside of Jordan. It seeks to provide a strong and secure environment to ensure the interaction of supply and demand forces for trading in proper and fair trading practices. It also aims to raise the awareness and knowledge of investing in the financial markets and defining the services provided by the ASE (ASE, 2023).

2. One-parameter linear-exponential distribution

Ghitany et al. (2008) introduced a single parameter distribution called the Lindley distribution. The pdf of this distribution is given by

$$f_l(x) = \frac{\theta^2(1+x)e^{-\theta x}}{1+\theta}; \;\; x>0, \;\theta>0. \tag{3}$$

Sah (2021) proposed the one-parameter linear-exponential distribution (OPLED). Its pdf and second moment are defined as

$$f_o(x) = \frac{\theta^2(\theta^2+x)e^{-\theta x}}{1+\theta^3}; \;\; x>0, \;\theta>0 \tag{4}$$

$$E\left(X^2\right) = \frac{2(\theta^3+3)}{\theta^2(1+\theta^3)}. \tag{5}$$

3. Area-biased one-parameter linear-exponential distribution

This section introduces the new proposed distribution, the area-biased one-parameter exponential distribution (ABOPLED). The pdf of this distribution is defined using (2), (4) and (5):

$$f(x) = \frac{\theta^4 x^2\left(x+\theta^2\right)e^{-\theta x}}{2\theta^3+6}; \;\; x>0, \;\theta>0. \tag{6}$$

The cumulative distribution function (CDF) of ABOPLED is defined as:

$$\begin{aligned} F(x) &= \int_0^x \frac{\theta^4 u^2\left(u+\theta^2\right)e^{-\theta u}}{2\theta^3+6}du \\ &= 1-\frac{\left(\theta^3x^3+\left(\theta^5+3\theta^2\right)x^2+\left(2\theta^4+6\theta\right)x+2\theta^3+6\right)e^{-\theta x}}{2\theta^3+6} \\ &= 1-\left(\frac{\theta^3x^3}{2\theta^3+6}+\frac{\theta^2x^2}{2}+\theta x+1\right)e^{-\theta x}. \end{aligned} \tag{7}$$

$F(x)$ satisfies the conditions of the CDF, since (1) $F(x)$ is right-continuous. (2) $\lim_{x\longrightarrow 0} F(x)=0$, and (3) $\lim_{x\longrightarrow\infty} F(x)=1$

(a) The pdf **(b)** The CDF

Figure 1. The pdf and CDF of ABOPLED for different values of θ.

Figures 1a and 1b show the pdf and CDF plots of ABOPLED for θ values of 1.5-9.5(step(1)). Figure 1a shows that the peak of the distribution gets sharper for smaller values of θ. Figure 1b shows that the CDF reaches 1 faster for larger values of θ.

4. Moments and related measures of the ABOPLED

The moment generating function along with the moments and related measures of the ABOPLED and some tables of the mean, standard deviation, coefficient of skewness, excess kurtosis, and coefficient of variation for certain values of the parameter will be derived in this section.

4.1. Moments

Theorem 1 . *Let X be a random variable following the ABOPLED. The r^{th} moment of X is*

$$E(X^r) = \frac{\left((r+3)!+\theta^3(r+2)!\right)}{2\theta^r(2\theta^3+6)}, \quad r=1,2,.... \tag{8}$$

Proof 1 *The r^{th} moment can be found as*

$$\begin{aligned} E(X^r) &= \int_0^\infty x^r f(x)dx = \int_0^\infty \frac{\theta^4 x^{r+2}\left(x+\theta^2\right)e^{-\theta x}}{2\theta^3+6}dx \\ &= \frac{\theta^4}{2\theta^3+6}\int_0^\infty x^{r+2}\left(x+\theta^2\right)e^{-\theta x}dx = \frac{\left((r+3)!+\theta^3(r+2)!\right)}{2\theta^r(2\theta^3+6)}. \end{aligned}$$

Thus, the first four moments can be calculated by substituting r with 1, 2, 3 and 4 in (8). Hence, we have

$$\mu = E(X) = \frac{\left(12+3\theta^3\right)}{\theta(\theta^3+3)}, \tag{9}$$

$$E(X^2) = \frac{\left(60+12\theta^3\right)}{\theta^2(\theta^3+3)}, \tag{10}$$

$$E(X^3) = \frac{\left(360+60\theta^3\right)}{\theta^3(\theta^3+3)}, \tag{11}$$

$$E(X^4) = \frac{\left(2520+360\theta^3\right)}{\theta^4(\theta^3+3)}. \tag{12}$$

4.2. Related measures

From (9), and (10), the variance (σ^2) and the standard deviation (σ) are as follows:

$$\sigma^2 = E(X^2) - \mu^2 = \frac{\left(60 + 12\theta^3\right)}{\theta^2(\theta^3+3)} - \left(\frac{\left(12+3\theta^3\right)}{\theta(\theta^3+3)}\right)^2 = \frac{3(\theta^3+2)(\theta^3+6)}{\theta^2\left(\theta^3+3\right)^2}$$

$$\sigma = \frac{\sqrt{3(\theta^3+2)(\theta^3+6)}}{\theta\left(\theta^3+3\right)}. \tag{13}$$

The coefficient of variation (CV) is defined using (9) and (13) as:

$$CV = \frac{\sqrt{3(\theta^3+2)(\theta^3+6)}}{(12+3\theta^3)}. \tag{14}$$

The skewness (Sk) is defined using (9), (10), (11) and (13) as:

$$Sk(X) = \frac{E(X^3) - 3\mu E(X^2) + 2\mu^3}{\sigma^3} = \frac{6\left(\theta^9 + 12\theta^6 + 36\theta^3 + 36\right)}{\left(\sqrt{3(\theta^3+2)(\theta^3+6)}\right)^3}. \tag{15}$$

Au et al. (2015) defined the excess kurtosis (*eKur*) as: $eKur(X) = Kur(X) - 3$. Thus, for ABOPLED it is defined using (9), (10), (11), (12) and (13) as:

$$\begin{aligned} eKur(X) &= \frac{E(X^4) - 4\mu E(X^3) + 6\mu^2 E(X^2) - 3\mu^4}{\sigma^4} - 3 \\ &= \frac{\left(5\theta^{12} + 80\theta^9 + 408\theta^6 + 864\theta^3 + 648\right)}{\left((\theta^3+2)(\theta^3+6)\right)^2} - 3. \end{aligned} \tag{16}$$

Table 1. Related moments measures for ABOPLED for different values of θ

θ	μ	σ	*Sk*	*eKur*	*CV*	θ	μ	σ	*Sk*	*eKur*	*CV*
1.25	**2.8845**	**1.5686**	**1.045**	**1.614**	**54.3793**	**4.50**	**0.6737**	**0.3889**	**1.154**	**1.9966**	**57.7223**
1.50	**2.3137**	**1.2858**	**1.0738**	**1.7005**	**55.5715**	**4.75**	**0.6373**	**0.3679**	**1.1542**	**1.9975**	**57.7257**
1.75	**1.9194**	**1.0825**	**1.0991**	**1.7846**	**56.4014**	**5.00**	**0.6047**	**0.3491**	**1.1543**	**1.9981**	**57.7281**
2.00	**1.6364**	**0.9315**	**1.1181**	**1.8527**	**56.9275**	**5.25**	**0.5753**	**0.3321**	**1.1544**	**1.9986**	**57.7298**
2.25	**1.426**	**0.8163**	**1.1311**	**1.9019**	**57.2455**	**5.50**	**0.5487**	**0.3168**	**1.1545**	**1.9989**	**57.731**
2.50	**1.2644**	**0.7262**	**1.1395**	**1.9354**	**57.4344**	**5.75**	**0.5244**	**0.3028**	**1.1545**	**1.9992**	**57.732**
2.75	**1.1368**	**0.6542**	**1.1448**	**1.9574**	**57.5469**	**6.00**	**0.5023**	**0.29**	**1.1546**	**1.9994**	**57.7326**
3.00	**1.0333**	**0.5954**	**1.1482**	**1.9716**	**57.6147**	**6.25**	**0.4819**	**0.2782**	**1.1546**	**1.9995**	**57.7332**
3.25	**0.9478**	**0.5465**	**1.1504**	**1.9809**	**57.6564**	**6.50**	**0.4632**	**0.2674**	**1.1546**	**1.9996**	**57.7335**
3.50	**0.8758**	**0.5052**	**1.1518**	**1.9869**	**57.6825**	**6.75**	**0.4459**	**0.2574**	**1.1546**	**1.9997**	**57.7338**
3.75	**0.8144**	**0.4699**	**1.1527**	**1.9909**	**57.6991**	**7.00**	**0.4298**	**0.2481**	**1.1546**	**1.9997**	**57.7341**
4.00	**0.7612**	**0.4393**	**1.1533**	**1.9936**	**57.71**	**7.25**	**0.4149**	**0.2395**	**1.1547**	**1.9998**	**57.7342**
4.25	**0.7147**	**0.4125**	**1.1537**	**1.9954**	**57.7173**	**7.50**	**0.4009**	**0.2315**	**1.1547**	**1.9998**	**57.7344**

Table 1 shows the mean, standard deviation, skewness, excess kurtosis, and coefficient of variation of the ABOPLED distribution for θ values of 0.25-7.5 (step 0.25). The distribution is right-skewed and the table makes this clear regardless of the values of θ as all skewness values are positive. The tails of the proposed distribution are heavier than the tails of the normal distribution, which is demonstrated by the fact that all excess kurtosis values are positive. Furthermore, Table 1 shows how the mean and standard deviation increases as the value of α increases and how they decrease as θ increases. The coefficient of variation increases as the values of θ increase.

4.3. Moment generating function

Another way of deriving the moments is called the moment generating function (MGF). It is defined by the theorem described below.

Theorem 2 *Assume that random variable X follows the ABOPLED, then the moment generating function of X is given by*

$$M_X(t) = \frac{\theta^4\left(\theta^2(\theta-t)+3\right)}{(\theta-t)^4(\theta^3+3)}, \theta > t. \tag{17}$$

Proof 2

$$M_X(t) = \int_0^\infty e^{tx} f(x)dx = \int_0^\infty e^{tx}\frac{\theta^4 x^2\left(x+\theta^2\right)e^{-\theta x}}{2\theta^3+6}dx = \frac{\theta^4\left(\theta^2(\theta-t)+3\right)}{(\theta-t)^4(\theta^3+3)}$$

The r^{th} derivative of $M_X(t)$ at $t=0$ gives the r^{th} central moment of random variable X, $i.e.\ M^{(r)}(0) = E(X^r)$.

4.4. Mode

The mode is the most frequent value that occurs in data (Manikandan, 2011). When data occur equally, then there is no mode. On the other hand, some data sets can have more than one mode. This happens when the data set has two or more values of an equal frequency which is greater than that of any other value in its neighbourhood. The mode of the ABOPLED is given by equating the derivative of the pdf (6), or equivalently the logarithm of the pdf, with respect to x to zero. Thus, from (6), we have:

$$f'(x) = \frac{\theta^4}{2\theta^3+6}\left(-\theta\left(x^3+\theta^2x^2\right)+\left(3x^2+2\theta^2x\right)\right)e^{-\theta x}. \tag{18}$$

When equating (18) to 0, we receive

$$\begin{aligned} 0 &= -\theta x^3+(3-\theta^3)x^2+2\theta^2x \\ x &= \frac{\theta^3-3-\sqrt{\theta^6+2\theta^3+9}}{-2\theta}. \end{aligned} \tag{19}$$

Hence, we have one mode only assured by the plot of (19) in Figure 2f. Figure 2f shows the plot of Equation (19). It indicates that the equation has only one solution regardless of the value of θ.

5. Reliability analysis of the ABOPLED

This section introduces the reliability, hazard rate, cumulative hazard rate, reversed hazard rate, and odds rate functions for the ABOPLED, as well as an explanation of their shapes for various values of the distribution parameters.

The reliability function of the ABOPLED can be calculated using (7):

$$R(t)=1-F(t) = \frac{\left(\theta^3t^3+\left(\theta^5+3\theta^2\right)t^2+\left(2\theta^4+6\theta\right)t+2\theta^3+6\right)e^{-\theta t}}{2\theta^3+6}. \quad (20)$$

The hazard rate function of ABOPLED, is defined using (6) and (20):

$$h(t)=\frac{f(t)}{1-F(t)} = \frac{\theta^4t^2\left(t+\theta^2\right)}{\left(\theta^3t^3+\left(\theta^5+3\theta^2\right)t^2+\left(2\theta^4+6\theta\right)t+2\theta^3+6\right)}.$$

Using (20), the cumulative hazard rate function of is defined as:

$$H(t)=-ln\left(R(t)\right) = -ln\left(\frac{\left(\theta^3t^3+\left(\theta^5+3\theta^2\right)t^2+\left(2\theta^4+6\theta\right)t+2\theta^3+6\right)e^{-\theta t}}{2\theta^3+6}\right).$$

The reversed hazard rate function is defined using (6), and (7):

$$RH(t)=\frac{f(t)}{F(t)} = \frac{\theta^4t^2\left(t+\theta^2\right)e^{-\theta t}}{2\theta^3+6-\left(\left(\theta^3t^3+\left(\theta^5+3\theta^2\right)t^2+\left(2\theta^4+6\theta\right)t+2\theta^3+6\right)e^{-\theta t}\right)}.$$

The odds rate function is defined using (7), and (20):

$$O(t) = \frac{F(t)}{1-F(t)}=\frac{2\theta^3+6-\left(\left(\theta^3t^3+\left(\theta^5+3\theta^2\right)t^2+\left(2\theta^4+6\theta\right)t+2\theta^3+6\right)e^{-\theta t}\right)}{\left(\left(\theta^3t^3+\left(\theta^5+3\theta^2\right)t^2+\left(2\theta^4+6\theta\right)t+2\theta^3+6\right)e^{-\theta t}\right)}.$$

Figure 2 shows the reliability analysis functions of ABOPLED for θ values of 1.5-9.5 (step1). Figure 2a shows the plot of the reliability function. Figure 2b represents the plot of the hazard rate function. The reversed hazard rate function is presented in Figure 2c. Whereas the cumulative hazard rate function is presented in Figure 2d. The odds rate function plot is shown in Figure 2e.

6. Some structural and statistical properties

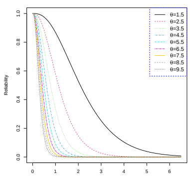
6.1. Order statistics

Let $X_{(1)},X_{(2)},\cdots,X_{(n)}$ be the order statistics of the random sample $X_1,X_2,\cdots,X_n$ obtained from ABOPLED, then the pdf of the k^{th} order statistics is:

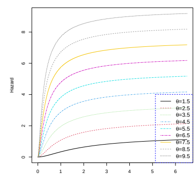
$$f_{(k)}(x) = \frac{n!}{(k-1)!(n-k)!}[F(x)]^{k-1}[1-F(x)]^{n-k}f(x)$$

$$= \frac{n!}{(k-1)!(n-k)!}\left(1-\left(\frac{\theta^3x^3}{2\theta^3+6}+\frac{\theta^2x^2}{2}+\theta x+1\right)e^{-\theta x}\right)^{k-1}$$
$$\times\left(\left(\frac{\theta^3x^3}{2\theta^3+6}+\frac{\theta^2x^2}{2}+\theta x+1\right)e^{-\theta x}\right)^{n-k}\left[\frac{\theta^4x^2\left(x+\theta^2\right)e^{-\theta x}}{2\theta^3+6}\right]. \quad (21)$$

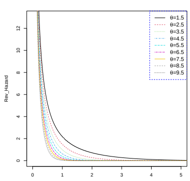
The minimum and maximum order statistics of ABOPLED can be found by replacing $k=1$ and $k=n$, respectively, in (21). As a result, we obtain what follows:



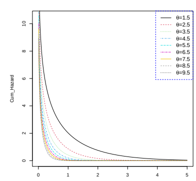
(a) The reliability function.



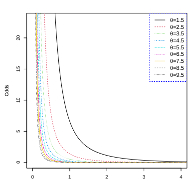
(b) The hazard rate function.



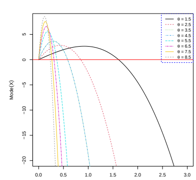
(c) The reversed hazard rate function.



(d) The cumulative hazard rate function.



(e) The odds rate.



(f) The plot of (19)

Figure 2. Reliability analysis functions of ABOPLED for different θ values and the plot of (19).

$$f_{(1)}(x) = n\left(\left(\frac{\theta^3x^3}{2\theta^3+6}+\frac{\theta^2x^2}{2}+\theta x+1\right)e^{-\theta x}\right)^{n-1}\left[\frac{\theta^4x^2\left(x+\theta^2\right)e^{-\theta x}}{2\theta^3+6}\right] \quad (22)$$

$$f_{(n)}(x) = n\left(1-\left(\frac{\theta^3x^3}{2\theta^3+6}+\frac{\theta^2x^2}{2}+\theta x+1\right)e^{-\theta x}\right)^{n-1}\left[\frac{\theta^4x^2\left(x+\theta^2\right)e^{-\theta x}}{2\theta^3+6}\right] \quad (23)$$

6.2. Quartiles

Quartiles are special cases of quantiles. The first quartile (Q1) corresponds to 25% of the values that are below a specific value in the distribution, while the second quartile (Q2) corresponds to 50% of the values that are below a specific value in the distribution. It represents the median of the distribution. The third quartile (Q3) corresponds to 75% of the values that are below a specific value in the distribution. The quantile function can be obtained by finding the inverse of (7), so it can be obtained as:

$$\begin{aligned} q &= F(x_q) = 1 - \frac{\left(\theta^3 x_q^3 + \left(\theta^5 + 3\theta^2\right) x_q^2 + \left(2\theta^4 + 6\theta\right) x_q + 2\theta^3 + 6\right) e^{-\theta x_q}}{2\theta^3 + 6} \\ 1 - q &= \frac{\left(\theta^3 x_q^3 + \left(\theta^5 + 3\theta^2\right) x_q^2 + \left(2\theta^4 + 6\theta\right) x_q + 2\theta^3 + 6\right) e^{-\theta x_q}}{2\theta^3 + 6}, \end{aligned} \tag{24}$$

where q is a random variable following the uniform distribution, *i.e.* $q \in (0,1)$. For $q = 0.5$, we obtain the median of the distribution. Equation (24) does not have an explicit solution. Figure 3 shows the plot of this equation for θ = 1.5 and q values of 0.05, 0.1, 0.25, 0.4, 0.5, 0.6, 0.75, 0.8 and 0.95. It shows that (3b) it has exactly one solution.

(a) The order statistics plot **(b)** The quantile function

Figure 3. The pdf of order statistics for θ =2.5 and n = 10 and the quantile function when θ =2.5.

6.3. Mean waiting time

Significantly, the mean waiting time is the measure used to verify the effectiveness of the service. The form, which is most effective in decreasing waiting time, can be determined by comparing many different services patterns. Pollaczek (1957) introduced a formula for the mean waiting time in a G/G/1 queue. Rosenberg (1968) introduced the mean waiting time to measure the effectiveness of the service because it is the easiest property of the waiting time distribution to calculate. Otsuka et al. (2010) proposed a theoretical analysis of the mean waiting time for message delivery in lattice ad hoc networks. Romero-Silva and Hurtado (2017) studied the difference in mean waiting times between two classes of customers in a single-server FIFO queue. For the ABOPLED, the mean waiting time can be written as:

$$
\begin{aligned}
mw(t) &= \frac{1}{F(t)}\int_0^t F(x)dx \\
&= \frac{\begin{pmatrix} \left(\left(2\theta^4+6\theta\right)t-6\theta^3-24\right)e^{\theta t}+\theta^3t^3+\left(\theta^5+6\theta^2\right)t^2 \\ +\left(4\theta^4+18\theta\right)t+6\theta^3+24 \end{pmatrix}}{\theta\left(2\theta^3+6-\left(\theta^3t^3+\left(\theta^5+3\theta^2\right)t^2+\left(2\theta^4+6\theta\right)t+2\theta^3+6\right)\right)}. \qquad (25)
\end{aligned}
$$

6.4. Mean residual life time

Several studies have attempted to find substitutes that do not depend on the entire right tail of the pdf, such as mean residual lifetime and median residual life, and the corresponding residual life quantiles. Several studies have addressed this issue. First, Schmittlein and Morrison (1981) introduced the median residual lifetime and characterisation theorem along with its application. Joe and Proschan (1984a) and Joe and Proschan (1984b) examined the comparison of two life distributions based on their percentile residual life functions. Lillo (2005) studied the median residual lifetime and its properties. Jeong et al. (2008) investigated the nonparametric inference on median residual life function. Recently, Zamanzade et al. (2024) analysed the estimation of the mean residual life based on ranked set sampling. For the ABOPLED, the mean residual lifetime can be written as"

$$
\begin{aligned}
MR(t) &= \frac{1}{\overline{F}(t)}\int_t^\infty \overline{F}(x)dx = \frac{2\theta^3+6}{\left(\theta^3t^3+\left(\theta^5+3\theta^2\right)t^2+\left(2\theta^4+6\theta\right)t+2\theta^3+6\right)e^{-\theta t}} \\
&\quad \times \int_t^\infty \frac{\left(\theta^3x^3+\left(\theta^5+3\theta^2\right)x^2+\left(2\theta^4+6\theta\right)x+2\theta^3+6\right)e^{-\theta x}}{2\theta^3+6}dx \\
&= \frac{2\theta^3+6}{\left(\theta^3t^3+\left(\theta^5+3\theta^2\right)t^2+\left(2\theta^4+6\theta\right)t+2\theta^3+6\right)e^{-\theta t}} \\
&\quad \times \frac{\left(\theta^2t^3+(\theta(\theta^3+6)t^2+(4\theta^3+18)t+6\theta^2+\frac{24}{\theta}\right)e^{-\theta t}}{2\theta^3+6} \\
&= \frac{\left(\theta^2t^3+(\theta(\theta^3+6)t^2+(4\theta^3+18)t+6\theta^2+\frac{24}{\theta}\right)}{\left(\theta^3t^3+\left(\theta^5+3\theta^2\right)t^2+\left(2\theta^4+6\theta\right)t+2\theta^3+6\right)}. \qquad (26)
\end{aligned}
$$

6.5. Entropy

Olbryś and Ostrowski (2021) introduced introduced a new procedure for the measurement of stock market depth and market liquidity. An algorithm inferring the initiator of a trade supports the proposed Shannon entropy-based market depth indicator. The findings of the empirical experiments for real high-frequency data specify that this new entropy-based method can be considered as a good market depth and liquidity proxy with an intuitive base for both theoretical and the empirical analyses in financial markets. In this section, we have derived the theoretical entropies with some numerical results.

The Shannon (Shannon, 1948) (S_ρ), Rényi (R_ρ) (Rényi, 1961) and Tsallis (T_ρ) (Tsallis, 1988) entropies of the ABOPLED are defined as:

$$\begin{aligned} S_\rho &= -\int_0^\infty f(x)\log(f(x))dx \\ &= -\int_0^\infty \left(\frac{(\theta^4)(\theta^2+x)x^2}{(2\theta^3+6)}e^{-\theta x}\right)\log\left(\frac{(\theta^4)(\theta^2+x)x^2}{(2\theta^3+6)}e^{-\theta x}\right)dx \end{aligned} \tag{27}$$

$$\begin{aligned} R_\rho &= \frac{1}{1-\rho}\log\left(\int_0^\infty (f(x))^\rho dx\right),\ \rho>0,\ \rho\neq 0 \\ &= \frac{1}{1-\rho}\log\left(\int_0^\infty \left(\frac{(\theta^4)(\theta^2+x)x^2}{(2\theta^3+6)}e^{-\theta x}\right)^\rho dx\right) \\ &= \frac{1}{1-\rho}\log\left(\left(\frac{(\theta^4)}{(2\theta^3+6)}\right)^\rho \int_0^\infty \sum_{k=0}^{\rho}\binom{\rho}{k}\theta^{2k}x^{3\rho-k}e^{-\rho\theta x}dx\right) \\ &= \frac{1}{1-\rho}\log\left(\left(\frac{(\theta^4)}{(2\theta^3+6)}\right)^\rho \sum_{k=0}^{\rho}\binom{\rho}{k}\frac{(3\rho-k)!}{\rho^{3\rho-k+1}\theta^{3\rho+k+1}}\right) \end{aligned} \tag{28}$$

$$\begin{aligned} T_\rho &= \frac{1}{\rho-1}\left(1-\int_0^\infty (f(x))^\rho dx\right),\ \rho>0,\ \rho\neq 0 \\ &= \frac{1}{\rho-1}\left(1-\left(\left(\frac{(\theta^4)}{(2\theta^3+6)}\right)^\rho \sum_{k=0}^{\rho}\binom{\rho}{k}\frac{(3\rho-k)!}{\rho^{3\rho-k+1}\theta^{3\rho+k+1}}\right)\right). \end{aligned} \tag{29}$$

Table 2 shows the numerical results for Shannon, Rényi and Tsallis entropies for ABOPLED using different values of θ of 0.05-4.25 (step(0.1)) when ρ = 5. It also shows the mean waiting time and the mean residual life time. The table clarifies that as the values of θ increase, all entropy values decrease. It shows the values of the mean waiting time and the mean residual life time for the same values of θ. The mean waiting time values rise as the values of θ grow. On the other hand, the values of the mean residual life time decrease as the values of θ increase.

Table 2. Numerical results for Shannon, Rényi and Tsallis entropies, the mean waiting time and mean residual life time for ABOPLED using different θ values with ρ=5.

θ	S_ρ	R_ρ	T_ρ	*mw*	*MR*	θ	S_ρ	R_ρ	T_ρ	*mw*	*MR*
0.05	**3.477**	**4.704**	**0.250**	**0.010**	**71.213**	**2.25**	**1.099**	**0.771**	**0.239**	**1.081**	**0.653**
0.25	**3.410**	**3.089**	**0.250**	**0.054**	**15.729**	**2.45**	**1.003**	**0.674**	**0.233**	**1.300**	**0.569**
0.45	**2.822**	**2.501**	**0.250**	**0.101**	**8.374**	**2.65**	**0.916**	**0.586**	**0.226**	**1.540**	**0.503**
0.65	**2.453**	**2.132**	**0.250**	**0.154**	**5.384**	**2.85**	**0.836**	**0.506**	**0.217**	**1.792**	**0.451**
0.85	**2.180**	**1.860**	**0.250**	**0.214**	**3.701**	**3.05**	**0.763**	**0.432**	**0.206**	**2.050**	**0.409**
1.05	**1.958**	**1.639**	**0.250**	**0.283**	**2.630**	**3.25**	**0.695**	**0.364**	**0.192**	**2.308**	**0.374**
1.25	**1.769**	**1.449**	**0.249**	**0.365**	**1.924**	**3.45**	**0.631**	**0.300**	**0.175**	**2.563**	**0.345**
1.45	**1.603**	**1.281**	**0.249**	**0.462**	**1.454**	**3.65**	**0.572**	**0.241**	**0.155**	**2.813**	**0.320**
1.65	**1.455**	**1.132**	**0.247**	**0.579**	**1.138**	**3.85**	**0.516**	**0.185**	**0.131**	**3.058**	**0.299**
1.85	**1.323**	**0.998**	**0.245**	**0.719**	**0.920**	**4.05**	**0.463**	**0.132**	**0.102**	**3.299**	**0.281**
2.05	**1.205**	**0.878**	**0.243**	**0.886**	**0.766**	**4.25**	**0.413**	**0.082**	**0.070**	**3.535**	**0.264**

7. Parameters estimation

7.1. Maximum likelihood method

Let $X_1, X_2, \cdots, X_n$ be a random sample from ABOPLED, where $x_1, x_2, \cdots, x_n$ are the observed values of the random sample. The likelihood function is:

$$L = \prod_{i=1}^{n}\left[\frac{\theta^4 x_i^2\left(x_i+\theta^2\right)e^{-\theta x_i}}{2\theta^3+6}\right] = \left[\frac{\theta^4}{2\theta^3+6}\right]^n\left(\prod_{i=1}^{n} x_i^2\right)\left(\prod_{i=1}^{n}(x_i+\theta^2)\right)\left(e^{-\theta\sum_{i=0}^{n} x_i}\right)$$

Thus, the log-likelihood function is:

$$\ell = ln(L) = 4nln(\theta) - nln(2\theta^3+6) + \sum_{i=0}^{n} ln\left(x_i^2\right) + \sum_{i=0}^{n} ln\left((x_i+\theta^2)\right) - \theta\sum_{i=0}^{n} x_i \quad (30)$$

With respect to θ, we receive

$$\frac{\partial\ell}{\partial\theta} = \frac{4n}{\theta} - \frac{6n\theta^2}{2\theta^3+6} + \sum_{i=1}^{n}\frac{2\theta}{x_i+\theta^2} - \sum_{i=0}^{n} x_i.$$

Nonlinear equation $\frac{\partial\ell}{\partial\theta} = 0$ can be solved numerically as there is no explicit solution, and the maximum likelihood estimate (MLE) of θ is this solution.

7.2. Least square methods

This subsection describes the least squares methods for estimating the ABOPLED parameters. These methods were summarised by Swain et al. (1988) as follows: Let $X_{(1)}, X_{(2)}, \cdots, X_{(n)}$ be the order statistics of random sample $X_1, X_2, \cdots, X_n$ obtained from ABOPLED with the observed ordered values of $x_{(1)}, x_{(2)}, \cdots, x_{(n)}$. The ordinary least squares (OLS) method is defined as:

$$R_{OLS} = \sum_{i=1}^{n}\left[F(x_{(i)}) - \frac{i}{n+1}\right]^2$$

$$= \sum_{i=1}^{n}\left[\frac{n+1-i}{n+1} - \left(\frac{\theta^3 x_{(i)}^3}{2\theta^3+6} + \frac{\theta^2 x_{(i)}^2}{2} + \theta x_{(i)} + 1\right)e^{-\theta x_{(i)}}\right]^2. \quad (31)$$

Thus, the OLS estimator of θ is the solution of $\frac{\partial R_{OLS}}{\partial\theta} = 0$.

The weighted least squares (WLS) estimate can be determined as:

$$W_{WLS} = \sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n+1-i)}\left[1 - \left(\frac{\theta^3 x_{(i)}^3}{2\theta^3+6} + \frac{\theta^2 x_{(i)}^2}{2} + \theta x_{(i)} + 1\right)e^{-\theta x_{(i)}} - \frac{i}{n+1}\right]^2. \quad (32)$$

Again, the WLS estimators of θ is the solution of $\frac{\partial W_{WLS}}{\partial\theta} = 0$.

7.3. Maximum product spacings method

The maximum product of spacings (MPS) (Cheng and Amin, 1983) estimator, $\hat{\theta}_{MPS}$, of θ can be obtained using

$$\begin{aligned} NL(\alpha,\beta|x) &= \frac{1}{n+1}\sum_{i=1}^{n+1} ln\left[F(x_{(i)};\alpha,\beta)-F(x_{(i-1)};\alpha,\beta)\right] \\ &= \frac{1}{n+1}\sum_{i=1}^{n+1} ln\left[\left[1-\left(\frac{\theta^3x_{(i)}^3}{2\theta^3+6}+\frac{\theta^2x_{(i)}^2}{2}+\theta x_{(i)}+1\right)e^{-\theta x_{(i)}}\right]\right. \\ &\quad \left.-\left[1-\left(\frac{\theta^3x_{(i-1)}^3}{2\theta^3+6}+\frac{\theta^2x_{(i-1)}^2}{2}+\theta x_{(i-1)}+1\right)e^{-\theta x_{(i-1)}}\right]\right]. \qquad (33) \end{aligned}$$

$\hat{\theta}_{MPS}$ can be obtained by solving nonlinear equation $\frac{\partial NL(\theta|x)}{\partial\theta}=0$ with respect to θ parameter.

7.4. Cramer-Von Mises and Anderson-Darling methods

The Cramer-Von Mises (CVM) method (Cramér, 1928 and Von Mises, 1928) for estimating ABOPLED parameters is defined as:

$$\begin{aligned} CVM &= \frac{1}{12n}+\sum_{i=1}^{n}\left[F(x_{(i)},\theta)-\frac{2i-1}{2n}\right]^2 \\ &= \frac{1}{12n}+\sum_{i=1}^{n}\left[1-\left(\frac{\theta^3x_{(i)}^3}{2\theta^3+6}+\frac{\theta^2x_{(i)}^2}{2}+\theta x_{(i)}+1\right)e^{-\theta x_{(i)}}-\frac{2i-1}{2n}\right]^2. \qquad (34) \end{aligned}$$

The estimator is the solution of the following system of nonlinear equation $\frac{\partial CVM}{\partial\theta}=0$.

The Anderson-Darling (AD) estimators (Anderson, 1962) of θ can be obtained as:

$$AD=-n-\frac{1}{n}\sum_{i=1}^{n}(2i-1)\left\{ln[F(x_{(i)};\alpha,\beta\theta)]+ln\overline{F}(x_{(n+1-i)};\alpha,\beta)\right\}$$

$$=-n-\sum_{i=1}^{n}\frac{(2i-1)}{n}\left[\begin{array}{c} ln\left(1-\left(\frac{\theta^3x_{(i)}^3}{2\theta^3+6}+\frac{\theta^2x_{(i)}^2}{2}+\theta x_{(i)}+1\right)e^{-\theta x_{(i)}}\right) \\ +ln\left(\left(\frac{\theta^3x_{(n+1-i)}^3}{2\theta^3+6}+\frac{\theta^2x_{(n+1-i)}^2}{2}+\theta x_{(n+1-i)}+1\right)e^{-\theta x_{(n+1-i)}}\right) \end{array}\right]. \qquad (35)$$

The AD estimator of θ is the solution of the non-linear equation $\frac{\partial AD}{\partial\theta}=0$.

The estimator presented in this section will be estimated using a simulation study in the next part of the article.

8. Simulation study

A simulation study is conducted to examine the efficiency of the estimators used and the precision of the methods applied to estimate the ABOPLED parameters is discussed in Section 7.

The following algorithm is used to estimate the distribution parameters:

- A Monte Carlo simulation study is carried out using different sample sizes: n = 30, 50, 100, 200, 300 and 500 to assess the performance of the ABOPLED via the *R* package (R Core Team, 2021)
- 1,000 samples are simulated using the true parameters values θ =2.
- The MLE, OLS, WLS, MPS, CVM and AD estimators are obtained through the non-linear equations by maximising or minimising Equations (30), (31), (32), (33), (34), (35), respectively, as required by the method with respect to θ.
- The AB and MSEs of all estimates are calculated.
- For each sample, the estimates of the parameter θ, MSE and the bias are obtained. Then, we calculate the AB and the MSE as follows: $AB(\hat{\theta}) = \frac{1}{N}\sum_{i=1}^{N}(\hat{\theta}-\theta),\ MSE = \frac{1}{N}\sum_{i=1}^{N}(\hat{\theta}-\theta)^2$. The results of this simulation are summarised in Table 3.

Table 3 shows the estimate of θ and its AB and MSE. It shows that these values decrease as the sample size increases, indicating that the estimate behaves consistently for $\hat{\theta}$, therefore it is unbiased and consistent. Based on AB and MSE, we recommend using the MLE method to estimate the parameter.

Table 3. AB and MSE for estimated ABOPLED parameters

Method	n	$\hat{\theta}$	AB	MSE	Method	n	$\hat{\theta}$	AB	MSE
MLE	30	2.0047	0.0047	0.0035	MLE	200	2.0017	0.0017	0.0006
OLS		2.1161	0.1161	0.2267	OLS		2.0621	0.0621	0.0313
WLS		2.0868	0.0868	0.2594	WLS		2.0186	0.0186	0.0305
CVM		2.2910	0.2910	0.3560	CVM		2.0310	0.0310	0.0860
MPS		2.4773	0.4773	0.7012	MPS		2.1164	0.1164	0.3551
AD		2.1014	0.1014	0.3807	AD		2.0301	0.0301	0.0860
MLE	50	2.0025	0.0025	0.0021	MLE	300	2.0016	0.0016	0.0003
OLS		2.0852	0.0852	0.1282	OLS		2.0577	0.0577	0.0222
WLS		2.0489	0.0489	0.1364	WLS		2.0145	0.0145	0.0207
CVM		2.0516	0.0516	0.2017	CVM		2.0305	0.0305	0.0746
MPS		2.2317	0.2317	0.5507	MPS		2.1148	0.1148	0.3138
AD		2.0516	0.0516	0.2017	AD		2.0250	0.0250	0.0746
MLE	100	2.0022	0.0022	0.0010	MLE	500	2.0008	0.0008	0.0002
OLS		2.0750	0.0750	0.0612	OLS		2.0568	0.0568	0.0137
WLS		2.0363	0.0363	0.0641	WLS		2.0134	0.0134	0.0113
CVM		2.0442	0.0442	0.1297	CVM		2.0299	0.0299	0.0660
MPS		2.1517	0.1517	0.5069	MPS		2.1066	0.1066	0.3011
AD		2.0452	0.0452	0.2017	AD		2.0130	0.0130	0.0660

9. Real data applications

This section compares the proposed distribution's goodness of fit to a few other existing distributions in order to demonstrate its flexibility based on three financial real data sets.

The data were collected from companies listed at the ASE. The sample selection criteria depended on data availability for three years: 2021, 2022, 2023. The population comprised of all companies listed at the ASE, i.e. a total of 203 companies, the final sample consisted of 92 companies, excluding those suspended from trading, with incomplete data and those that incurred losses during the study periods. The total number of observations of EPS and ROE was 273 each.

Earnings per share is the amount of income earned on a share of common stock during an accounting period. It is calculated by dividing the profit or the loss attributable to ordinary equity holders of the parent entity (the numerator) by the weighted average number of ordinary shares outstanding (the denominator) during the period (IAS, 2023). The ratio indicated the company's ability to produce a profit for common shareholders. It is widely used by analysts and other external users of financial statements, as well as by management.

The following distributions are used for this comparison:

- Gharaibeh distribution (Gharaibeh, 2021):

$$f(x) = \frac{\theta^6}{120(\theta^6+\theta^4+\theta^2+1)}\left(x^5+20x^3+120x+120\theta\right)e^{-\theta x}; x>0,\, \theta>0;$$

- Exponential distribution (Exp) (Kingman, 1982):

$$f(x) = \alpha e^{-\theta x},\ x>0,\ \theta>0;$$

- Lindley distribution (Ghitany et al., 2008):

$$f(x) = \frac{\theta^2(1+x)e^{-\theta x}}{1+\theta}; x>0, \theta>0;$$

- Length bias Benrabia distribution (LBBD) (Almakhareez and Alzoubi, 2024a):

$$f_l(x;\alpha,\beta) = \frac{\beta(\alpha\Gamma(\alpha-1)\beta x+\beta^{\alpha}x^{\alpha-1})e^{-\beta x}}{(\alpha-\beta+\alpha\beta)\Gamma(\alpha-1)}, x>0, \alpha>1, \beta>0;$$

- Area bias Benrabia distribution (ABBD) (Almakhareez and Alzoubi, 2024b):

$$f_a(x;\alpha,\beta) = \frac{(\alpha\Gamma(\alpha-1)\beta^3x^2+\beta^{\alpha+2}x^{\alpha})e^{-\beta x}}{(\alpha^2\beta-\alpha\beta+2\alpha)\Gamma(\alpha-1)};\ x,\beta>0, \alpha>1;$$

- Karam distribution (Gharaibeh and Sahtout, 2022):

$$f(x)=\frac{\theta^6\left(x^5+x^4+x^2+1\right)}{\theta^5+2\theta^3+24\theta+120}e^{-\theta x};\ x>0;\ \theta>0;$$

- OPLED (See (4));
- size-biased Ishita distribution (SBID) (Al-Omari et al., 2019b):

$$f_s(x)=\frac{\theta^4}{\theta^3+6}x(\theta+x^2)e^{-\theta x},\ x>0,\ \theta>0.$$

Tables 4-6 show the data sets used in this section. They show the financial ratios per year for financial, industry and service sectors during the 2021-2023 period.

Table 4. Dataset I: EPS for the financial sector during the 2021-2023 period

Company Name	**Financial Ratios/Year**		
	2023	**2022**	**2021**
JORDAN ISLAMIC BANK	**0.31151**	**0.30555**	**0.29529**
SAFWA ISLAMIC BANK	**0.1751**	**0.15112**	**0.1406**
ISLAMIC INTERNATIONAL ARAB BANK	**0.35326**	**0.35497**	**0.4**
JORDAN KUWAIT BANK	**0.39405**	**0.12455**	**0.05159**
JORDAN COMMERCIAL BANK	**0.09571**	**0.0945**	**0.05837**
THE HOUSING BANK FOR TRADE AND FINANCE	**0.43406**	**0.41111**	**0.33531**
ARAB JORDAN INVESTMENT BANK	**0.12384**	**0.12039**	**0.11394**
BANK AL ETIHAD	**0.23569**	**0.21455**	**0.20312**
ARAB BANKING CORPORATION /(JORDAN)	**0.04546**	**0.06063**	**0.08729**
INVEST BANK	**0.24629**	**0.19826**	**0.17812**
CAPITAL BANK OF JORDAN	**0.27317**	**0.33007**	**0.39407**
CAIRO AMMAN BANK	**0.18571**	**0.18218**	**0.17263**
BANK OF JORDAN	**0.22012**	**0.2007**	**0.18004**
JORDAN AHLI BANK	**0.09266**	**0.08422**	**0.07092**
ARAB BANK	**0.58648**	**0.51113**	**0.2436**
MIDDLE EAST INSURANCE	**0.03929**	**0.10864**	**0.07403**
AL-NISR AL-ARABI INSURANCE	**0.63802**	**0.231**	**0.29263**
JORDAN INSURANCE	**0.06033**	**0.00036**	**0.02921**
DELTA INSURANCE	**0.10801**	**0.0777**	**0.01752**
JERUSALEM INSURANCE	**0.18775**	**0.20698**	**0.16436**
THE UNITED INSURANCE	**0.21804**	**0.1675**	**0.16527**
GULF INSURANCE GROUP/ JORDAN	**0.36495**	**0.28091**	**0.25762**
NATIONAL INSURANCE	**0.16729**	**0.13781**	**0.11505**
EURO ARAB INSURANCE GROUP	**0.19688**	**0.19041**	**0.11778**
THE MEDITERRANEAN & GULF INSURANCE COMPANY-JORDAN P.L.C	**0.0208**	**0.03486**	**0.00737**
FIRST INSURANCE	**0.10379**	**0.07202**	**0.07**
THE ISLAMIC INSURANCE	**0.11616**	**0.11066**	**0.13**
AL-AMAL FINANCIAL INVESTMENTS	**0.01476**	**0.02963**	**0.09023**
BABELON INVESTMENTS	**0.07179**	**0.0246**	**0.00509**
DARAT JORDAN HOLDINGS	**0.04654**	**0.01134**	**0.10751**

FIRST FINANCE	**0.03256**	**0.04269**	**0.02772**
INMA INVESTMENT AND FINANCIAL ADVANCES	**0.03367**	**0.04155**	**0.02566**
JORDAN LOAN GUARANTEE CORPORATION	**0.06282**	**0.03956**	**0.03886**
JORDAN MORTGAGE REFINANCE	**0.3677**	**0.35538**	**0.43181**
JORDANIAN MANAGEMENT AND CONSULTING COMPANY	**0.29049**	**0.17916**	**0.19246**
AD-DULAYL INDUSTRIAL PARK & REAL ESTATE COMPANY P.L.C	**0.06236**	**0.05714**	**0.04944**
AL-TAJAMOUAT FOR TOURISTIC PROJECTS CO PLC	**0.02698**	**0.02374**	**0.00992**
AMAD INVESTMENT & REAL ESTATE DEVELOPMENT	**0.00437**	**0.07656**	**0.01058**
CONTEMPRO FOR HOUSING PROJECTS	**0.00308**	**0.01992**	**0.0209**
JORDAN MASAKEN FOR LAND & INDUSTRIAL DEVELOP-MENT PROJECTS	**0.0148**	**0.00919**	**0.00289**
NOOR CAPITAL MARKTS FOR DIVERSIFIED INVESTMENTS	**0.12639**	**0.19163**	**0.317**
THE PROFESSIONAL COMPANY FOR REAL ESTATE INVEST-MENT AND HOUSING	**0.02423**	**0.0337**	**0.02976**
THE REAL ESTATE & INVESTMENT PORTFOLIO CO.	**0.00286**	**0.05566**	**0.02754**
NOOR ASSETS MANAGEMENT AND LEASING CO.	**0.2158**	**0.13914**	**0.12803**

Table 5. Dataset II: EPS for the industry sector during the 2021-2023 period

Company Name	**Financial Ratios/Year**		
	2023	**2022**	**2021**
THE INDUSTRIAL COMMERCIAL & AGRICULTURAL	**0.05789**	**0.03362**	**0.06274**
THE ARAB PESTICIDES & VETERINARY DRUGS MFG. CO.	**0.27107**	**0.26055**	**0.23649**
JORDAN CHEMICAL INDUSTRIES	**0.02470**	**0.14547**	**0.05689**
UNITED CABLE INDUSTRIES	**0.02930**	**0.01857**	**0.01208**
READY MIX CONCRTE AND CONSTRUCTION SUPPLIES	**0.17116**	**0.07234**	**0.02440**
ARABIAN STEEL PIPES MANUFACTURING	**0.13898**	**0.09263**	**0.06792**
AL-QUDS READY MIX	**0.15671**	**0.01634**	**0.02434**
ASSAS FOR CONCRETE PRODUCTS CO. LTD	**0.05061**	**0.07530**	**0.04958**
JORDAN DAIRY	**0.16668**	**0.05058**	**0.08475**
GENERAL INVESTMENT	**0.21088**	**0.24084**	**0.20878**
UNIVERSAL MODERN INDUSTRIES	**0.07028**	**0.11759**	**0.13885**
NUTRI DAR	**0.00842**	**0.04065**	**0.02242**
JORDAN VEGETABLE OIL INDUSTRIES	**0.35863**	**0.25515**	**0.25914**
SINIORA FOOD INDUSTRIES PLC	**0.15817**	**0.19213**	**0.27924**
ARAB ALUMINIUM INDUSTRY /ARAL	**0.00100**	**0.06988**	**0.12114**
JORDAN PHOSPHATE MINES	**1.80000**	**8.67159**	**4.05965**
NORTHERN CEMENT CO.	**0.07000**	**0.11382**	**0.13499**
THE ARAB POTASH	**3.51000**	**7.21611**	**2.60108**
INVESTMENTS & INTEGRATED INDUSTRIES CO. PLC (HOLD-ING CO)	**0.04000**	**0.03953**	**0.01875**
DAR AL DAWA DEVELOPMENT & INVESTMENT	**0.09033**	**0.06637**	**0.03005**
HAYAT PHARMACEUTICAL INDUSTRIES CO.	**0.24502**	**0.37144**	**0.38339**
PHILADELPHIA PHARMACEEUTICALS	**0.10763**	**0.09348**	**0.05470**
THE JORDAN WORSTED MILLS	**0.12776**	**0.14262**	**0.09987**

Table 6. Dataset III: EPS for the service sector during the 2021-2023 period

Company Name	Financial Ratios/Year		
	2023	2022	2021
BINDAR TRADING & INVESTMENT	0.262608	0.40806	0.170304
COMPREHENSIVE LEASING COMPANY	0.243656	0.22314	0.388221
JORDAN INTERNATIONAL TRADING CENTER	0.07173	0.093165	0.057167
JORDAN TRADE FACILITIES	0.374112	0.242446	0.238253
JORDANIAN DUTY FREE SHOPS	0.336582	0.375173	0.00718
AL-ISRA FOR EDUCATION AND INVESTMENT	0.205352	0.332692	0.263241
AL-ZARQA EDUCATIONAL & INVESTMENT	0.046396	0.039709	0.033073
PETRA EDUCATION COMPANY	0.140622	0.167452	0.230803
PHILADELPHIA INTERNATIONAL EDUCATIONAL INVESTMENT	0.11588	0.005117	0.019367
THE ARAB INTERNATIONL FOR EDUCATION & INVESTMENT	0.118594	0.056378	0.108322
JORDAN PETROLEUM REFINERY	1.226189	0.961974	0.520464
AFAQ FOR ENERGY CO. P.L.C	4.914741	3.491291	0.211068
NATIONAL PETROULEUM	2.495661	5.240506	0.378521
THE CONSULTANT & INVESTMENT GROUP	0.041827	0.044151	0.041361
ARAB INTERNATIONAL HOTELS	0.026987	0.008047	0.004793
AL-FARIS NATIONAL COMPANY FOR INVESTMENT & EXPORT	0.009391	0.0315	0.044468
JORDAN TELECOM	0.243991	0.234832	0.13933
JORDAN NATIONAL SHIPPING LINES	0.078482	0.257447	0.193284
SALAM INTERNATIONL TRANSPORT & TRADING	0.013732	0.117071	0.076555
TRUST INTERNATIONAL TRANSPORT	0.080481	0.007918	0.013924
IRBID DISTRICT ELECTRICITY	0.657753	2.070165	0.690224
JORDAN ELECTRIC POWER	0.187703	0.170691	0.114224
ELECTRICITY DISTRIBUTION	0.507459	1.578454	0.618777
CENTRAL ELECTRICITY GENERATING	4.424406	0.822969	0.398052

Table 7 shows the summary of the three data sets used in this study.

Table 7. Summary of the datasets

Dataset	Min.	1st Qu	Median	Mean	3rd Qu	Max.
I	0.00036	0.03950	0.11449	0.14826	0.20887	0.63802
II	0.00100	0.05058	0.09987	0.51149	0.21088	8.67159
III	0.004793	0.053883	0.190493	0.538412	0.380946	5.240506

Tables 8 - 10 show that the suggested distribution has the lowest values of $-ln(L)$, AIC, CAIC, BIC, HQIC and KS with the highest p-value. Therefore, the suggested distribution is preferred over the competence distributions. The 95% CIs of the parameter θ are calculated in these tables.

Table 8. Application I

Distribution	$-ln(L)$	AIC	$CAIC$	BIC	$HQIC$	KS	p-value	parameter	Estimate	SE	95%CI	
Gharaibeh	268.9182	498.5167	498.6809	504.66	501	0.79057	2.23E-05	θ	1.40952	0.07806	1.256519	1.562515
ABOPLED	214.7046	247.6625	247.8083	249.848	248.5	0.108	0.5671	θ	2.45937	0.148823	2.16768	2.751065
Karam	220.2601	254.0115	254.1574	256.197	254.8	0.1224	0.409	θ	0.926	0.162	0.60848	1.24352
OPLED	240.4058	246.3666	248.4974	250.326	249.1	0.19233	0.043	θ	0.0924	0.009725	0.07334	0.111461
Exponential	254.1606	412.3661	412.4869	415.313	413.5	0.523845	0.000126	θ	0.20058	0.027815	0.146062	0.255096
Lindley	283.1807	493.5926	493.9845	499.837	496	0.835538	2.99E-07	θ	0.89039	0.055254	0.782089	0.998684
ABBD	339.3911	546.5599	546.8783	551.633	548.5	0.952397	2.42E-12	α	1.32772	0.131967	1.069067	1.586376
								β	0.59652	0.047479	0.503466	0.689583
LBBD	292.5332	519.4088	519.8325	526.16	522	0.42493	4.91E-09	α	13.4936	1.916729	9.736853	17.25043
								β	0.79805	0.093192	0.615392	0.980704
SBID	342.8647	555.0833	555.1819	557.489	556	0.910601	1.46E-12	θ	0.8271	0.215	0.405702	1.248502

Table 9. Application II

Distribution	$-ln(L)$	AIC	$CAIC$	BIC	$HQIC$	KS	p-value	parameter	Estimate	SE	95%CI	
Gharaibeh	131.4238	264.8477	264.9074	267.0818	265.734	0.40153	4.35E-10	θ	1.100993	0.062543	0.978409	1.223577
ABOPLED	629.0371	1262.074	1262.198	1267.284	1264.183	0.095631	0.3198	θ	0.129883	0.011658	0.107032	0.152733
Karam	91.09243	184.1849	184.2446	186.419	185.0712	0.26513	0.000123	θ	1.97627	0.106528	1.767475	2.185065
OPLED	89.82951	181.659	181.7187	183.8931	182.5454	0.28699	2.32E-05	θ	1.408618	0.078005	1.255727	1.561509
Exponential	130.9789	263.9578	264.0175	266.1919	264.8441	0.4477	1.94E-12	θ	0.407271	0.049029	0.311173	0.503368
Lindley	119.311	240.622	240.6817	242.8561	241.5083	0.40044	4.90E-10	θ	0.653587	0.049124	0.557304	0.74987
ABBD	396.0404	796.0808	796.2547	800.6341	797.8935	0.15922	0.05196	α	4.546753	1.956473	0.712066	8.38144
								β	0.032501	0.001715	0.02914	0.035863
LBBD	116.3911	234.7822	234.9201	236.2161	235.2496	0.14041	0.1951	α	2.444609	1.21023	0.072558	4.816659
								β	0.008821	0.000642	0.007563	0.010078
SBID	96.31876	194.6375	194.6972	196.8716	195.5239	0.29669	1.06E-05	θ	1.353036	0.072632	1.210677	1.495395

Table 10. Application III

Distribution	$-ln(L)$	AIC	$CAIC$	BIC	$HQIC$	KS	p-value	parameter	Estimate	SE	95%CI	
Gharaibeh	190.437	382.016	382.04	383.505	382.619	0.15386	1.76E-02	θ	0.53498	0.13245	0.27537	0.79458
ABOPLED	63.473	127.746	127.771	128.788	128.168	0.04221	0.6136	θ	0.40425	0.05325	0.29988	0.50862
Karam	268.019	537.53	537.56	539.474	538.317	0.2158	0.000123	θ	0.59548	0.16864	0.26495	0.92602
OPLED	333.624	669.248	669.289	671.854	670.303	0.92175	$< 2.2E-16$	θ	0.63064	0.15134	0.33401	0.92727
Exponential	243.719	488.92	488.95	490.85	489.701	0.17301	5.02E-03	θ	0.10125	0.04523	0.0126	0.1899
Lindley	74.541	149.55	149.56	150.159	149.797	0.06768	0.5881	θ	0.18657	0.06413	0.06089	0.31226
ABBD	321.539	647.077	647.201	652.287	649.186	0.44059	$< 2.2E-16$	α	4.87299	0.75101	3.401	6.34498
								β	0.41435	0.05011	0.31614	0.51256
LBBD	144.212	290.243	290.299	292.611	291.201	0.12309	9.66E-02	α	3.95669	1.67379	0.67606	7.23733
								β	0.23114	0.03585	0.16087	0.3014
SBID	188.221	377.572	377.595	379.046	378.169	0.15845	1.32E-02	θ	0.40413	0.13112	0.14713	0.66112

10. Conclusions

In this paper, we propose the area-biased one-parameter linear exponential distribution. The main properties of this distribution are derived such as the moments and the related measures, the harmonic mean and the mode. The reliability analysis functions are derived along with the pdfs of the minimum, maximum and the k^{th} order statistics; additionally, the quantile function; additionally, the mean absolute deviations from the mean and the median jointly with the mean waiting and residual lifetime. A simulation study using the MLE, OLS, WLS, MPS, CVM and AD methods of estimating parameters is conducted showing that the estimators are unbiased and consistent. Three real financial data applications prove the goodness of fit for this distribution. They show that the suggested distribution fits the real data better than the competence distributions.

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References

Al-Omari, A., Al-Nasser, A., and Ciavolino, E., (2019). A Size-Biased Ishita Distribution and Application to Real Data. *Quality & Quantity: International Journal of Methodology*, 53(1), pp. 493–512.

Al-Omari, A., Alanzi, A., (2021). Inverse Length Biased Maxwell Distribution: Statistical Inference with an application . *Computer Systems Science and Engineering*, 39(1), pp. 147–164.

Al-Omari, A., Alhyasat, K., Ibrahim, K. and Abu-Bakar, M., (2019). Power length biased Suja distribution: properties and application . Electronic Journal of App.ied Statistical Analysis, 12(2), pp. 429–452.

Almakhareez, S., Alzoubi, L., (2024). Extension to Benrabia Distribution with applications and Parameter Estimation. *RESEARCH IN STATISTICS*, 2(1).

Almakhareez, S. and Alzoubi, L., (2024). Length-Biased Benrabia Distribution with applications. *Electronic Journal of App.ied Statistical Analysis (EJASA)*, 17(2), pp. 255–277.

Alzoubi, L., Gharaibeh, M., Alkhazaalh, A. and Benrabia, M., (2022). Estimation of Sameera Distribution Parameters with applications to Real Data. *App.ied Mathematics Information Sciences*, 16, pp. 1057–1071.

Alzoubi, L., Gharaibeh, M., Alkhazaalh, A. and Berabia, M., (2022). Loai distribution: Properties, parameters estimation and application to covid-19 real data. *Mathematical Statistician and Engineering applications*, 7(4), pp. 1231–1255.

Anderson, T., (1962). Distribution of the two-sample cramer–von mises criterion. *Annals of Mathematical Statistics*, 33(3), pp. 1148–1159.

ASE, (2021-2023). Amman stock exchange. Online.

Au, M., Agba, B. and Gagnon, F., (2015). Fast identification of partial discharge sources using blind source separation and kurtosis. *Electronics Letters*, 51(25), pp. 2132–2134.

Benrabia, M., Alzoubi, L., (2022). Alzoubi Distribution: Properties and applications. *Journal of Statistics applications & Probability, An International Journal*, 11(2), pp. 625–640.

Benrabia, M., Alzoubi, L., (2022). Benrabia Distribution: Properties and applications. *Electronic Journal of App.ied Statistical Analysis*, 15(2), pp. 300–317.

Cheng, R. and Amin, A., (1983). Estimating Parameters in Continuous Univariate Distributions with a Shifted Origin. *Journal of the Royal Statistical Society. Series B (Methodological)*, 45(3), pp. 394–403.

Cox, D., (1968). *Some Sampling Problems in Technology*. In: Johnson, N.L. and Smith, H., Eds., New Developments in Survey Sampling.

Cramér, H., (1928). On the Composition of Elementary Errors. *Scandinavian Actuarial Journal*, 1, pp. 13–74.

Dang, N. , Dam, H., Dinh, A. and Le, T., (2024). The factors influencing the stock prices of construction industry enterprises: An empirical study in Vietnam. Revista de Gestão Social e Ambiental, 18(3), pp. 1–17.

Das, K., Roy, T., (2011). On Some Length-Biased Weighted Weibull Distribution. *Advances in App.ied Science Research*, 2, pp. 465–475.

Ferniawan, M., Kusumawati, A. and Afdal, (2024). The influence of earnings per share (eps), price earnings ratio (per), price to book value (pbv), and debt equity ratio(der) on the stock return. *Akrual: Jurnal Bisnis dan Akuntansi Kontemporer*, 17(1), pp. 114–130.

Fisher, R., (1934). The Effect of Methods of Ascertainment upon the Estimation of Frequencies. *The Annals of Human Genetics*, 6, pp. 13–25.

Gharaibeh, M., Sahtout, M., (2022). Karam distribution: A new lifetime distribution with real data application. *Journal of App.ied Probability & Statistics*, 17(1), pp. 71–85.

Gharaibeh, M., (2021). Gharaibeh Distribution and Its applications. *Journal of Statistics applications & Probability*, 10(2), pp. 441–452.

Ghitany, M., Atieh, B. and Nadarajah, S., (2008). Lindley Distribution and its applications. *Mathematics and Computers in Simulation*, 78, pp. 493–506.

IAS, (2021–2023). International accounting standards (ias,33).

Jeong, J., Jung, S., and Costantino, J., (2008). Nonparametric inference on median residual life function. *Biometrics*, 64(1), pp. 157–163.

Joe, H., Proschan, F. (1984a). Comparison of Two Life Distributions on the Basis of Their Percentile Residual Life Functions. *The Canadian Journal of Statistics / La Revue Canadienne de Statistique*, 12(2), pp. 91–97.

Joe, H., Proschan, F., (1984). Percentile Residual Life Functions. *Operations Research*, 32(3), pp. 668–678.

Kingman, J., (1982). The coalescent. *Stochastic Processes and their applications*, 13(3), pp. 235–248.

Kristanti, F. , Nurhayati, I. and Fariska, P., (2024). Financial performance's impact on firm value: How managerial ownership mediates the relationship. *Journal of Ecohumanism*, 3(6), pp. 204–216.

Lillo, R., (2005). On the Median Residual Lifetime and its Aging Properties: A Characterization Theorem and applications. *Naval Research Logistics (NRL)*, 52(4), pp. 370–380.

Manikandan, S., (2011). Measures of central tendency: Median and mode. *Journal of Pharmacology and Pharmacotherapeutics*, 2(3), pp. 214–215.

Olbryś, J., Ostrowski, K., (2021). An Entropy-Based App.oach to Measurement of Stock Market Depth. *Entropy*, 23(5).

Otsuka, A., Nakano, K. and Miyakita, K. (2010). Theoretical Analysis of Mean Waiting Time for Message Delivery in Lattice Ad Hoc Networks. *Journal of Circuits, Systems and Computers*, 19(08), pp. 1711–1741.

Patil, G., Ord, J., (1976). On size-biased sampling and related form-invariant weighted distributions. Sankhya: The Indian Journal of Statistics, 38(1), pp. 48–61.

Pollaczek, F., (1957). *Problémes stochastiques posés par le phénoméne de formation d´une queue d´attente á un guichet et par des phénoménes app.rentés*. Gauthier-Villars.

R Core Team, (2021). *R: A Language and Environment for Statistical Computing*. R Foundation for Statistical Computing, Vienna, Austria.

Rao, C., (1965). On Discrete Distributions Arising out of Methods of Ascertainment. *Sankhya: The Indian Journal of Statistics, Series A*, (1961-2002), 27(2/4), pp. 311–324.

Rényi, A., (1961). On Measures of Entropy and Information. Proceedings of the 4th Berkeley *Symposium on Mathematical Statistics and Probability*, 1, pp. 547–561.

Romero-Silva, R., Hurtado, M., (2017). The difference of mean waiting times between two classes of customers in a single-server fifo queue: An experimental study. *Cogent Engineering*, 4(1), pp. 500–517.

Rosenberg, L. (1968). Mean Waiting Time as a Measure of Effectiveness. *The American Statistician*, 22(4), pp. 31–34.

Saghir, A., Khadim, A. and Lin, Z., (2017). The Maxwell-length-biased distribution: Properties and Estimation. *Journal of Statistical Theory and Practice*, 11(1), pp. 26–40.

Sah, B., (2021). One-Parameter Linear-Exponential Distribution. *International Journal of Statistics and App.ied Mathematics*, 6(6), pp. 6–15.

Schmittlein, D., Morrison, D., (1981). The Median Residual Lifetime: A Characterization Theorem and an application. *Operations Research*, 29(2), pp. 392–399.

Shannon, C., (1948). A Mathematical Theory of Communication. *The Bell System Technical Journal*, 27:379–423, pp. 623–656.

Sharma, V., Dey, S., Singh, S. and Manzoor, U., (2018). On length and area biased maxwell distributions . *Communications in Statistics - Simulation and Computation*, 47(5):1506– 1528.

Shen, Y., Ning, J. and Qin, J. (2009). Analyzing length-biased data with semiparametric transformation and accelerated failure time models. *Journal of the American Statistical Association*, 104(487), pp. 1192–1202.

Swain, J., Venkatraman, S. and Wilson, J., (1988). Least-squares estimation of distribution functions in johnson's translation system. Journal of Statistical Computation and Simulation, 29(4), pp. 271–297.

Taubah, I., Roslina, N. Y., Damayanti, I., and Tansar, I., (2024). The influence of earning per share (eps) and price earnings ratio (per) on stock prices in the food and beverage sector on the indonesian stock exchange (bei) for the 2015-2019 period. *Journal of Economics, Management, and Entrepreneurship*, 2(1), pp. 49–63.

Tsallis, C., (1988). Possible generalization of Boltzmann-Gibbs statistics. Journal of Statistical Physics, 52, pp. 479–487.

Von Mises, R., (1928). *Wahrscheinlichkeit, Statistik und Wahrheit*. Julius Springer.

Zamanzade, E., Zamanzade, E. and Parvardeh, A., (2024). Estimation of a decreasing mean residual life based on ranked set sampling with an application to survival analysis. *The international journal of biostatistics*, 20(2), pp. 571–583. https://doi.org/10.1515/ijb-2023-0051

Zelen, M., (1974). Problems in cell kinetics and the early detection of disease. *Reliability and Biometry, SIAM*, 56(3), pp. 701–706.